Contents

# INTRODUCTION TO STATISTICAL THEORY

## Part 1

(A text book for Degree and Post-Graduate Students)

By

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24th February 1972

Dear Professor Sher Mohammad Chowdhry,

I am very thankful to you for a gift of your books which I received a few days ago. I have found the books very well written and of great use to the student community and would like to congratulate you for producing such standard text books on "statistics".

With kind regards.

Yours sincerely,

A. M. Rajput

A. M. Rajput.
Senior Lecturer

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# CHAPTER 1

# INTRODUCTION

# INTRODUCTION

## 1.1 MEANING OF STATISTICS

People view Statistics in many different ways. Generally it is considered to be a subject that deals with percentages, charts, graphs, averages and tables. Some people think that Statistics is a subject consisting of rules, methods and techniques of collecting and presenting large amount of numerical information, while other people think that it is a subject of making inferences about the population on the basis of sample information.

The word "Statistics" which comes from the Latin word *status*, meaning a political state, originally meant information useful to the state, for example, information about the sizes of populations and armed forces. But this word has now acquired different meanings.

In the *first place*, the word *statistics* refers to **"numerical facts systematically arranged"**. In this sense, the word statistics is always used in the plural. We have, for instance, statistics of prices, statistics of road accidents, statistics of crimes, statistics of births, statistics of educational institutions, etc. In all these examples, the word statistics denotes a set of numerical data in the respective fields. This is the meaning the man in the street gives to the word *Statistics* and most people usually use the word *data* instead.



**Example 1.1** In the following examples, the facts and figures usually called Statistics presented in the media almost every day are given:

i) Children who brush their teeth with brand XYZ toothpaste have 60% fewer countries.

ii) The Bureau of census projects the population of Pakistan to be 170.1 million in the year 2010.

iii) Eight out of ten Pakistanis do not have skills.

iv) The prevalence of diabetes is nearly 3 times as high in over weight people as compared to normal weight people.

v) In 1980 it was estimated that 0.1% of people had tried any sort of drug; where as in 2008 it was estimated that 10% had done so.

In the *second place*, the word *statistics* is defined as a discipline that includes procedures and techniques used to collect, process and analyse numerical data to make inferences and to reach decisions in the face of uncertainty. It should of course be borne in mind that uncertainty does not imply ignorance but it refers to the incompleteness and the instability of data available. In this sense, the word statistics is used in the singular. As it embodies more or less all stages of the general process of learning, sometimes called *scientific method*, statistics is characterized as a science. Thus the word *statistics* used in the plural refers to a set of numerical information and in the singular, denotes the science of basing decision on numerical data. It should be noted that statistics as a subject is mathematical in character.

*Thirdly*, the word *statistics* are numerical quantities calculated from sample observations; a single quantity that has been so calculated is called a *statistic*. The mean of a sample for instance is a statistic. The word statistics is plural when used in this sense.

**1.1.1 Use of Statistical Information.** The statistical information are and can be used for a variety of purposes. Some of them are:

i) to inform general public;

ii) to explain things that have happened;

iii) to justify a claim;

iv) to provide general comparisons;

v) to predict the decision regarding future outcomes;.

vi) to estimate the unknown quantities;

vii) to establish association / relationship between factors.

Hence Statistics is a subject which is much more than just numbers. It tells us what is done to e with numbers. The following three examples further explain how Statistics may be used:

**Example 1.2** Suppose we want to determine the best teacher at Govt. College University, Lahor How should we decide this? This could be done by asking Govt. College University students who the bes teacher is. To do so, we collect the data, analyze the results and make the decision. Now variou questions are:

i) should we survey every student?

ii) how will the survey be conducted?

iii) how will the data be analyzed?

iv) how will the best teacher be determined? etc.

In order to answer these and other questions, Statistical techniques are used.



**Example 1.3** A TV station claims that an advertisement of a product on their channel attracts mo customers compared to all other TV channels. Now if this claim is based on data, there it can be used market the TV channel. Suppose we have some doubts about the claim. In order to remove the doubts, w might gather relevant information, analyze the results using appropriate statistical technique and make decision regarding the claim.

**Example 1.4** Suppose University of the Punjab is planning an expansion program of its physic facilities. To draw up an effective course of action, the University authorities decide that it needs answer this question, how many college students will we need to accommodate over the next ten year The question can be further broken down into many smaller questions. How many college students wi then be in the Punjab? How many will want to attend the University of the Punjab? etc. Once aga Statistical methods can assist in evaluating and planning of expansion program.

**1.1.2 Characteristics of Statistics.** The definition stated above indicates that statistics is a subje in its own right. It may therefore be desirable to know the characteristic features of statistics in order appreciate and understand its general nature. Some of its important characteristics are given below:

i) Statistics deals with the behaviour of aggregates or large groups of data. It has nothing to with what is happening to a particular individual or object of the aggregate.

ii) Statistics deals with aggregates of observations of the same kind rather than isolated figures.

iii) Statistics deals with variability that obscure underlying patterns. No two objects in the universe are exactly alike. If they were, there would have been no statistical problem.

iv) Statistics deals with uncertainties as every process of getting observations whether controlled or uncontrolled, involves deficiencies or chance variation. That is why we have to talk in terms of probability.

v) Statistics deals with those characteristics or aspects of things which can be described numerically either by counts or by measurements.

vi) Statistics deals with those aggregates which are subject to a number of random causes, *e.g.* the heights of persons are subject to a number of causes such as race, ancestry, age, diet, habits, climate and so forth.

vii) Statistical laws are valid *on the average* or in the long run. There is no guarantee that a certain law will hold in all cases. Statistical inference is therefore made in the face of uncertainty.

viii) Statistical results might be misleading and incorrect if sufficient care in collecting, processing and interpreting the data is not exercised or if the statistical data are handled by a person who is not well versed in the subject matter of statistics.

1.1.3 **Descriptive and Inferential Statistics.** Statistics as a subject, may be divided into *descriptive statistics and inferential statistics.*

*Descriptive statistics* is that branch of statistics which deals with concepts and methods concerned with summarization and description of the important aspects of numerical data. This area of study consists of the condensation of data, their graphical displays and the computation of a few numerical quantities that provide information about the centre of the data and indicate the spread of the observations.

*Inferential statistics* deals with procedures for making inferences about the characteristics that describe the large group of data or the whole, called the *population*, from the knowledge derived from only a part of the data, known as *sample.* This area includes the estimation of population parameters and testing of statistical hypotheses. This phase of statistics is based on probability theory as the inferences which are made on the basis of sample evidence, cannot be absolutely certain.



**Comparison**

| Descriptive Statistics | Inferential Statistics |
|---|---|
| i) A cricket player wants to find his score average for the last 20 games. | i) A cricket player wants to estimate his chance of scoring based on his current season average. |
| ii) Aamir wants to describe the variation in his four test scores in Statistics. | ii) Based on the first four test scores, Aamir would like to predict the variation in his final Statistics test scores. |
| iii) Mrs. Rashid wants to determine the average weekly amount she spent on groceries in the past 6 months. | iii) Based on last six months grocery bills, Mrs. Rashid would like to predict the average amount she will spend on groceries for the upcoming year. |

**1.1.4 Populations and Samples.** A population or a statistical population is a collection or set of all possible observations whether finite or infinite, relevant to some characteristic of interest. A statistical population may be real such as the heights of all college students or hypothetical such as all the possible outcomes from the toss of a coin. The number of observations in a finite population is called the *size* of the population and is denoted by the letter N. Numerical quantities describing a population are called *parameters,* customarily represented by Greek letters. It is important to note that in statistics the word *population* is a technical term not necessarily referring to all the people in a specified area, rather denoting the aggregate of measurements or counts of some characteristic for the entire group of objects or individuals.

A *sample* is a part or a subset of a population. Generally it consists of some of the observations but in certain situations, it may include the whole of the population. The number of observations included in a sample is called the *size* of the sample and is denoted by the letter $n$. A numerical quantity computed from a sample, is called a *statistic,* which is usually represented by ordinary Latin letter. The information derived from sample data is used to draw conclusions about the population.

**Example 1.5** State whether each of the following is a population or a sample.

i) Total number of absentees by all students in a college during the last month.

ii) Number of colour TV sets owned by all families in Lahore.

iii) Monthly salaries of all employees of a company.

iv) Wheat yield per acre for 5 pieces of land.

v) Number of computers sold during the last month at all the computer stores in Lahore.



**Solution**

i) Population

ii) Population

iii) Population

iv) Sample

v) Population

**1.1.5 Importance of Statistics.** Statistics is perhaps a subject that is used by everybody. The following functions and uses of statistics in most diverse fields serve to indicate its importance.

i) Statistics assists in summarizing the larger sets of data in a form that is easily understandable.

ii) Statistics assists in the efficient design of laboratory and field experiments as well as surveys.

iii) Statistics assists in a sound and effective planning in any field of inquiry.

iv) Statistics assists in drawing general conclusions and in making predictions of how much of a thing will happen under given conditions.

v) Statistical techniques being powerful tools for analysing numerical data, are used in almost every branch of learning. In the biological and physical sciences, Genetics, Agronomy, Anthropometry, Astronomy, Physics, Geology, etc. are the main areas where statistical techniques have been developed and are increasingly used.

vi) A businessman, an industrialist and a research worker all employ statistical methods in their work. Banks, Insurance companies and Governments all have their statistics departments.

vii) A modern administrator whether in public or private sector, leans on statistical data to provide a factual basis for decision.

viii) A politician uses statistics advantageously to lend support and credence to his arguments while elucidating the problems he handles.

ix) A social scientist uses statistical methods in various areas of socio-economic life of a nation. It is sometimes said that "*a social scientist without an adequate understanding of statistics, is often like the blind man groping in a dark room for a black cat that is not there*".

## 1.2 OBSERVATIONS AND VARIABLES

In statistics, an *observation* often means any sort of numerically recording of information, whether it is a physical measurement such as height or weight; a classification such as heads or tails, or an answer to a question such as yes or no.

**1.2.1 Variables.** A characteristic that varies with an individual or an object, is called a *variable*. For example, age is a variable as it varies from person to person. A variable can assume a number of values. The given set of all possible values from which the variable takes on a value, is called its *domain*. If for a given problem, the domain of a variable contains only one value, then the variable is referred to as *constant*.

Variables may be classified into quantitative and qualitative according to the form of the characteristic of interest. A variable is called a *quantitative variable* when a characteristic can be expressed numerically such as age, weight, income or number of children. On the other hand, if the characteristic is non-numerical such as education, gender, eye-colour, quality, intelligence, poverty, satisfaction, etc. the variable is referred to as a *qualitative variable*. A qualitative characteristic is also called an *attribute*. An individual or an object with such a characteristic can be counted or enumerated after having been assigned to one of the several mutually exclusive classes or categories.

**1.2.2 Discrete and Continuous Variables.** A quantitative variable may be classified as discrete or continuous. A *discrete* variable is one that can take only a discrete set of integers or whole numbers, that is the values are taken by jumps or breaks. A discrete variable represents *count* data such as the number of persons in a family, the number of rooms in a house, the number of deaths in an accident, the income of an individual, etc.

A variable is called a *continuous* variable if it can take on any value--fractional or integer--within a given interval, *i.e.* its domain is an interval with all possible values without gaps. A continuous variable represents *measurement* data such as the age of a person, the height of a plant, the weight of a commodity, the temperature at a place, etc.

A variable whether countable or measurable, is generally denoted by some symbol such as $X$ or $Y$ and $X_i$ or $Y_j$ represents that $ith$ or $jth$ value of the variable. The subscript $i$ or $j$ is replaced by a number such as 1,2,3,… when referred to a particular value.

**Example 1.6** Identify each of the following as examples of (1) Attribute, (2) Discrete, or (3) Continuous variables.

a) the hair colour of children;
b) length of time required for a wound to heal;
c) the number of telephone calls arriving at a switch board per 1 hour period;
d) the breaking strength of a given type of a string;
e) the number of questions answered correctly on a test;
f) the number of stop signs in the city of Lahore;
g) the colour of your eye;
h) the number of children in your family.

**Solution**

a) Attribute
b) Continuous
c) Discrete
d) Continuous
e) Discrete
f) Discrete
g) Attribute
h) Discrete



**1.2.3 Measurement Scales.** By *measurement*, we usually mean the assigning of numbers to observations or objects and *scaling* is a process of measuring. The four scales of measurements are briefly mentioned below:

**Nominal Scale.** The classification or grouping of the observations into mutually exclusive qualitative categories or classes is said to constitute a *nominal scale*. For example, students are classified as male and female. Number 1 and 2 may also be used to identify these two categories. Similarly, rainfall may be classified as heavy, moderate and light. We may use number 1, 2 and 3 to denote the three classes of rainfall. The numbers when they are used only to identify the categories of the given scale, carry no numerical significance and there is no particular order for the grouping.

**Ordinal or Ranking Scale.** It includes the characteristic of a nominal scale and in addition has the property of *ordering* or *ranking* of measurements. For example, the performance of students (or players) is rated as excellent, good, fair or poor, etc. Number 1, 2, 3, 4, etc. are also used to indicate ranks. The only relation that holds between any pair of categories is that of "greater than" (or more preferred).

**Interval Scale.** A measurement scale possessing a constant interval size (distance) but not a true zero point, is called an *interval scale*. Temperature measured on either the Celcius or the Fahrenheit scale is an outstanding example of interval scale because the same difference exists between 20°C (68°F) and 30°C (86°F) as between 5°C (41°F) and 15°C (59°F). It cannot be said that a temperature of 40 degrees is twice as hot as a temperature of 20 degree, i.e. the ratio 40/20 has no meaning. The arithmetic operation of addition, subtraction, etc. are meaningful.

**Ratio Scale.** It is a special kind of an interval scale where the scale of measurement has a true *zero* point as its origin. The ratio scale is used to measure weight, volume, length, distance, money, etc. The key to differentiating interval and ratio scale is that the *zero* point is meaningful for ratio scale.

**Example of Measurement Scales**

| Nominal-level data | Ordinal-level data | Internal-level data | Ratio-level |
|---|---|---|---|
| Gender (Male, Female) | Grades (A, B, C, D, F) | Temperature | Age |
| Eye colour | Position (1st, 2nd, 3rd etc.) | IQ score | Weight |
| Religion | Ranking of cricket player | SAT score | Height |
| Specialization | Rating (poor, good, excellent) | | Time |
| Nationality | Socio-economic status (poor, middle class, rich) | | Salary |
| | | | Distance |

**1.2.4 Errors of Measurement.** Experience has shown that a continuous variable can never be measured with perfect fineness because of certain habits and practices, methods of measurements, instruments used, etc. The measurements are thus always recorded correct to the nearest units and hence are of limited accuracy. The *actual* or *true* values are, however, assumed to exist. For example, if a student's weight is recorded as 60 kg (correct to the nearest kilogram), his true weight in fact lies between 59.5 kg and 60.5 kg, whereas a weight recorded as 60.00 kg means the true weight is known to lie between 59.995 and 60.005 kg. Thus there is a difference, however small it may be, between the measured value and the true value. This sort of departure from the true value is technically known as the *error of measurement*. In other words, if the observed value and the true value of a variable are denoted by $x$ and $x+\varepsilon$ respectively, then the difference $(x+\varepsilon)-x$, *i.e.* $\varepsilon$ is the error. This error involves the unit of measurement of $x$ and is therefore called an *absolute error*. An absolute error divided by the true value is called the *relative error*. Thus the relative error $=\dfrac{\varepsilon}{x+\varepsilon}$, which when multiplied by 100, is *percentage error*. These errors are independent of the units of measurement of $x$. It ought to be noted that an error has both magnitude and direction and that the word *error* in statistics does not mean mistake which is a chance inaccuracy.

An error is said to be *biased* when the observed value is consistently and constantly higher or lower than the true value. Biased errors arise from the personal limitations of the observer, the imperfection in the instruments used or some other conditions which control the measurements. These errors are not revealed by repeating the measurements. They are cumulative in nature, that is, the greater the number of measurements, the greater would be the magnitude of error. They are thus more troublesome. These errors are also called *cumulative* or *systematic errors*.

An error, on the other hand, is said to be *unbiased* when the deviations, *i.e.* the excesses and defects, from the true value tend to occur equally often. Unbiased errors are revealed when measurements are repeated and they tend to cancel out in the long run. These errors are therefore *compensating* and are also known as *random errors* or *accidental errors*.

A measurement free from all classes of errors is considered as an accurate measurement. This is why efforts are made to reduce the magnitude of errors to a minimum so that the level of accuracy at

which the measurements are recorded, is increased. To achieve this end, a clear understanding of the meaning of *significant digits* and the process of *rounding off* the numbers is very important in statistical computations.

**1.2.5 Significant Digits.** Accuracy in measurements is related to significant digits. The significant digits in a number, are those that represent accurate and meaningful information. For instance, the number 35 representing a continuous variable has two significant digits. In recorded measurements, all digits except zeros are always significant. For zeros, we may state as:

i) Zeros are significant if they follow a decimal point and conclude a number, e.g. the measurement 2.500 has four significant digits.

ii) Zeros are non-significant when they follow a decimal point but commence a number, *e.g.* the measurements .04 and .000237 contain only 1 and 3 significant digits respectively.

iii) Zeros may or may not be significant when they lie entirely to the left of the decimal point, where they may not represent measurement but may be used to simply locate the decimal point. In such a case, a definite specification such as standard notation, becomes necessary. When any number is expressed as a product of a power of 10 and a number between 1 and 10, it is said to be written in standard notation. For example, the number 75400 can have 3 significant digits when written in standard notation as $7.54 \times 10^4$. It can also have 5 significant digits if written as $7.5400 \times 10^4$.

iv) Zeros are always significant when occur within a series of significant digits, *e.g.* the numbers 20.3, .1001, 4.00507, etc., have 3, 4 and 6 significant digits respectively.

It should be remembered that

a) significant digits in a number are not disturbed by the location of the decimal point, *e.g.* the measurements recorded as 269., 26.9, .269 or .000269 have only 3 significant digits;

b) in case of discrete data which are generated by the process of counting, the number of significant digits is considered indefinite because the level of accuracy cannot be improved, *e.g.* the number 15700 has indefinite significant digits;

c) the rules regarding the determination of the number of significant digits, are applicable to continuous variables;

d) in the operations of addition and subtraction, all digit positions which are not significant in any of the values being added or subtracted, are not significant in the total or difference;

e) in the operations of multiplication and division, the number of significant digits in the result is determined by the value with the smallest number of significant digits that enters into the calculations.



**1.2.6 Rounding off a Number.** The process of *rounding off* or simply *rounding* a number means that a certain number of digits counted from the left, are to be retained and the last few digits are to be (i) dropped in a decimal number or (ii) replaced with zeros in a whole number. The rules generally used for rounding decimal numbers are as follows:

i) The last significant digit is increased by 1, if the first digit of the remainder to be dropped is more than 5 or 5 followed by digits not all of which are zero, *e.g.* the numbers 2.145001 and 5.3772 are rounded off to three significant digits as 2.15 and 5.38 respectively.

ii) The last significant digit remains unaltered, if the first digit of the remainder to be dropped is 4 or less, *e.g.* the numbers 2.1548 and 7.3627 are rounded off to three significant digits as 2.15 and 7.36 respectively.

iii) When the digit to be dropped is exactly 5, the accepted practice is to increase the last significant digit by 1, if it is odd and to leave unaltered if it is even *e.g.* the number 4.535 and 2.745 are rounded off to three significant digits as 4.54 and 2.74 respectively.

For rounding whole numbers, we can change the word "the first digit to be dropped" to "the first digit to be replaced by zero" in the rules stated above.

The point to be made here is the rules for identifying significant digits and the process of rounding the numbers should be applied to final calculations and not to the intermediate results.

## 1.3 COLLECTION OF DATA

The most important part of statistical work is perhaps the collection of data. Statistical data are collected either by a *complete* enumeration of the whole field, called *census*, which in many cases would be too costly and too time consuming as it requires large number of enumerators and supervisory staff, or by a *partial* enumeration associated with a sample which saves much time and money. The sampling methods explained at length in later chapters, are increasingly employed both in official and in private inquires to collect data.

When data are classified according to *source*, it is customary to make the following distinction.

Data that have been originally collected (*raw data*) and have not undergone any sort of statistical treatment, are called *Primary data*, while data that have undergone any sort of treatment by statistical methods at least once, *i.e.* the data have been collected, classified, tabulated or presented in some form for a certain purpose, are called *Secondary data*.

The survey research includes the following important steps:

a) to define the objectives of the survey.

b) to define the variable(s) and the population of interest.

c) to define the data collection and data measuring schemes.

d) to determine the appropriate descriptive and inferential data analysis techniques.

A brief description of the methods generally adopted either on census basis or on sample basis for collecting data, is given below.

**1.3.1 Collection of Primary Data.** One or more of the following methods are employed to collect primary data:

**Survey Research**

i) **Direct Personal Investigation.** In this method, an investigator collects the information personally from the individual concerned. Since he interviews the informants himself, the information collected is generally considered quite accurate and complete. This method may prove very costly and time consuming when the area to be covered is vast. However it is useful for laboratory experiments or localized inquires. Errors are likely to enter the results due to personal bias of the investigator.

ii) **Indirect Investigation or Personal Interviews.** Sometimes the direct sources do not exist or the informants hesitate to respond for some reasons or other. In such a case, third parties or witnesses having information are interviewed. As some of the informants are likely to deliberately give wrong information, so the reliance is not placed on the evidence of one witness only. Moreover, due allowance is to be made for the personal bias. This method is useful when the information desired is complex or there is reluctance or indifference on the part of the informants. It can be adopted for extensive inquires.

iii) **Collection through Questionnaires.** A questionnaire is an inquiry form comprising of a number of pertinent questions with space for entering information asked. The questionnaires are usually sent by mail and the informants are requested to return the questionnaires to the investigator, after doing the needful within a certain period. This method is cheap, fairly expeditious and good for extensive inquires. But the difficulty is that the majority of respondents (persons who are required to answer the questions) does not care to fill the questionnaires in and to return them to the investigators. Sometimes, the questionnaires are returned incomplete and full of errors. In spite of these drawbacks, the method is considered as the standard method for routine business and administrative inquires. The answers to the questionnaires are very often recorded by trained enumerators to overcome the difficulties these days. It is important to note that the questions should be few, brief, very simple, easy for all respondents to answer, clearly worded and not offensive to certain respondents.

iv) **Collection through Enumerators.** Under this method, the information is gathered by employing trained enumerators who assist the informants in making the entries in the schedules or questionnaires correctly. This method gives the most reliable information if the enumerator is well trained, experienced and tactful. It is considered the best method when a large scale governmental inquiry is to be conducted. This method cannot be adopted by private individual or institution as its cost would be prohibitive to them.

v) **Collection through Local Sources.** In this method, there is no formal collection of data but the agents or local correspondents are directed to collect and to send the required information, using their own judgment as to the best way of obtaining it. This method is cheap and expeditious, but gives only the estimates.

vi) **Computer interviews.** Respondents enter data directly into a computer in response to questions presented on the monitor.

**Experimental Research**

i) **Laboratory experiments.** Manipulation of the independent variable(s) in an artificial situation. Basic designs consider the impact of only one independent variable.

ii) **Field experiments.** Manipulation of the independent variable(s) in a natural situation.

**1.3.2 Collection of Secondary Data.** The secondary data may be obtained from the following sources:

**Secondary Research**

**Internal Secondary Data.** Data generated within the organization itself, such as salesperson reports, sales invoices, accounting records.

**External Secondary Data**

i) **Official,** *e.g.* the publications of the Statistical Division, Ministry of Finance, the Federal and Provincial Bureaus of Statistics, Ministries of Food, Agriculture, Industry, Labour, etc.

ii) **Semi-Official,** *e.g.* State Bank of Pakistan, Railway Board, Central Cotton Committee, Boards of Economic Inquiry, District Councils, Municipalities, etc.

iii) Publications of Trade Association, Chambers of Commerce, etc.

iv) Technical and Trade journals and newspapers.

v) Research organisation such as universities, and other institutions.

**1.3.3 Editing of Data.** The primary data should be intensively checked at an early stage in order to locate incomplete or inconsistent entries. If possible, the incomplete and defective questionnaires should be returned to the respondents for amendments. In order to accept the secondary data as authoritative, one should critically examine the reliability of the compiler and the suitability of the data. The scope and object of the inquiry, sources of information and the degree of accuracy should also be carefully scrutinized.

**1.3.4 Uses and Misuses of Statistics.** Statistics has numerous uses. It is difficult to find a field in which Statistics is not used. Statistics plays integral part in many disciplines, viz: Economics, Health, Planning, Astronomy, Management, Business, Psychology, Agriculture, Sociology, Education etc.

A few examples of how and where Statistics is used is as under:

i) In experimental science, the experiments generate data, it must be collected and analyzed.

ii) In Government, many types of statistical data are collected all the time. This data can be used for various types of planning and also to inform the general public.

iii) In education, Statistics are used to describe the results and standards of education.

Statistical techniques are many times misused: to sell products that don't work; to prove something that is not really true, to get the attention of public by evoking fear and shock etc. There are two sayings about Statistics which explains the misuses of Statistics.

a) Statistics can prove anything.

b) 'There are three types of lies – lies, damned lies, and Statistics'

Statistics can also be misused in many ways such as using Not Representative Samples, Small Sample Size, Ambiguous Averages and dispersions, Detached facts, Implied Connections, Wrong and Misleading Graphs, Wrong use of Statistical techniques, Serious violation of assumptions behind the Statistical techniques and Faculty Surveys etc.

## EXERCISES

### OBJECTIVE

Answer 'True' and 'False'. If the statement is not true then replace the underlined words with words that make the statement true:

i) All numerical data <u>are not</u> Statistics.

ii) A <u>Statistic</u> is a summary measure computed for a population.

iii) A <u>Parameter</u> is a summary measure computed for a sample.

iv) <u>Descriptive</u> Statistics are used to make projections or estimates about the population.

v) <u>Inferential</u> Statistics is the study and description of data.

vi) A <u>sample</u> is typically a very large collection of individuals or objects of our interest.

vii) The thickness of the glass is an example of <u>attribute</u> data.

viii) The number of students in a class is an example of <u>continuous</u> data.

ix) The make of a car is an example of <u>discrete</u> data.

x) The main objective of <u>Statistics</u> is to collect a sample, analyze it and make inferences about the unknown characteristics of the population from which the sample has been drawn.

## SUBJECTIVE

1.1 Explain what is meant by Statistics. Give the important uses and limitations of statistics.

1.2 Define Statistics. Discuss, giving examples, the importance of the study of statistics and show how it can help the extension of scientific knowledge. (P.U., B.A. (Hons.), 1960)

1.3 a) What do you understand by the term Statistics? Give its chief characteristics.

b) Give a brief account of the importance of statistics in different fields. (P.U., B.A./B.Sc. 1971)

1.4 Comment upon the following statement of Sir Ronald Fisher: "Statistics may be regarded as (i) the study of populations, (ii) the study of variation and (iii) the study of the methods of reduction of data". (P.U., B.A. (Hons.), Part-I, 1964)

1.5 Comment on the statement given below:

"Statistics is concerned with understanding the real world through the information that we derive from classification and measurement. Its distinctive characteristic is that it deals with variability and uncertainty which is everywhere".

1.6 a) Define Statistics and explain its characteristics.

b) What are the uses of Statistics? (P.U., B.A. (Hons. in Econ.), 1960)

1.7 Explain the difference between the following?

a) Statistics and Statistic.

b) Population and Sample.

c) Descriptive and Inferential Statistics.

d) Quantitative and Qualitative Variables.

e) Discrete and Continuous Variables.

f) Biased and Unbiased errors.

g) Primary data and Secondary data.

h) Nominal and ordinal scale, Interval and ratio scale.

1.8 What is a statistical error? In what way does it differ from a mistake? Explain the difference between absolute and relative errors.

1.9 a) Define a Variable. Differentiate between a discrete and a continuous variable.

b) Classify the following variables as discrete or continuous:

i) The number of students attending a class.

ii) The amount of milk produced by a cow.

iii) The number of heads in the toss of 6 coins.

iv) The yearly income of a College Professor.

v) The age of a shopkeeper.

vi) The weight of a college student.

vii) The number of petals on a flower.

viii) The life times of television tubes produced by a company.

ix) Temperature recorded every half hour at a weather bureau.

x) The number of shares sold each day in the stock market.



1.10 Classifv each variable as qualitative or quantitative

i) Colour of eyes.

ii) Number of computers sold in the last month.

iii) Marital status of faculty members.

iv) Student's weight.

v) Lifetime of car batteries.

vi) Number of burgers sold by a fast food shop.

vii) Brand of cars.

1.11 Classify each as nominal-level, ordinal-level, internal-level or ratio-level measurement.

i) Weights of cars

ii) Rankings of squash players.

iii) Temperature of the city.

iv) Salaries of the top five executives in bank.

v) Marital status.

vi) Ages of students.

vii) Ratings of five players (Poor, Fair, Good, Excellent)

viii) IQ of student.

ix) Rating of movies

x) Weights of suitcases on plane.

1.12 Round off the following continuous data to four significant digits each.
(i) 32.21705, (ii) 937.05002, (iii) 0.003599499,
(iv) 1.003599499, (v) 0.07000455, (vi) 22.2500001.

1.13 a) Distinguish between Primary and Secondary data, giving examples of each.

b) Describe the methods which can be used in the collection of statistical data, stating the advantages and disadvantages of each method.

c) Enumerate the main sources of errors in Statistics and give their effects.

(P.U., B.A./B.Sc., 1982-S)

1.14 What methods would you employ in the collection of statistical data when the field of inquiry is (i) small, (ii) fairly large and (iii) very large, if you are to pay due regard to accuracy, labour, cost and time?

1.15 What are the different methods employed in the collection of data for statistical enquiries? In what type of inquiry should each one of them be used? (P.U., B.A./B.Sc., 1961)

1.16 Select a newspaper or a magazine article that involves a statistical study and answer the following questions.

a) Is this study descriptive or inferential? Explain your answer.

b) What variables are used in the study?

c) What level of measurements was used to obtain data?

d) Is population defined in the article? If not, how could it be defined?

e) How the data might have been collected?

f) Do you agree with the conclusions given in the article?

◆◆◆◆◆◆◆◆◆◆◆

# CHAPTER 2

# PRESENTATION OF DATA

# PRESENTATION OF DATA

## 2.1 INTRODUCTION

The device of gathering data often results in a massive volume of statistical data, which are in the form of individual measurements or counts. It is difficult to learn anything by examining the unorganised data which is more often confusing than clarifying. The mass of data is therefore to be organised and condensed into a form that can be more rapidly and easily understood and interpreted. For this purpose, techniques of classification, tabulation and graphic displays are presented in this chapter.

## 2.2 CLASSIFICATION

The term *classification* is defined as the process of dividing a set of observations or objects into classes or groups in such a way that (i) observations or objects in the same class or group are similar, (ii) observations or objects in each class or group are dissimilar to observations or objects in other class or group. Classification is thus the sorting of data into homogeneous classes or groups according to their being alike or not. When the data are sorted according to one criterion only, it is called a *simple classification* or a *one-way classification*. Classification is called a *two-way classification* when the data are sorted according to two criteria. A manifold classification or cross-classification is made according to several criteria.

Data may also be classified according to qualitative, temporal and geographical characteristic. Arrangement of data according to the values of a *variable* characteristic is called a *distribution*. When the classifying variable is expressed in terms of location, we get a *spatial* or *geographical* distribution. Temporal arrangement of values is referred to as a *time series*.



**2.2.1 Aims of Classification.** The main aims of classification are:

(i) to reduce the large sets of data to an easily understood summary;

(ii) to display the points of similarity and dissimilarity;

(iii) to save mental strain by eliminating unnecessary details;

(iv) to reflect the important aspects of the data; and

(v) to prepare the ground for comparison and inference.

**2.2.2 Basic Principles of Classification.** While classifying large sets of data, the following points should be taken into consideration.

(i) The classes or categories into which the data are to be divided, should be *mutually exclusive* and no overlap should exist between successive classes. In other words, classes should be arranged so that each observation or object can be placed in one and only one class.

(ii) The classes or categories should be all inclusive. All inclusive classes are classes that include all the data.

(iii) As far as possible, the conventional classification procedure should be adopted.

(iv) The classification procedure should not be so elaborate as to lead to trivial classes nor it should be so crude as to concentrate all the data in one or two classes.

## 2.3 TABULATION

By *tabulation*, we mean a systematic presentation of data classified under suitable heads and subheads, and placed in columns and rows. This sort of logical arrangement makes the data easy to

understand, facilitates comparisons and provides an effective way to convey information to a reade British statistician, Professor Bowley (1869-1957), refers to *tabulation* as "*the intermediate pr between the accumulation of data, in whatever form they are obtained, and the final reasoned acco the results shown by the statistics.*"

**2.3.1 Types of Tables.** Statistical tables classified according to purpose, are of two types General purpose (primary) tables and Specific purpose (derived or text) tables. The general purpose are large in size, are extensive with vast coverage and are constructed for reference purposes. The spe purpose tables are simpler in structure and deal with one or two criteria of classification only. Such are used to analyse or to assist in analysing data.

When the classification corresponds to one, two or many criteria or characteristics, the tabula called a *single, double* or *manifold tabulation* respectively. Tabulation of a dependent variable number of students) against the independent variable (say, weight) provides an example of a s tabulation. Tables with two criteria of classification, *e.g.* gender and marital status or height and we etc. are examples of double tabulation. An example of manifold tabulation is the presentation of population of a country by age, by gender, by residence, by literacy, by livelihood classes, etc.

The main parts of a statistical table are the title, the boxhead, the stub, the body, one or prefatory notes, footnotes and a source, etc. They are described in the next section.

**2.3.2 Main Parts of a Table and its Construction.** The main parts of a table and the general to be observed in constructing any table are described below.

a) **Title.** A table must have a self-explanatory title which should usually tell us the "*what, w how classified* and *when*" of the data, in that order. Some other important points are stated below:

i) Titles should be brief in the form of phrases. Complete sentences are unnecessary.
ii) Abbreviations should not be used.
iii) Main titles should be in capitals throughout. Sub-titles, if any, should be in lower case le with major words capitalized and should indicate clearly what the table describes.
iv) The different parts of a title should be separated by commas but no full-stop at the end.
v) Words in titles should not be hyphenated except when really necessary.
vi) If a title necessitates the use of two or more lines, an *inverted pyramid* arrangement of lines should be used.

b) **Column Captions and Boxhead.** The heading of each column is called a *Column Capt* while the section of a table that contains the column captions, is referred to as *Boxhead*. Points to n here are given below:

i) The heading should be clear but concise.
ii) They should be arranged in such a way that the most important characteristic is placed in first column. The column of totals is usually placed at extreme right, but some people pre the totals on the left.
iii) Only the first word in each column caption should be capitalized. No full-stop should be pu the end.
iv) Abbreviations, when clear, may be used.
v) Main caption should be centred over the column it is to span.
vi) Extra lines should be used to avoid crowding in caption box.

vii) Whenever possible, caption width should be made roughly proportional to the size of numbers to be inserted.

c) **Row Captions and Stub.** The heading or title for a row, is called the *Row Caption* and the on containing the row captions is known as *Stub*. The necessary points in this respect are given ow:

i) The principles for column captions apply to row captions in stub.

ii) If the stub is long and has several levels of classification, the major classification should be capitalized to separate the table into parts.

iii) Whenever the figures have more than four or five significant digits, the digits should be grouped in threes or fours. For example, one should write 23 178 327, not 23178327.

iv) In long tables, some space should be left after every five or ten rows.

v) Totals should usually be placed at the bottom, but some prefer to place them at the top.

vi) Items in the stub should be arranged so as to facilitate easy reading.

vii) Every stub should have an appropriate heading describing its contents. This heading should be centred in the upper left box of the table.

d) **Prefatory Notes and Footnotes.** Explanatory notes incorporated in the table beneath the title below the body, are called *prefatory notes* and *footnotes* respectively.

*Prefatory notes* give additional specifications of the data indicative of items included or excluded all data of the table, statements of the box, etc. They are placed between the title and the boxhead. The ing should be in lower case alphabet. *Footnotes* are used to clarify anything in the table by giving a description, by drawing attention to incompleteness or by stating any special circumstances ting the data. The footnotes should be specific in nature. They are placed immediately below the m line of the table, above the source. Footnote symbols should be placed as follows:

i) If they refer to an entire column or a set of columns, place them at the end of the appropriate caption.

ii) If they refer to an entire row or a set of rows, place them at the end of the appropriate stub title.

iii) If they refer to a single cell in the table, place beside the cell entry in the body of the table.

The footnotes should be indicated either by lower case alphabet enclosed in parentheses or by ols as *, †, ‡, etc.; never by a number.

e) **Source Notes.** Every table should have a source note, unless the table is an original tabulation its source is clear from the context. It is placed immediately below the table and below the footnotes, y. The source notes must include the compiling agency, publication, date of publication and page as are used as a means of verification and reference.

f) **Body and Arrangement of Data.** The body of a table is the most important part,, which ns the entire data arranged in columns and rows. A *rough-sketch* enables us to have an idea about mber of columns and rows required.

Arrangement of the data is made by taking into consideration the *basis of classification* and the *e of the table*. Thus the data may be arranged either (i) according to the alphabetical order or according to the time of occurrence or (iii) according to location or (iv) according to magnitude or

importance, or (v) by a customary classification, *e.g.* classifying as men, women and children, etc. Whatever arrangements are used, the table should be neat, simple and attractive to the eye.

g) **Spacing and Rulings.** A proper and judicious use of spacing and ruling enhances the effectiveness of a table and helps in separating or emphasizing certain items in it. Thick or double lines (rulings) are used for emphasis and for separating the title, the boxhead, the stub, etc., while parts under captions and related columns are separated by thin or single liens.

h) **General.** There are some other considerations too, that are enumerated below:

i) A table should be simple. A complex table if possible, may be broken into relatively simple tables.

ii) Units of measurements and nature of the data should be specified in title,, captions, etc. in parentheses.

iii) Percentages should be clearly indicated as 'per cent of total' etc. and their total should be shown as 100.0.

iv) If the figures entered in the table are rounded off, this should be indicated in the prefatory note or in the stub or caption.

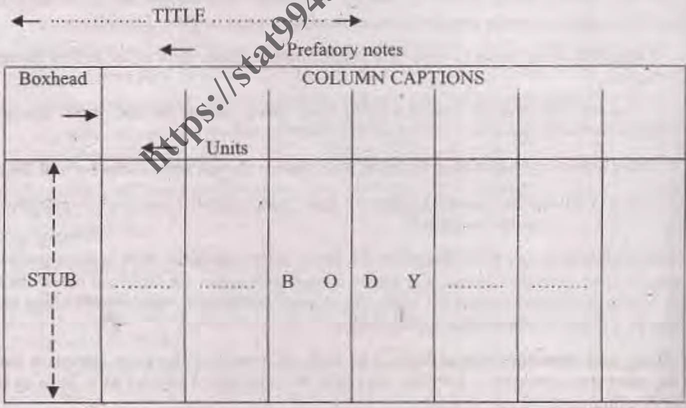
v) Zeros need not be entered.

vi) Minus signs are a part of the table and precede the number.

vii) The relationship of the parts to the whole should be shown by thin or heavy rulings.

viii) The item or items to be emphasized should be placed in the most prominent position of the table.

The general sketch of a table is given below:




Footnotes .....

Source notes.....

**Example 2.1** A district is divided into two areas, *viz*. Urban area and Rural area. Total population of the district is 271,076 out of which only 46,740 live in the urban area. Total male population of

district is 139,699 and that of urban area is 23,083. Total unmarried population of the district is 112,352 out of which 36,864 are rural females. In the urban area, unmarried people number 21,072 out of which 12,149 are males.

Prepare a table showing the population of the district by marital status, by residence and by gender.

A rough table which will probably need amending later, might look as follows:

| AREA | BOTH GENDERS | | MALE | | FEMALE | |
|---|---|---|---|---|---|---|
| | *Married* | *Unmarried* | *Married* | *Unmarried* | *Married* | *Unmarried* |
| Urban | | | | | | |
| Rural | | | | | | |

We first compute the relevant figures as below:

Rural population = Total population – Urban population
= 271,076 – 46,740 = 224,336

Female population = Total population – Male population
= 271,076 – 139,699 = 131,377

Rural male population = District male population – Urban males
= 139,699 – 23,083 = 116,616

Similarly, Urban females = 46,740 – 23,083 = 23,657

Rural females = 224,336 – 116,616 = 107,720

Married population of District = 271,076 – 112,352 = 158,724

Rural unmarried population = 112,352 – 21,072 = 91,280

Rural unmarried males = 91,280 – 36,864 = 54,416

Urban unmarried females = 21,072 – 12,149 = 8,923 etc.

Having computed all these figures, they are presented in the final table that appears below:

Title:

**POPULATION OF DISTRICT "A" BY GENDER, MARITAL STATUS AND RESIDENCE**

| | Areas | Both Genders | | | Male | | | Female | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Boxhead (Captions) | | Total | Married | Un-married | Total | Married | Un-married | Total | Married | Un-married |
| | District | 271,076 | 158,724 | 112,352 | 139,699 | 73,134 | 66,565 | 131,377 | 85,590 | 45,787 |
| Stub: | Urban | 46,740 | 25,668 | 21,072 | 23,083 | 10,934 | 12,149 | 23,657 | 14,734 | 8,923 |
| | Rural | 224,336 | 133,056 | 91,280 | 116,616 | 54,200 | 54,416 | 107,720 | 70,856 | 36,864 |

Source:

## 2.4 FREQUENCY DISTRIBUTION

The organization of a set of data in a table showing the distribution of the data into classes or groups together with the number of observations in each class or group is called a *Frequency Distribution*. The number of observations falling in a particular class is referred to as the *class frequency* or simply *frequency* and is denoted by *f*. Data presented in the form of a frequency distribution are also called *grouped data* while the data in the original form are referred to as *ungrouped data*. The data are said to be arranged in an *array* when arranged in ascending or descending order of magnitude. The purpose of a frequency distribution is to produce a meaningful pattern for the overall distribution of the data from which conclusions can be drawn. A fairly common frequency pattern is the rising to a peak and then declining. In terms of its construction, each class or group has lower and upper limits, lower and upper boundaries, an interval and a middle value.

**2.4.1 Class-limits.** The *class-limits* are defined as the numbers or the values of the variables which describe the classes; the smaller number is the *lower* class limit and the larger number is the *upper* class limit. Class-limits should be well defined and there should be no overlapping. In other words, the limits should be *inclusive, i.e.* the values corresponding exactly to the lower limit or the upper limit be included in that class. The class-limits are therefore selected in such a way that they have the same number of significant places as the recorded values. Suppose the data are recorded to the nearest integers. Then an appropriate method for defining the class limits without overlapping, for example, may be 10 – 14, 15 – 19, 20 – 24, etc. The class limits may be defined as 10.0 – 14.9, 15.0 – 19.9, 20.0 – 24.9, etc. when the data are recorded to nearest tenth of an integer. Sometimes, a class has either no lower class limit or no upper class-limit. Such a class is called an *open-end class*. The open-end classes, if possible, should be avoided as they are a hindrance in performing certain calculations. A class indicated as 10 – 15 will include 10 but not 15, *i.e.* $10 \leq X < 15$.

**2.4.2 Class-boundaries.** The *class-boundaries* are the precise numbers which separate one class from another. The selection of these numbers removes the difficulty, if any, in knowing the class to which a particular value should be assigned. A class-boundary is located midway between the upper limit of a class and the lower limit of the next higher class, *e.g.* 9.5 – 14.5, 14.5 – 19.5, 19.5 – 24.5, or 9.95 – 14.95, 14.95 – 19.95, etc. The class-boundaries are thus always defined more precisely than the level of measurements being used so that the possibility of any observation falling exactly on the boundary is avoided. That is why the class boundaries carry one more decimal place than the class limits or the observed values. The upper class boundary of a class coincides with the lower boundary of the next class.



**2.4.3 Class Mark.** A class mark, also called class midpoint, is that number which divides each class into two parts. In practice, it is obtained by dividing either the sum of the lower and upper limits of a class, or the sum of the lower and upper boundaries of the class by 2 but in a few cases, it does not hold, particularly in modern practice of age grouping. For purposes of calculations, the frequency in a particular class is assumed to have the same value as the class-mark or midpoint. This assumption may introduce an error, called the *grouping error*, but statistical experience has shown that such errors usually tend to counterbalance over the entire distribution. The grouping error may also be minimized by selecting a *class* (group) in such a way that its midpoint corresponds to the mean of the observed values falling in that class.

**2.4.4 Class Width or Interval.** The *class-width* or *interval* of a class is equal to the difference between the class boundaries. It may also be obtained by finding the difference either between two successive lower class limits, or between two successive class marks. The lower limit of a class should not be subtracted from its upper limit to get the class interval. An equal class interval, usually denoted by *h* or *c*, facilitates the calculations of statistical constants such as the mean, the standard deviation,

moments, etc. That is why in practice, it is desirable to have equal class-intervals. But in some types of economic and medical data, it is wise to use unequal class-intervals on account of greater concentration of measurements in certain classes. Such class intervals usually become uniform when logarithms of class marks are taken. It should be noted that some people use the terms "class" and "class-interval" interchangeably and the width of the class is referred to as the *size* or length of the class-interval.

**2.4.5 Constructing a Grouped Frequency Distribution.** The following are some basic rules that should be kept in mind when constructing a grouped frequency distribution:

i) **Decide on the number of classes into which the data are to be grouped.** There are no hard and fast rules for deciding on the number of classes which actually depends on the size of data. Statistical experience tells us that no less than 5 and no more than 20 classes are generally used. Use of too many classes will defeat the purpose of condensation and too few will result in too much loss of information. H.A. Sturges has proposed an empirical rule for determining the number of classes into which a set of observations should be grouped. The rule is

$$k = 1 + 3.3 \log N,$$

where $k$ denotes the number of classes and $N$ is the total number of observations. For example, if there are 100 observations, then by applying Sturges' rule, we should have

$$k = 1 + 3.3\,(2.0000) = 7.6, \textit{ i.e. } \; 8 \text{ classes}$$

Thus eight classes are required but this rule is rarely used in practice.

ii) **Determine the range of variation in the data**, i.e. the difference between the largest and the smallest values in the data.

iii) **Divide the range of variation by the number of classes** to determine the approximate width or size of the equal class-interval. In case of fractional results, the next higher whole number is usually taken as the size or width of class-interval. If equal class-intervals are inconvenient or may be undesirable, then classes of unequal size are used. But in practice, intervals that are multiple of 5 or 10, are commonly used as people can understand them more readily.

iv) **Decide where to locate the class-limit** of the lowest class and then the lower class boundary. The lowest class usually starts with the smallest data value or a number less than it. It is better if it is a multiple of class-interval. Find the upper class boundary by adding the width of the class-interval to the lower class-boundary and write down the upper class limits too. The open-end classes, *i.e.* classes with the lowermost or uppermost class boundary unknown, should be avoided if possible.

v) **Determine the remaining class-limits** and class boundaries by adding the class-interval repeatedly. The lowest class should be placed at the top and the rest should follow according to size. In some cases, the highest class is placed at the top.

vi) **Distribute the data into the appropriate classes.** This is best done by using a "Tally-Column" where values are tabulated against appropriate classes by merely making short bars or tally marks to represent them. It is customary for convenience in counting to place the first four bars vertically and the fifth one diagonally and to leave a space. The number of tallies is then written in the frequency column. The tally column is usually omitted in the final presentation of the frequency distribution. But in case of small number of values, the actual values should be shown against each class to mitigate chances of error.

vii) **Finally, total the frequency column** to see that all the data have been accounted for.

These rules are applied to group raw data which are assumed to be continuous. *In case of discrete data which carry only integral values,* the concept *of a class boundary is unrealistic as there can be no points where the adjoining classes meet. In spite of this logical difficulty, when the discrete data are sufficiently large, they are treated for convenience of calculations as continuous and hence are grouped in the same way as the continuous data.*

**Example 2.2** Make a grouped frequency distribution from the following data, relating to the weight recorded to the nearest grams of 60 apples picked out at random from a consignment.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 106 | 107 | 76 | 82 | 109 | 107 | 115 | 93 | 187 | 95 | 123 | 125 |
| 111 | 92 | 86 | 70 | 126 | 68 | 130 | 129 | 139 | 119 | 115 | 128 |
| 100 | 186 | 84 | 99 | 113 | 204 | 111 | 141 | 136 | 123 | 90 | 115 |
| 98 | 110 | 78 | 185 | 162 | 178 | 140 | 152 | 173 | 146 | 158 | 194 |
| 148 | 90 | 107 | 181 | 131 | 75 | 184 | 104 | 110 | 80 | 118 | 82 |

By scanning the data, we find that the largest weight is 204 grams and the smallest weight is 68 grams so that the range is 204 – 68 = 136 grams.

Suppose we decide to take 7 classes of equal size. Then size or width of the equal class interval would be $\frac{136}{7} = 19.47$. But we take $h = 20$, the next integral value higher than 19.47 to facilitate the numerical work.

Let us decide to locate the lower limit of the lowest class at 65. With this choice, the class limits will be 65 – 84, 85 – 104, 105 – 124, …, the class boundaries become 64.5 – 84.5, 84.5 – 104.5, 104.5 – 124.5, …, and the class-marks are 74.5, 94.5, 114.5…. The grouped frequency distribution is then constructed as follows:

i) **By listing the actual values:**

**FREQUENCY DISTRIBUTION OF WEIGHTS OF 60 APPLES**

| Weight | Entries | Frequency |
|---|---|---|
| 65 – 84 | 76, 82, 70, 68, 84, 78, 75, 80, 82 | 9 |
| 85 – 104 | 93, 95, 92, 86, 100, 99, 90, 98, 90, 104 | 10 |
| 105 – 124 | 106, 107, 109, 107, 115, 123, 111, 119, 115, 113, 111, 123, 115, 110, 107, 110, 118 | 17 |
| 125 – 144 | 125, 126, 130, 129, 139, 128, 141, 136, 140, 131 | 10 |
| 145 – 164 | 162, 152, 146, 158, 148 | 5 |
| 165 – 184 | 178, 173, 181, 184 | 4 |
| 185 – 204 | 187, 186, 204, 185, 194 | 5 |
| **Total** | | **60** |

This table is sometimes known as an *entry table*. The values against each class may be arranged in an array.

ii) **By using a Tally-Column:**

**FREQUENCY DISTRIBUTION OF WEIGHTS OF 60 APPLES**

| Classes (weight) | Class-boundaries | Mid-points or Class-Marks | Tally | Frequency |
|---|---|---|---|---|
| 65 – 84 | 64.5 – 84.5 | 74.5 | 卌 IIII | 9 |
| 85 – 104 | 84.5 – 104.5 | 94.5 | 卌 卌 | 10 |
| 105 – 124 | 104.5 – 124.5 | 114.5 | 卌 卌 卌 II | 17 |
| 125 – 144 | 124.5 – 144.5 | 134.5 | 卌 卌 | 10 |
| 145 – 164 | 144.5 – 164.5 | 154.5 | 卌 | 5 |
| 164 – 184 | 164.5 – 184.5 | 174.5 | IIII | 4 |
| 185 – 204 | 184.5 – 204.5 | 194.5 | 卌 | 5 |
| **Total** | .... | .... | .... | **60** |

Practical

**Example 2.3** Given below are the mean annual death rates per 1,000 at ages 20 – 65 in each of 88 occupational groups. Construct a grouped frequency distribution.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7.5 | 8.2 | 6.2 | 8.9 | 7.8 | 5.4 | 9.4 | 9.9 | 10.9 | 10.8 | 7.4 | 4 |
| 9.7 | 11.6 | 12.6 | 5.0 | 10.2 | 9.2 | 12.0 | 9.9 | 7.3 | 7.3 | 8.4 | 12 |
| 10.3 | 10.1 | 10.0 | 11.1 | 6.5 | 12.5 | 7.8 | 6.5 | 8.7 | 9.3 | 12.4 | 18 |
| 10.4 | 9.1 | 9.7 | 9.3 | 6.2 | 10.3 | 6.6 | 7.4 | 8.6 | 7.7 | 9.4 | 23 |
| 7.7 | 12.8 | 8.7 | 5.5 | 8.6 | 9.6 | 11.9 | 10.4 | 7.8 | 7.6 | 12.1 | 18 |
| 4.6 | 14.0 | 8.1 | 11.4 | 10.6 | 11.6 | 10.4 | 8.1 | 4.6 | 6.6 | 12.8 | 12 |
| 6.8 | 7.1 | 6.6 | 8.8 | 8.8 | 10.7 | 10.8 | 6.0 | 7.9 | 7.3 | 9.3 | 1 |
| 9.3 | 8.9 | 10.1 | 3.9 | 6.0 | 6.9 | 9.0 | 8.8 | 9.4 | 11.4 | 10.9 | |

(B.I.S.E. Lahore, 1971)

A scan of the data shows that the largest value is 14.0 and the smallest value is 3.9 so that the range = 14.0 – 3.9 = 10.1.

As the data are recorded to one decimal place, we may therefore locate the lower limit of the first group at 3.5. Let us choose a class interval of 1.0. Then the class limits are specified as 3.5 – 4.4,

4.5 – 5.4, 5.5 – 6.4, ... With this choice, the class-boundaries are 3.45 – 4.45, 4.45 – 5.45, 5.45 – 6.45, ..., which do not coincide with the given values.

The following table shows the required frequency distribution:

**FREQUENCY DISTRIBUTION OF MEAN DEATH RATES**

| Death Rates | Class-boundaries | Tally | Frequency |
|---|---|---|---|
| 3.5 – 4.4 | 3.45 – 4.45 | 𝍷 | 1 |
| 4.5 – 5.4 | 4.45 – 5.45 | 𝍷𝍷𝍷𝍷 | 4 |
| 5.5 – 6.4 | 5.45 – 6.45 | 𝍸 | 5 |
| 6.5 – 7.4 | 6.45 – 7.45 | 𝍸 𝍸 𝍷𝍷𝍷 | 13 |
| 7.5 – 8.4 | 7.45 – 8.45 | 𝍸 𝍸 𝍷𝍷 | 12 |
| 8.5 – 9.4 | 8.45 – 9.45 | 𝍸 𝍸 𝍸 𝍷𝍷𝍷𝍷 | 19 |
| 9.5 – 10.4 | 9.45 – 10.45 | 𝍸 𝍸 𝍷𝍷𝍷 | 13 |
| 10.5 – 11.4 | 10.45 – 11.45 | 𝍸 𝍸 | 10 |
| 11.5 – 12.4 | 11.45 – 12.45 | 𝍸 𝍷 | 6 |
| 12.5 – 13.4 | 12.45 – 13.45 | 𝍷𝍷𝍷𝍷 | 4 |
| 13.5 – 14.4 | 13.45 – 14.45 | 𝍷 | 1 |
| **Total** | | | **88** |



**Example 2.4** Construct a frequency distribution for the data below. Indicate the class boundaries and class limits clearly.

| | | | | | |
|---|---|---|---|---|---|
| 41.78 | 29.32 | 31.47 | 35.35 | 32.82 | 39.42 |
| 61.65 | 28.31 | 44.63 | 22.78 | 44.44 | 48.12 |
| 81.71 | 33.47 | 50.35 | 29.19 | 51.26 | 50.32 |
| 26.84 | 18.95 | 48.19 | 43.72 | 43.89 | 47.15 |
| 60.20 | 44.43 | 41.17 | 37.50 | 22.35 | 29.17 |

By scanning the data, we find that the largest value is 81.71 and the smallest value is 18.95 so tha the range is 81.71 – 18.95 = 62.76.

Suppose we decide to take 5 classes of equal size. Then size or width of the equal class interva would be $\frac{62.76}{5} = 12.55$. But we take h = 13.00, the next integral value higher than 12.55 to take numerical work.

As the data are recorded to two decimal places, we may locate the lower limit of the first group at 18.00. With this choice, the class limits will be 18.00 – 30.99, 31.00 – 43.99, ……., the class boundaries become 17.995 – 30.995, 30.995 – 43.995. The grouped frequency distribution is then constructed as follows:

| Classes | Class boundaries | Tally | f |
|---|---|---|---|
| 18.00 – 30.99 | 17.995 – 30.995 | 𝍸 \|\|\| | 8 |
| 31.00 – 43.99 | 30.995 – 43.995 | 𝍸 𝍸 | 10 |
| 44.00 – 56.99 | 43.995 – 56.995 | 𝍸 \|\|\|\| | 9 |
| 57.00 – 69.99 | 56.995 – 69.995 | \|\| | 2 |
| 70.00 – 82.99 | 69.995 – 82.995 | \| | 1 |
| **Total** | **….** | **….** | **30** |

**Example 2.5** A survey of 50 retail establishments had assistants,, excluding proprietors, as follows:

2, 3, 9, 0, 4, 4, 1, 5, 4, 8, 5, 3, 6, 6, 0, 2,
2, 7, 6, 4, 8, 4, 3, 3, 1, 0, 8, 7, 5, 1, 3, 4, 2, 4, 7,
5, 2, 6, 3, 1, 7, 5, 4, 6, 4, 2, 5, 3, 4

Arrange the values as a frequency distribution.

By scanning the data, we find that the number of assistants is a *discrete variable* and the range is small, so the data can be conveniently sorted by taking the values of classes as 0, 1, 2, etc. The frequency, distribution is then constructed as shown below:

**FREQUENCY DISTRIBUTION OF ASSISTANTS IN 50 RETAIL ESTABLISHMENTS**

| Number of Assistants ($x$) | Tally | Number of Establishments ($f$) |
|---|---|---|
| 0 | \|\|\| | 3 |
| 1 | \|\|\|\| | 4 |
| 2 | 𝍸 \| | 6 |
| 3 | 𝍸 \|\| | 7 |
| 4 | 𝍸 𝍸 | 10 |
| 5 | 𝍸 \| | 6 |
| 6 | 𝍸 | 5 |
| 7 | 𝍸 | 5 |
| 8 | \|\|\| | 3 |
| 9 | \| | 1 |
| **Total** | **….** | **50** |



Such a frequency distribution in which each class consists of a single value is sometimes called a discrete or ungrouped frequency distribution.

**2.4.6 Cumulative Frequency Distribution.** The total frequency of a variable from its one end to a certain value (usually upper class boundary in grouped data), called the *base,* is known as the *cumulative frequency, less than or more than* the base of the variable. A table that shows the cumulative frequencies, is called a *cumulative frequency distribution.* The cumulative frequency of the last class is the sum of all frequencies in the distribution. If the cumulation process is from the lowest value to the highest, it is referred to as "*a less than*" type cumulative frequency distribution. For example, let us consider a frequency distribution having $k$ classes, each of width $h$. Let us denote the midpoint of the *ith* class by $x_i$ with frequency $f_i$ such that $\sum_{i=1}^{k} f_i = n$. Now the lower class-boundary of the first group is $x_1 - h/2$ and the upper class boundaries are $x_i + h/2$, $(i = 1, 2, \ldots, k)$. The cumulative frequency distribution is then obtained by adding each successive frequency to the cumulative total of frequencies for the preceding classes as shown below:

| Class-boundary | Cumulative Frequency |
|---|---|
| less than $x_1 - h/2$ | 0 |
| less than $x_1 + h/2$ | $f_1$ |
| less than $x_2 + h/2$ | $f_1 + f_2$ |
| less than $x_3 + h/2$ | $f_1 + f_2 + f_3$ |
| . | . |
| . | . |
| . | . |
| less than $x_k + h/2$ | $\sum f_i = n$ |

It should be noted that a *less than* type cumulative frequency distribution starts with the lower class boundary of the first group indicating that there is no frequency below $x_1 - h/2$.

When the frequencies are cumulated from the highest value to the lowest value, it is called a "*more than*" type cumulative frequency.

If the class frequencies against various classes are divided by the total frequency, we get the *relative frequencies* which always add to one. The class frequencies may also be expressed as percentages, the total of which would be 100. A percentage cumulative distribution is useful to read off the percentage of values falling between certain specified values.

**Example 2.6** Construct (i) a "less than" type cumulative distribution, and (ii) a "more than" type cumulative distribution from the frequency distribution of weights of 60 apples of Example 2.2.

i) A "less than" type cumulative frequency distribution is shown below:

| Weight (grams) | Cumulative Frequency ($F$) |
|---|---|
| Less than 64.5 | 0 |
| Less than 84.5 | 9 |
| Less than 104.5 | 19 |
| Less than 124.5 | 36 |
| Less than 144.5 | 46 |
| Less than 164.5 | 51 |
| Less than 184.5 | 55 |
| Less than 204.5 | 60 |

ii) A "more than" type cumulative frequency distribution is given below:

| Weight (grams) | Cumulative Frequency *(F)* |
|---|---|
| More than 64.5 | 60 |
| More than 84.5 | 51 |
| More than 104.5 | 41 |
| More than 124.5 | 24 |
| More than 144.5 | 14 |
| More than 164.5 | 9 |
| More than 184.5 | 5 |
| More than 204.5 | 0 |

## 2.5 STEM-AND-LEAF DISPLAY

A clear disadvantage of using a frequency table is that the identity of individual observations is lost in grouping process. To overcome this drawback, John Tukey (1977) introduced a technique known as the *Stem-and-Leaf Display*. This technique offers a quick and novel way for simultaneously sorting and displaying data sets where each number in the data set is divided into two parts, a *Stem* and a *Leaf*. A stem is the leading digit(s) of each number and is used in sorting, while a *leaf* is the rest of the number or the trailing digit(s) and shown in display. A vertical line separates the leaf (or leaves) from the stem. For example, the number 243 could be split two ways:

| leading digit | trailing digits | **OR** | leading digit | trailing digits |
|---|---|---|---|---|
| 2 | 43 | | 24 | 3 |
| stem | leaf | | stem | leaf |

All possible stems are arranged in order from the smallest to the largest and placed on the left hand side of the line.

*The stem-and-leaf display* is a useful step for listing the data in an *array*, leaves are associated with the stem to know the numbers. The *stem-and-leaf* table provides a useful description of the data set and can easily be converted to a frequency table. It is a common practice to arrange the trailing digits in each row from smallest to highest.

**Example 2.7** The ages of 30 patients admitted to a certain hospital during a particular week were as follows:

48, 31, 54, 37, 18, 64, 61, 43, 40, 71, 51, 12, 52, 65, 53
42, 39, 62, 74, 48, 29, 67, 30, 49, 68, 35, 57, 26, 27, 58

Construct a stem-and-leaf display from the data and list the data in an array.

A scan of the data indicates that the observations range (in age) from 12 to 74. We use the first (or leading) digit as the *stem* and the second (or trailing) digit as the *leaf*. The first observation is 48, which has a stem of 4 and a leaf of 8, the second a stem of 3 and a leaf of 1, etc. Placing the leaves in the order in which they appear in the data, we get the *stem-and-leaf* display as shown on next page:

| Stem (leading digit) | Leaf (trailing digit) |
|---|---|
| 1 | 8 2 |
| 2 | 9 6 7 |
| 3 | 1 7 9 0 5 |
| 4 | 8 3 0 2 8 9 |
| 5 | 4 1 2 3 7 8 |
| 6 | 4 1 5 2 7 8 |
| 7 | 1 4 |

To get the array, we associate the leaves in order of size with the stems as shown below:

12, 18, 26, 27, 29, 30, 31, 35, 37, 39, 40, 42, 43, 48, 48, 49,

51, 52, 53, 54, 57, 58, 61, 62, 64, 65, 67, 68, 71, 74

**Example 2.8** Construct a stem-and-leaf display for the data of annual death rates given in Example 2.3.

Using the decimal part in each number as the *leaf* and the rest of the digits as the *stem,* we get the following stem-and-leaf display (leaves are ordered):



| Stem | Leaf |
|---|---|
| 3' | 9 |
| 4 | 6 6 |
| 5 | 0 4 5 |
| 6 | 0 0 2 2 5 5 6 6 6 8 9 |
| 7 | 1 3 3 3 4 4 5 6 7 7 8 8 8 9 |
| 8 | 1 1 2 4 6 6 7 7 8 8 8 9 9 |
| 9 | 0 1 2 3 3 3 3 4 4 4 6 7 7 9 9 |
| 10 | 0 1 1 2 3 3 4 4 6 6 7 8 8 9 9 |
| 11 | 1 4 4 6 6 9 |
| 12 | 0 1 4 5 6 8 8 |
| 14 | 0 |

## 2.6 GRAPHICAL REPRESENTATION

Tabulation, we know, is a good method of condensing and representing statistical data in a readily understandable form, but many people have no taste for figures. They would prefer a way of representation where figures could be avoided. This purpose is achieved by the presentation of statistical data in a visual form. The visual display of statistical data in the form of points, lines, areas and other geometrical forms and symbols, is in the most general terms known as *Graphical Representation.* Statistical data can be studied with this method without going through figures, presented in the form of tables.

Such visual representation can be divided into two main groups, *graphs and diagrams* to be described in the sections that follow. The basic difference between a graph and a diagram is that a graph is a representation of data by a continuous curve, usually shown on a graph paper while a diagram is any other one, two or three--dimensional form of visual representation.

## 2.7 DIAGRAMS

Diagrammatic representation is best suited to spatial series and data split into different categories. Whenever a comparison of the same type of data at different places is to be made, diagrams will be the best way to do that. Diagrammatic representation has several advantages over tabular representation of figures. Beautifully and neatly constructed diagrams are more attractive than simple figures. Diagrams, being a visual display, leave more effective and long lasting impression on the mind of a reader. They make unwieldy data intelligible at a glance. Comparison is made easier with diagrams. Diagrams have some disadvantages too. Diagrams are less accurate than tables; cost money and time and the amount of information conveyed is limited. However, this method of representation is excessively used in business and administration.

Different types of diagrams or charts commonly used for displaying statistical data are described below:

i) *Linear or One-Dimensional Diagrams.* They consist of Simple Bars, Multiple Bars and Component Bar charts. Here the values are represented only by one dimension, generally the length of the bar.

ii) *Areal or Two-Dimensional Diagrams.* They consist of Rectangles, Sub-divided Rectangles and Squares, the areas of which are proportional to the values of the given quantities. This device is used to represent data having moderately large variations.

iii) *Cubic or Three-Dimensional Diagrams.* They are in the form of Cubes and cylinders, whose volumes are proportional to the values they represent. These diagrams are used when the variation among the values of the data to be portrayed is so large that even the square roots of the values concerned fail to reduce the variation appreciably.

iv) *Pie-Diagrams.* They are in the form of Circles and Sectors. Here the areas of circles or sectors are in proportion to the values they represent or compare.

v) *Pictograms.* They consist of pictures or small symbolic figures representing the statistical data. A pictogram is an effective way of visual comparisons. For example, we can compare the armed strength of various countries by drawing pictures of the number of soldiers, where each pictorial soldier may denote, say, 1,000 soldiers. In a similar way, the production of wheat can be compared by means of the pictures of wheat bags of a specified size. It is essential to repeat the pictures a number of times to represent the differences in magnitudes.

While drawing diagrams, the following points should be kept in mind:

i) An appropriate scale consistent with the size of paper available and the size of the data to be represented, should be chosen and indicated either at the side or at the bottom of the diagram. This scale *must* start at zero.

ii) A diagram like a table, must have a title, which should be brief and self-explanatory. A key, footnote or source will also be necessary.

iii) A diagram should be shaded, coloured or cross-hatched to show the different parts, if any.

iv) Lettering should be shown horizontally.

**2.7.1 Simple Bar Chart.** A *simple bar chart* consists of horizontal or vertical bars of equal wid and lengths proportional to the values they represent. As the basis of comparison is linear or on dimensional, the widths of these bars have no significance but are taken to make the chart look attracti *The space separating the bars should not exceed the width of the bar and should not be less than half its width. The bars should neither be exceedingly long and narrow nor short and broad. The vertical b chart is an effective way for presenting a time series and qualitatively classified data whereas horizon bars are useful for geographical or spatial distributions.* The data when do not relate to time, should arranged in ascending or descending order before charting.

**Example 2.9** Draw a simple bar diagram to represent the turnover of a company for 6 years.

| | | | | | | |
|---|---|---|---|---|---|---|
| *Years*: | 1980 | 1981 | 1982 | 1983 | 1984 | 1985 |
| *Turnover (Rupees)*: | 38,000 | 45,000 | 48,000 | 52,500 | 55,000 | 58,000 |

The bar chart is drawn below:

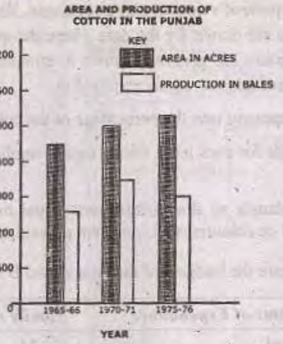

**2.7.2 Multiple Bar Chart.** A *multiple bar chart* shows two or more characteristics correspo to the values of a common variable in the form of grouped bars, whose lengths are proportional t values of the characteristics, and each of which is shaded or coloured differently to aid identifica This is a good device for the comparison of two or three kinds of information. For example, imp exports and productions of a country can be compared from year to year by grouping the three together.

**Example 2.10** Draw multiple bar charts to show the area and production of cotton in the P from the following data:

| Year | Area (000 acres) | Production (000 bales) |
|---|---|---|
| 1965 – 66 | 2866 | 1588 |
| 1970 – 71 | 3233 | 2229 |
| 1975 – 76 | 3420 | 1937 |

(Source: Statistical Wing, Agriculture Deptt. Lahore)

The multiple bar charts are drawn below:




**2.7.3 Component Bar Chart.** A *component bar chart* is an effective technique in which each bar is divided into two or more sections, proportional in size to the component parts of a total being displayed by each bar. The various component parts shown as sections of the bar, are shaded or coloured differently to increase the overall effectiveness of the diagram. Component bar charts are used to represent the relation of the various components of data and the percentages. They are also known as *sub-divided bars*.

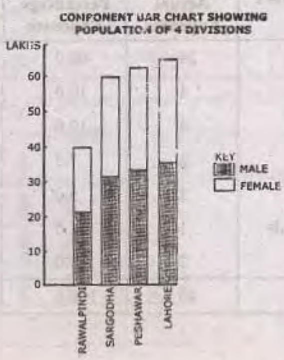
lac

**Example 2.11** Draw a component bar chart for the following data.

(Population in Lakhs)

| Division | Both Genders | Male | Female |
|---|---|---|---|
| Peshawar .... | 64 | 33 | 31 |
| Rawalpindi .... | 40 | 21 | 19 |
| Sargodha .... | 60 | 32 | 28 |
| Lahore .... | 65 | 35 | 30 |

The appropriate component bar chart after arranging the population figures in ascending order is shown below:



**2.7.4 Rectangles and Sub-divided Rectangles.** The area of a rectangle is equal to the product of its length and breadth. To represent a quantity by a rectangle, both length and breadth of the rectangle are used. Sub-divided rectangles are drawn for the data where the quantities along with their components are to be compared. These diagrams are generally drawn to compare the budgets of various families. In the construction of sub-divided rectangles, we are required to

i) change each component into the percentage of the corresponding total,

ii) draw one rectangle for each total, taking equal lengths (100 units) and breadths proportional to the totals,

iii) divide every rectangle so drawn into parts equal in number to the number of components. Each part shaded or coloured will represent percentage size of one component.

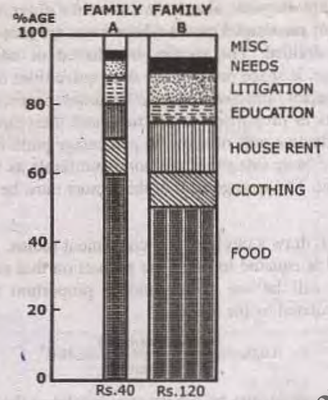
**Example 2.12** Compare the budgets of families A and B with a suitable diagram.

| *Items of Expenditure* | *Family A* | *Family B* |
|---|---|---|
| Food | 24 | 60 |
| Clothing | 4 | 14 |
| House Rent | 4 | 16 |
| Education | 3 | 6 |
| Litigation | 2 | 10 |
| Conventional Needs | 1 | 6 |
| Miscellaneous | 2 | 8 |
| Total | 40 | 120 |



The necessary computations required for the drawing of sub-divided rectangles are given below and the diagram is shown on page 33.

| **Items of Expenditure** | **Family A** | | **Family B** | |
|---|---|---|---|---|
| | **Actual Expenses** | **Percentage Expenses** | **Actual Expenses** | **Percentage Expenses** |
| Food | 24 | 60.0 | 60 | 50.0 |
| Clothing | 4 | 10.0 | 14 | 11.7 |
| House Rent | 4 | 10.0 | 16 | 13.3 |
| Education | 3 | 7.5 | 6 | 5.0 |
| Litigation | 2 | 5.0 | 10 | 8.3 |
| Conventional Needs | 1 | 2.5 | 6 | 5.0 |
| Miscellaneous | 2 | 5.0 | 8 | 6.7 |
| Total | 40 | 100.0 | 120 | 100.0 |

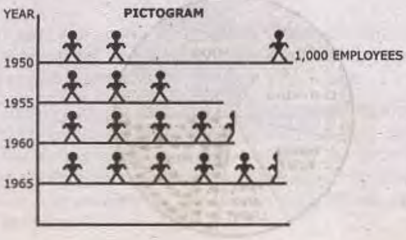



**2.7.5 Pictograms.** A *pictogram* is a popular device for portraying the statistical data by means of pictures or small symbols. It is said that *a picture is worth ten thousand words.* It is customary to represent a unit value of the data by a standard symbol or a picture and the whole quantity by an appropriate number of repetitious of symbol concerned. This means *the larger quantities should be represented by a larger number of symbols and not by larger symbols.* A quantity smaller than the unit is represented by a part of the picture or symbol used. The symbols or pictures to be used, must be simple and clear. A pictogram is virtually a bar chart constructed in pictorial way as the number of symbols or pictures corresponds to the length of a bar.

**Example 2.13** The following table shows the number of employees in a certain Textile Mills. Represent the data by means of a pictogram.

| Year | No. of Employees |
|---|---|
| 1950 | 2,004 |
| 1955 | 2,990 |
| 1960 | 4,240 |
| 1965 | 5,380 |

Representing 1,000 employees by one picture, the pictogram is drawn below:



**2.7.6 Pie Diagrams.** A *pie-diagram,* also known as *sector diagram,* is a graphic device consisti of a circle divided into sectors or pie-shaped pieces whose areas are proportional to the various parts in which the whole quantity is divided. The sectors are shaded or coloured differently to show t relationship of parts to the whole. If space permits, the descriptive titles of the constituent parts should be placed horizontally on each sector, otherwise a key becomes necessary. It is a convenient way displaying the component parts in proportion to the total and therefore is used as an alternative to component bar chart. It is an effective way of showing percentage parts when the whole quantity is tak as 100. It is also used when the basic categories are not quantifiable as with expenditure, classified in food, clothing, fuel and light, etc. The arrangement of the sectors must be made uniform in comparing p charts.

**To construct a pie chart,** draw a circle of any convenient radius. As a circle consists of 360°, t whole quantity to be displayed is equated to 360. The proportion that each component part or categ bears to the whole quantity will be the corresponding proportion of 360°. These correspondi proportions, *i.e.* angles, are calculated by the formula

$$\text{Angle} = \frac{\text{component part}}{\text{whole quantity}} \times 360^{\circ}$$

Then divide the circle into different sectors by constructing angles at the centre by means of a protrac and draw the corresponding radii.

**Example 2.14** Represent the total expenditure and expenditures on various items of a family by pie diagram.

| *Items:* | Food | Clothing | House Rent | Fuel and Light | Misc. |
|---|---|---|---|---|---|
| *Expenditure:* (*in Rs.*) | 50 | 30 | 20 | 15 | 35 |

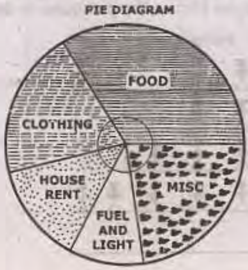
The corresponding angles needed to draw the chart are computed below.

| Items | Expenditure (in Rs.) | |
|---|---|---|
| Food | 50 | |
| Clothing | 30 | |
| House Rent | 20 | Angles of the Sectors (in Degrees) |
| Fuel and Light | 15 | 120 |
| Miscellaneous | 35 | 72 |
| **Total** | **150** | **360** |



"The pie diagram consisting of a circle divided into five sectors defined by angles 120°, 72°, 48 36° and 84°, is drawn below:"

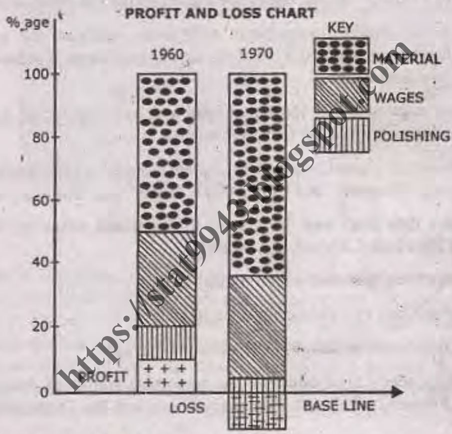
PIE DIAGRAM



**2.7.7 Profit and Loss Chart.** This is virtually a percentage component bar chart in which profits can be shown above the normal base line and losses below the base line. Since the bars are to be extended from the zero line to show losses, we start from the top. For an illustration, the following data are represented:

**COST, PROCEEDS, PROFIT OR LOSS PER CHAIR**

| Particulars | 1960 | 1970 |
|---|---|---|
| i) Materials | Rs. 10 | Rs. 16 |
| ii) Wages | 6 | 8 |
| iii) Polishing, etc. | 2 | 4 |
| Total cost | 18 | 28 |
| Proceeds | 20 | 25 |
| Profit ( + ) or loss ( − ) | +2 | -3 |




A pie chart may also be used for this purpose.

## 2.8 GRAPHS

As already stated, diagrams are useful for representing spatial series. Diagrams fail when we want to represent a statistical series spread over a period of time, or a frequency distribution or two related variables in visual form. For such representations, graphs are employed.

Graphs present the data in a simple, clear and effective manner, facilitate comparison between two or more than two statistical series, and help us in appreciating their significance readily. Another advantage of graphs is that they provide an overall picture of a statistical series. Graphs are also sometimes used to make predictions and forecasts. Certain partition values can also be located graphically. But graphs are less accurate as they do not give minute details. Moreover, they cost considerable expenditure and time.

**Construction of Graphs.** In the construction of a graph, the first step is to take a starting point, known as the origin, in the left-hand bottom corner of the graph paper. Two straight lines perpendicular to each other are drawn through the origin. The horizontal line is called the *X*-axis or abscissa and the vertical line is labeled as *Y*-axis or ordinate. The two lines together are known as co-ordinate axes. Some suitable scales are selected along *X*-axis and *Y*-axis. Independent variable is taken along *X*-axis and dependent variable along *Y*-axis. Points are plotted and joined to get the required graph. While constructing a graph, the following points should be kept in mind:

i) A scale and the form of representation is to be selected in such a way that the true impression of the data to be represented is given by the graph.

ii) Every graph must have a clear and comprehensive title at top. Where necessary, sub-titles should be added.

iii) The source of the data must be given. A key and footnotes should be provided when necessary.

iv) The independent variable should always be placed on the horizontal axis.

v) The vertical scale should always begin with zero, otherwise the graph will give a false impression. If, however, the first item of the data is quite large, a scale-break should be shown between zero and next member.

vi) The horizontal axis does not have to begin with zero unless of course, the independent variable or the lower limit of the first class interval is zero.

vii) The axes of the graph should be properly labelled. Labels should clearly state both the variable and the units, *e.g.* "Distance" and "Kilometer", "Sales" and "Rupees", etc.

viii) Curves if more than one, must be clearly distinguished either by different colours or by differentiated lines (solid, dashed, dot-dashed).

ix) The graph should not be loaded with too many curves.

**Graphs** can be divided into two main categories, namely:

a) Graphs of Time-Series or Graphs of Historical Data, and

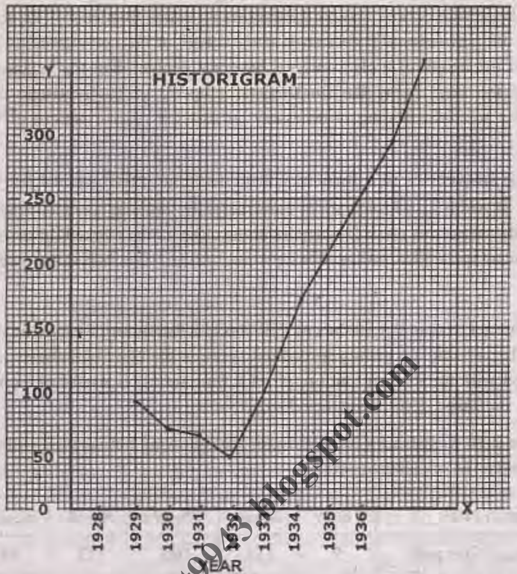
b) Graphs of Frequency Distributions. The important graphs of frequency distributions are Histogram, Frequency Polygon, Frequency Curve and the Cumulative Frequency Curve or Ogive.

**2.8.1 Graph of Time Series–Historigram.** A curve showing changes in the value of one or more items from one period of time to the next is known as the graph of a time series. This curve is also called a *Historigram*. Thus a historigram displays the variations in time series dealing with prices, production, imports, population, etc. To construct a historigram, time is taken along *X*-axis and the values of the variables along *Y*-axis. Points are plotted and are then connected by means of straight line segments to get the "Historigram".

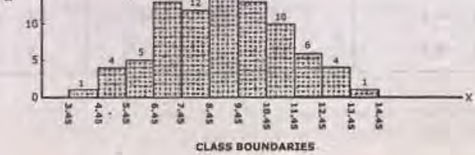
**Example 2.15** The following table gives the number of cars produced in Germany during the years 1929-1936. Draw a suitable graphs, *i.e.* Historigram of the series.

| Years: | 1929 | 1930 | 1931 | 1932 | 1933 | 1934 | 1935 | 1936 |
|---|---|---|---|---|---|---|---|---|
| No. of Cars: | 98 | 74 | 68 | 50 | 99 | 172 | 245 | 302 |

The historigram is drawn for the data by taking years on horizontal axis and the number of cars on vertical axis as below:




**2.8.2 Histogram.** A histogram consists of a set of adjacent rectangles whose bases are marked off by class boundaries (not class limits) on the X-axis and whose heights are proportional to the frequencies associated with respective classes. The area of each rectangle represents the respective class frequencies. This is one of the most important graphical representation of a frequency distribution. When the class-intervals are equal, the rectangles all have the same width and their heights directly represent the class frequencies, that is they are numerically proportional to the frequencies in the respective classes. The following figure shows the histogram for the frequency distribution of Example 2.3.



If the class-intervals are *not* all equal, the height of the rectangle over an unequal class-interval is to be adjusted because it is *area* and not height that measures frequency. This means that the *height of a rectangle must be proportionally decreased if the length of the corresponding class-interval increases.* For example, if the length of a class-interval becomes double, then the height of the rectangle is to be halved so that the area, being the fundamental property of the rectangle of a histogram, remains unchanged. This sort of rescaling is necessary so that the correct pattern of the distribution is to be conveyed.

When the frequencies in a frequency distribution are given against the class-marks $x_i$ of equal class-intervals of width $h$, a histogram is constructed by drawing vertical lines (dotted) whose heights correspond to the respective class-frequencies at the class-marks marked off on the axis of $X$ and erecting a series of adjacent rectangles with widths equal to $x_i \pm h/2$ (i.e. half of the width is taken on either side of $x_i$).

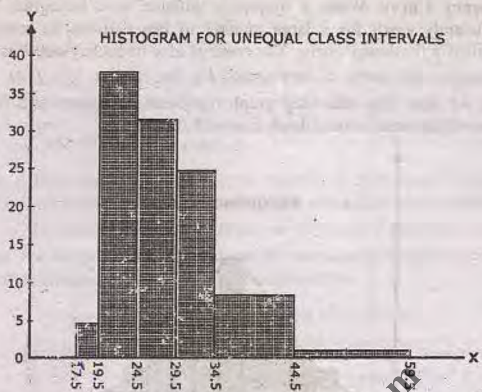
It is important to note that in the construction of a histogram, we assume that within any one class the values of the variable are evenly spread out between the class-boundaries. A histogram which must not be confused with the *historigram* (graph of a time series) is useful in forming a rough idea of the overall pattern and shape of the frequency distribution.

**Example 2.16** Construct a Histogram for the following frequency distribution relating to the ages (to nearest birthday) of telephone operators.

| Age (Years) | 18–19 | 20–24 | 25–29 | 30–34 | 35–44 | 45–59 |
|---|---|---|---|---|---|---|
| No. of Operators | 9 | 188 | 160 | 123 | 84 | 15 |

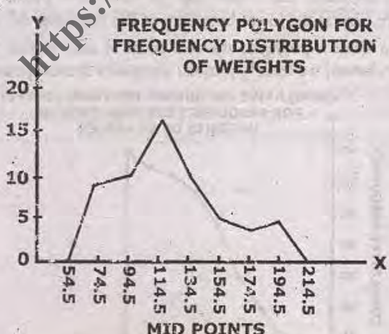
As the class-intervals are unequal, the height of each rectangle cannot be made equal to the frequency. The height of a rectangle is therefore calculated by dividing the frequency (the area) by the corresponding class interval (the width). The necessary calculations and the histogram follow:

| Class-boundaries | Class-Interval ($h$) | Frequency | Proportional Heights |
|---|---|---|---|
| 17.5 – 19.5 | 2 | 9 | $9 \div 2 = 4.5$ |
| 19.5 – 24.5 | 5 | 188 | $188 \div 5 = 37.6$ |
| 24.5 – 29.5 | 5 | 160 | $160 \div 5 = 32.0$ |
| 29.5 – 34.5 | 5 | 123 | $123 \div 5 = 24.6$ |
| 34.5 – 44.5 | 10 | 84 | $84 \div 10 = 8.4$ |
| 44.5 – 59.5 | 15 | 15 | $15 \div 15 = 1.0$ |



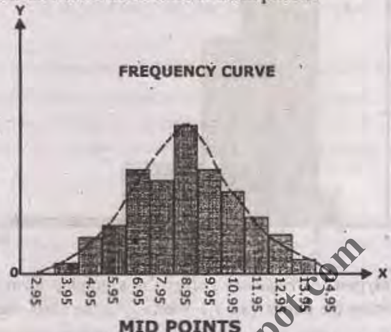

**2.8.3 Frequency Polygon.** A *frequency polygon* is a graphic form of a frequency distribution, which is constructed by plotting the points $(x_i, f_i)$ where $x_i$ is the class-mark of the *i*th class and $f_i$ is the corresponding frequency, and then connecting them by straight line segments provided the class-intervals are equal. In case of unequal class-intervals, heights of unequal classes are adjusted by using the same technique that was used for histogram. It can also be obtained by joining the tops of the successive rectangles in the histogram by means of straight line segments. The graph drawn in this way does not touch the horizontal axis. But a polygon, as we know, is a closed figure having many sides. It is therefore customary to add "extra" class marks at both ends of the distribution with zero class frequencies so that the polygon does form a closed figure with the horizontal axis. This should be done even if the curve ends in the minus part of the graph. The frequency polygon for the frequency distribution of weights in Example 2.2 is given below:



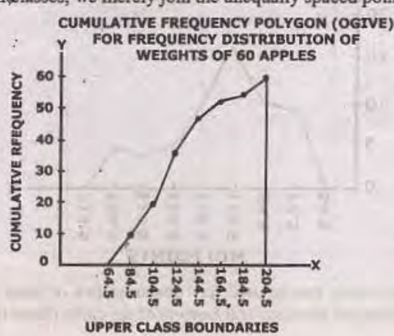



A frequency polygon which can be used for comparing two or more data sets, gives roughly the position of the *mode*, some idea of *skewness* and *kurtosis* of the curve (these terms are defined later).

**2.8.4 Frequency Curve.** When a frequency polygon or a histogram constructed over class intervals made sufficiently small for a large number of observations, is smoothed, it approaches a continuous curve, called a *frequency curve*. The concept of a frequency curve is of great importance in statistics. Mathematically, the curve is represented by the relation $y = f(x)$ and has an important property concerning its area. The following graph represents histogram and frequency curve for the frequency distribution of the mean annual death rates of Example 2.3.



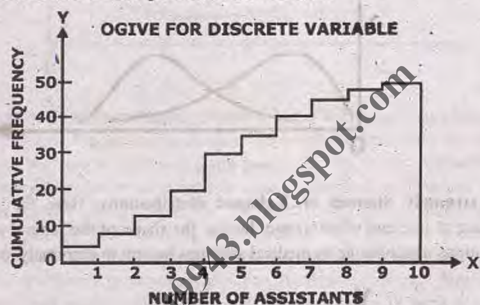

**2.8.5 Cumulative Frequency Polygon or Ogive.** A *cumulative frequency polygon*, popularly known as *Ogive* (rhymes with "alive" and pronounced *o' jiv*) is a graph obtained by plotting the cumulated frequencies of a distribution against the upper or lower class boundaries depending upon whether the cumulation is of the "less than" or "more than" type, and the points are joined by straight line segments. Because of its likeness to an architectural moulding called an *ogee*, a cumulative frequency polygon is called an *Ogive*. An Ogive, when the cumulation is of *less-than* type, is constructed by plotting the points $x_i + h/2, F_i)$ where $x_i + h/2$ is the upper class-boundary of the *ith* class and $F_i$ is the cumulative frequency for the *ith* class, and connecting the successive points by straight line segments. The polygon should start from zero at the lower boundary of the first interval, *i.e.* the point $(x_i - h/2, 0)$ is plotted and joined, and to have a polygon, the last point is also joined with the last upper class-boundary. In case of unequal classes, we merely join the unequally spaced points.



If relative frequencies are used, the cumulative frequency polygon rises from the value 0 at the left to the value 1 at the right. A smoothed Ogive is called an *Ogive curve,* which is often used to locate the partition values such as the median, quartiles, percentiles, etc. of a frequency distribution.

**A percentage cumulative frequency polygon or curve** may also be drawn by expressing the cumulative frequencies as percentages of the total frequency and then connecting the plotted percentages against upper class boundaries. This graphic device is useful for comparing two or more frequency distributions as they are adjusted to a uniform standard.

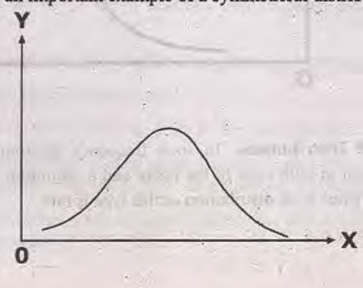
**2.8.6 Ogive for a Discrete Variable.** When a variable $X$ is discrete, its cumulative frequency polygon consists of horizontal line segments between any two successive values and has a jump of height $f_i$ at each value of $x_i$. In other words, the cumulative distribution increases only in jumps and is constant between jumps. For the purpose of illustration, the cumulative frequency polygon drawn for the frequency distribution of assistants in Example 2.5, is shown below:



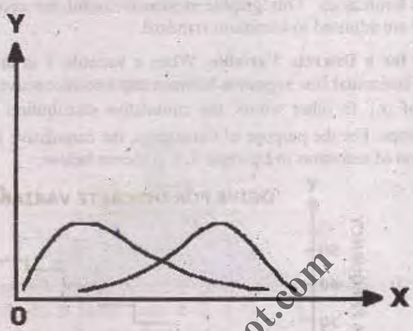

This graph shows that the cumulative frequency polygon is stepped. Such a function is called a *step-function.*

**2.8.7 Types of Frequency Curves.** The frequency distributions occurring in practice, usually belong to one of the following four types:

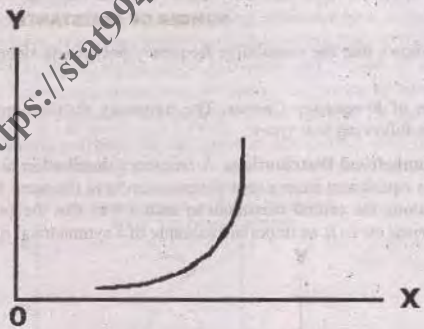
i) **The Symmetrical Distributions.** A frequency distribution or curve is said to be symmetrical if values equidistant from a central maximum have the same frequencies, i.e. the curve can be folded along the central maximum in such a way that the two halves of the curve coincide. The *Normal curve* is an important example of a symmetrical distribution.



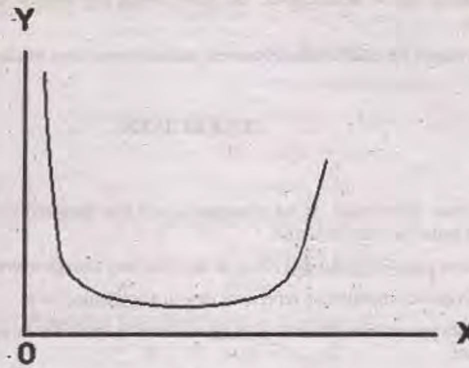
ii) **The Moderately Skewed or Asymmetrical Distributions.** A frequency distribution or curve is said to be *skewed* when it departs from symmetry. Here the frequencies tend to pile up at one end or the order end of the distribution or curve. This is the most common pattern encountered in practice.




iii) **The Extremely Skewed or J-shaped distributions.** Here the frequencies run up to a maximum at one end of the range, having the shape of the letter J or its reverse. Most of the distributions in economic or medical statistics belong to extremely skewed distributions.




iv) **The U-shaped Distributions.** In such frequency distributions or curves, the maximum frequencies occur at both ends of the range and a minimum towards the centre, shaped more or less like the letter U. A distribution of this type is rare.




**2.8.8 Ratio Charts or Semi-logarithmic Graphs.** In the ordinary types of graph, the scales used are called the *natural scales* or the *arithmetic scales*. These graphs can only be used to compare the absolute changes in values because the ordinary graph paper, also known as *arithmetic paper*, is so ruled that equal intervals anywhere on the paper represent equal differences or amounts. More often we are interested in studying the relative changes or ratios. The relative changes or ratios can be displayed and compared by the slope of straight line when the logarithms of the values are plotted on an arithmetic paper. In practice, the difficulty of looking up logarithms can be dispensed with by using another type of graph paper, called *Semi-logarithmic paper* or *ratio paper*. A semi-logarithmic paper or ratio paper is so constructed that equal intervals on the vertical axis indicate equal ratios or rates of change, while equal intervals on the horizontal axis represent equal differences or amounts of change. Thus the essential feature of a Semi-logarithmic chart is that one axis has a logarithmic scale and the other has arithmetic scale.



Graphs obtained by plotting the values on a semi-logarithmic paper or ratio paper and joining the successive points by means of straight line segments are called *Semi-logarithmic graphs* or *Ratio charts*. They are generally used when

i) the relative rates of change are to be compared;

ii) visual comparisons are to be made between two or more series which differ widely in magnitude: and

iii) the data are to be examined to see whether they are characterized by a constant rate of change.

A *ratio chart* possesses the following characteristics:

i) There is no zero line on the logarithmic scale as the logarithm of zero is minus infinity.

ii) A geometric progression when plotted on semi-logarithmic paper, forms a straight line, as the logarithms of a geometric progression form an arithmetic progression.

iii) The slope of the logarithmic scale variable indicates the rate at which the variable is changing (i.e. increasing or decreasing).

iv) In case of two or more curves, the curve having the steepest slope, has the largest rate of change.

v) Equal slopes (in case of parallel curves) indicate equal rates of change.

## EXERCISES

### OBJECTIVE

a) Answer 'True' and 'False'. If the statement is not true then replace the underlined words with words that make the statement true:

i) The term cross-sectional data refers to data that may change overtime.

ii) The frequency distribution represents data in a condensed form.

iii) The data presented in an array does not allow us to locate the largest and smallest values in the data set.

iv) The classes in any frequency distribution are generally not mutually exclusive.

v) For nominally or ordinary scaled data the frequency distribution cannot be constructed.

vi) Frequency distribution of continuous data may be represented diagrammatically.

vii) Frequency distribution can be presented graphically by using both histogram and historigram.

viii) Time series data can be tabulated using frequency distributions.

ix) Simple bar diagram is used for two-dimensional comparisons.

x) The width of a bar in histogram represents the frequency rather than the value of a variable.

xi) Class marks are the lower limits of each class.

xii) The lower class limit is the middle possible data value for a class.

xiii) The sum of relative frequencies in a relative frequency distribution should always equal 100.

xiv) A pie chart can be used to display quantitative data.

xv) A shape of the frequency distribution and the relative frequency distribution always will be different.



b) MULTIPLE CHOICE QUESTIONS.

i) Which of following is not an example of condensed data?

a) frequency distribution b) data array c) histogram d) polygon

ii) In the construction of a frequency distribution the steps are to:

a) decide the number of classes

b) arranging the data in ascending / descending order

c) locate the smallest and largest values in a data set

d) all of above

iii) The number of classes in a frequency distribution generally should be

a) less than five b) more than twenty

c) between five and twenty d) between ten and twenty

iv) As the number of observations and classes increase, the shape of the frequency polygon:

a) remains same b) tends to smooth

c) become more erratic d) none of them

v) A cumulative frequency distribution is graphically represented by:

a) frequency curve b) frequency polygon

c) pie chart d) ogive

vi) A relative frequency distribution presents frequencies in terms of:

a) whole numbers b) percentages

c) fractions d) all of above

vii) A diagram that presents properties that look like slices of a pizza is known as:

a) a bar diagram b) a component bar diagram

c) a histogram d) a pie diagram

viii) Observed data organized into tabular form is called:

a) a bar chart b) a pie chart

c) a frequency polygon d) a frequency distribution

ix) The number of occurrences of a data value is called:

a) the frequency b) the cumulative frequency

c) the relative frequency d) all of above

x) In the following stem-and-leaf diagram:

| Stem | Leaf |
|---|---|
| 3 | 2 3 |
| 4 | 1 2 2 2 3 |
| 5 | 1 1 3 5 5 5 5 6 |
| 6 | 4 5 6 7 |
| 7 | 2 8 |
| 8 | 6 |

The number that occurred the most is

a) 2 b) 55 c) 42 d) 5



## SUBJECTIVE

1) Explain what is meant by classification. What are its basic principles?

2.2 Define the terms "Classification" and "Tabulation". Outline the main steps in tabulation. Wh do you mean by *captions, stubs, title* and *prefatory notes*? (P.U., B.A./B.Sc. 19

2.3 What is a statistical table? What are different types of tables? Explain the different parts of a table and the main points to be kept in mind in their construction. (P.U., B.A./B.Sc. 19

2.4 Represent the data given in the following paragraph in the form of a table, so as to bring clearly all the facts, indicating the source and bearing suitable title:

"According to the census of Manufacturers Report 1945, the John Smith Manufact Company employed 400 non-union and 1,250 union employees in 1941. Of these 220 females of which 140 were non-union. In 1942, the number of union employees increase 1,475 of which 1,300 were males. Of the 250 non-union employees 200 were males. In 19 1,700 employees were union members and 50 were non-union. Of all the employees in 19 250 were females of which 240 were union members. In 1944, the total number of emplo was 2,000 of which one percent were non-union. Of all the employees in 1944, 300 females of which only 5 were non-union."

2.5 a) Write short notes on:

Class-frequency, Class-Interval, Class limits, Class Marks, Size of Class-Interval Sturges' rule.

b) Determine class boundaries, class limits and class marks for the first and last classe respect of the following:

i) Weights of 300 entering freshmen ranged from 98 to 226 pounds, correct nearest pound.

ii) The thickness of 460 washers ranged from 0.421 to 0.563 inches.

c) A sample consists of 84 observations, each recorded as correct to the nearest in ranging in value from 201 to 337. If it is decided to use seven classes of width 20 units to begin the first one at 199.5, find the class boundaries, limits and marks of the se classes.

(I.U. M.A. Econ., 1



2.6 a) What is meant by a frequency distribution? Describe briefly the main steps preparation of a frequency table from raw data.

(P.U., B.A./B.Sc. 1

b) Prepare a frequency table for the price data given below, taking 5 units as the wid class-interval.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 100 | 96 | 92 | 88 | 86 | 84 | 82 | 80 | 78 | 91 |
| 87 | 83 | 79 | 77 | 75 | 73 | 71 | 69 | 58 | 56 |
| 73 | 50 | 57 | 55 | 53 | 51 | 48 | 46 | 63 | 59 |
| 55 | 51 | 49 | 47 | 45 | 43 | 41 | 58 | 54 | 50 |
| 56 | 44 | 42 | 40 | 38 | 36 | 46 | 53 | 50 | 43 |

2.7 a) Why are frequency distributions constructed? What are the rules to be observed in making a frequency distribution from ungrouped data?

b) A record was made of the number of absences per day from a factory over 35 days with the following results:

| Absentees ($x$) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| No. of days ($f$) | 5 | 7 | 9 | 6 | 4 | 2 | 1 | 1 |

i) On how many days were there *fewer than* 4 *people* absent? 27

ii) On how many days were there *at least* 4 *people* absent? 10

iii) What is the total number of absences over the whole 35 days? 28

(M.A. Econ. II Semester, 1980)

2.8 a) Describe the steps you would take to construct a frequency distribution.

b) Tabulate the following marks in a grouped frequency distribution.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 74 | 49 | 103 | 95 | 90 | 118 | 52 | 88 | 101 | 96 | 72 | 56 | 64 | 110 | 97 |
| 59 | 62 | 96 | 82 | 65 | 85 | 105 | 116 | 91 | 83 | 99 | 52 | 76 | 84 | 89 |
| 77 | 104 | 96 | 84 | 62 | 58 | 66 | 100 | 80 | 54 | 75 | 55 | 99 | 104 | 78 |
| 66 | 96 | 83 | 57 | 60 | 51 | 114 | 120 | 121 | 92 | 88 | 64 | 63 | 95 | 78 |



2.9 The following data give the index numbers of 100 commodities in a certain year. Form a grouped frequency distribution, taking 5 as class interval.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 91 | 124 | 109 | 129 | 141 | 107 | 86 | 76 | 118 | 111 |
| 99 | 99 | 114 | 100 | 88 | 108 | 87 | 101 | 101 | 71 |
| 63 | 121 | 122 | 111 | 119 | 77 | 127 | 61 | 133 | 68 |
| 77 | 177 | 110 | 95 | 96 | 96 | 86 | 106 | 119 | 79 |
| 81 | 127 | 86 | 153 | 79 | 129 | 151 | 89 | 143 | 147 |
| 90 | 142 | 105 | 94 | 125 | 96 | 99 | 138 | 145 | 113 |
| 129 | 87 | 113 | 110 | 144 | 91 | 106 | 104 | 97 | 115 |
| 100 | 117 | 73 | 134 | 108 | 102 | 123 | 106 | 119 | 104 |
| 101 | 120 | 112 | 138 | 140 | 103 | 96 | 136 | 78 | 83 |
| 75 | 100 | 113 | 114 | 109 | 116 | 109 | 116 | 104 | 128 |

2.10 Arrange the data given below in an array and construct a frequency distribution, using a class interval of 5.00. Indicate the class boundaries and class limits clearly.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 79.4 | 71.6 | 95.5 | 73.0 | 74.2 | 81.8 | 90.6 | 55.9 |
| 75.2 | 81.9 | 68.9 | 74.2 | 80.7 | 65.7 | 67.6 | 82.9 |
| 88.1 | 77.8 | 69.4 | 83.2 | 82.7 | 73.8 | 64.2 | 63.9 |
| 68.3 | 48.6 | 83.5 | 70.8 | 72.1 | 71.6 | 59.4 | 77.6 |

(B.I.S.E. Lahore, 1972)

2.11 The following figures give the number of children born to 50 women:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 6 | 1 | 5 | 4 | 3 | 3 | 8 | 3 | 1 |
| 4 | 3 | 3 | 0 | 5 | 2 | 1 | 4 | 3 | 3 |
| 5 | 3 | 3 | 6 | 3 | 3 | 2 | 2 | 7 | 3 |
| 1 | 4 | 2 | 4 | 4 | 4 | 6 | 8 | 10 | 7 |
| 7 | 5 | 6 | 5 | 3 | 2 | 3 | 9 | 2 | 2 |

Construct an ungrouped frequency distribution of these data.

2.12 Count the number of letters in each word of the following passage, hyphenated words being treated as single words and make a frequency distribution of word length.

"To forgive an injury is often considered to be a sign of weakness; it is really a strength. It is easy to allow oneself to be carried away by resentment and hate into an vengeance; but it takes a strong character to restrain those natural passions. The man forgives an injury proves himself to be the superior of the man who wronged him, and wrong-doer to shame. Forgiveness may even turn a foe into a friend. So mercy is the form of revenge."

2.13 The weights of 50 football players are listed below:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 193 | 240 | 217 | 283 | 268 | 212 | 251 | 263 | 275 | 208 |
| 230 | 288 | 259 | 225 | 252 | [illegible] | 243 | 247 | 280 | 234 |
| 250 | 236 | 277 | 218 | 245 | [illegible] | 231 | 269 | 224 | 259 |
| 258 | 231 | 255 | 228 | [illegible] | 245 | 246 | 271 | 249 | 255 |
| 265 | 235 | 243 | 219 | 255 | 245 | 238 | 257 | 254 | 284 |

Make a stem-and leaf display for the data and convert it to a frequency table with 10 beginning with 190.

2.14 Make a stem-and-leaf table for the following data. Using 8.0 as the lower limit of class and with a width of 1 unit, convert it to a frequency distribution.

70=N

| | | | | | | |
|---|---|---|---|---|---|---|
| 9.0 | 10.2 | 11.3 | 12.1 | 10.7 | 13.8 | 10.8 |
| [illegible] | 13.6 | 16.4 | 11.0 | 15.8 | 9.3 | 13.7 |
| 11.7 | 11.0 | 8.0 | 12.0 | 11.5 | 9.7 | 11.6 |
| 10.1 | 14.1 | 10.0 | 9.9 | 13.4 | 15.7 | 11.5 |
| 12.3 | 9.8 | 13.0 | 9.1 | 8.3 | 12.9 | 14.0 |
| 10.5 | 13.2 | 10.5 | 10.6 | 12.5 | 15.1 | 12.8 |
| 10.4 | 11.2 | 9.3 | 11.7 | 17.7 | 13.9 | 16.9 |
| 13.4 | 11.8 | 16.8 | 14.2 | 11.8 | 9.6 | 11.9 |
| 8.7 | 14.7 | 10.9 | 17.9 | 11.5 | 14.7 | 15.9 |
| 11.8 | 10.6 | 12.6 | 12.6 | 15.7 | 14.9 | 9.9 |

2.15 Describe the advantages and disadvantages of diagrammatic representation. Describe the important types of diagrams.

2.16 Describe each of the diagrams listed below and give an illustration in each case.
Bar diagrams; Multiple bar diagrams;
Pie-diagrams; Pictograms, and Profit and Loss charts.

(P.U., B.A./B.Sc.

2.17 Give a description of various graphic and pictorial aids for representing data. Mention particular uses of some methods. (P.U., B.A./B.Sc. 1961)

2.18 Describe briefly the different types of diagrams generally used for presenting statistical data. State advantages and disadvantages of any three of them giving illustrations where possible.

2.19 Represent the following yield per acre data by a bar diagram.

| Years: | 1940 | 1941 | 1942 | 1943 | 1944 | 1945 | 1946 | 1947 | 1948 | 1949 |
|---|---|---|---|---|---|---|---|---|---|---|
| Yield Per acre: | 5 | 7 | 9 | 6 | 10 | 12 | 8 | 11 | 12 | 10 |

2.20 Following table gives the birth rates and death rates per thousand of a few countries. Represent them by multiple bar charts.

| Country | Birth Rate | Death Rate |
|---|---|---|
| India | 33 | 24 |
| Japan | 32 | 19 |
| Germany | 16 | 10 |
| Egypt | 44 | 24 |
| Australia | 20 | 9 |
| New Zealand | 18 | 8 |
| France | 21 | 10 |
| Russia | 38 | 16 |

Represent the following data by rectangular diagram showing percentage of Income spent by two families on different items of expenditure.

*Family-budgets of two families*

| Items of Expenditure | Family A Income Rs.80 | Family B Income Rs.40 |
|---|---|---|
| | Actual Expenses | Actual Expenses |
| Food | Rs. 32 | Rs. 20 |
| Clothing | Rs. 20 | Rs. 8 |
| Shelter | Rs. 8 | Rs. 4 |
| Fuel and Light | Rs. 4 | Rs. 2 |
| Miscellaneous | Rs. 16 | Rs. 6 |
| **Total** | **Rs. 80** | **Rs. 40** |



The following table gives the details of monthly expenditure of three families. Represent the data by a suitable diagram on percentage basis.

component

| Items of Expenditure | Family A (Rs.) | Family B (Rs.) | Family C (Rs.) | |
|---|---|---|---|---|
| Food Articles | 43 | 87 | 120 | 250 |
| Clothing | 18 | 17 | 25 | 60 |
| Recreation | 3 | 10 | 12 | 25 |
| Education | 5 | 9 | 15 | 29 |
| Rent | 10 | 21 | 17 | 48 |
| Miscellaneous | 6 | 15 | 11 | 32 |

2.23 Represent the following data by means of a pictogram:

(a)

| Industry | No. of Employees (000) |
|---|---|
| Marine | 96 |
| Forest | 187 |
| Mineral | 290 |
| Farm | 635 |

(b)

| Year | Production of Vans |
|---|---|
| 1982 | 2040 |
| 1983 | 2996 |
| 1984 | 4319 |
| 1985 | 6324 |

✓2.24 a) Draw a Pie-diagram and also a Component Bar-diagram for the following data:

| Item | Expenditure in Rs. |
|---|---|
| Food | 190 |
| Clothing | 64 |
| Rent | 100 |
| Medical care | 46 |
| Other items | 80 |

b) Graph the following data showing the areas in millions of square miles of the oceans of the world, using (i) a bar chart, (ii) a pie chart.

| Ocean | Pacific | Atlantic | Indian | Antarctic | Arctic |
|---|---|---|---|---|---|
| Area | 70.8 | 41.2 | 28.5 | 7.6 | 4.8 |

2.25 a) The area sown in *Rabi Crop* is as follows: Prepare a Pie-chart.

| | | |
|---|---|---|
| Wheat | 106 | lakh acres |
| Gram | 30 | lakh acres |
| Barley | 15 | lakh acres |
| Pulses | 10 | lakh acres |
| Fodder | 25 | lakh acres |
| Other crops | 14 | lakh acres |

b) Calculate the per cent contribution of each crop to the total *Rabi* crops.

(P.U., B.A./B.Sc., 19

2.26 Represent the following data by sub-divided bars drawn on a percentage basis or by a Pie-diagram.

*Cost per ton disposed commercially*

| Particulars | 1924 | 1928 |
|---|---|---|
| Wages | 12.74 | 7.95 |
| Other costs | 5.46 | 4.51 |
| Royalties | 0.56 | 0.50 |
| Total | 18.76 | 12.96 |
| Sale proceeds per ton | 19.91 | 12.16 |
| Profit ( + ) or loss ( − ) per ton | +1.15 | -0.80 |

2.27 a) Describe the Graphical Methods used in Statistics, giving their respective advantages and disadvantages. (P.U., M.A. Econ. 1981)

b) Draw up a list of rules for the construction of graphs. (P.U., B.A. (Optional), 1969)

2.28 State the general rules which should be borne in mind in the construction of graphs. Draw a suitable graph of the following time series.



| Year | Gross Profit (Rs.) | Expenses (Rs.) |
|---|---|---|
| 1931 | 7,9000 | 2,700 |
| 1932 | 5,550 | 1,700 |
| 1933 | 4,800 | 1,500 |
| 1934 | 4,500 | 1,150 |
| 1935 | 6,550 | 4,300 |
| 1936 | 9,100 | 5,800 |
| 1937 | 8,500 | 3,700 |
| 1938 | 7,300 | 3,900 |
| 1939 | 6,500 | 4,800 |
| 1940 | 6,200 | 3,650 |

2.29 Show graphically the following monthly imports and exports of a particular commodity during the year 1960-61. Also show graphically the balance of trade. Imports and exports are given in crores of rupees.

| Month | Imports | Exports |
|---|---|---|
| April | 14 | 12 |
| May | 13 | 19 |
| June | 10 | 9 |
| July | 11 | 12 |
| August | 12 | 10 |
| September | 13 | 9 |
| October | 10 | 13 |
| November | 9 | 12 |
| December | 10 | 14 |
| January | 11 | 12 |
| February | 12 | 10 |
| March | 11 | 13 |



2.30 a) Explain with the help of diagrams the difference between a frequency polygon, a histogram and an ogive. (P.U., B.A. (Part-I), 196

b) Construct (i) a Histogram, (ii) a Relative frequency polygon and (iii) an Ogive for the following frequency distribution of the heights of 100 male students at Islamia University Bahawalpur.

| Height (inches) | 60–62 | 63–65 | 66–68 | 69–71 | 72–74 |
|---|---|---|---|---|---|
| No. of Students | 5 | 18 | 42 | 27 | 8 |

(I.U. M.A., Econ. 1989)

2.31 What is meant by a *Histogram*? Draw a histogram for the distribution of earnings (Rs.) given below:

| Earnings | 180–184 | 185–189 | 190–194 | 195–199 | 200–204 | 205–209 | 210–214 | 215–219 |
|---|---|---|---|---|---|---|---|---|
| Workers | 10 | 24 | 30 | 36 | 40 | 29 | 23 | 8 |

State how you would construct the histogram if the class-intervals were unequal in size.

(Engg. University, B.Sc. Final, 1977)

2.32 a) Define the statistical term *Histogram*.

b) Explain the method of constructing histograms when the class intervals are unequal

c) In a savings group, there are 400 members and the number of savings certificates held by them are shown in the following table.

| No. of certificate held | No. of members |
|---|---|
| 1 – 50 | 10 |
| 51 – 100 | 15 |
| 101 – 150 | 30 |
| 151 – 200 | 40 |
| 201 – 300 | 120 |
| 301 – 400 | 100 |
| 401 – 500 | 85 |

Construct a histogram of the distribution of savings certificates. (P.U., B. Com., 1961)

2.33 Draw a Histogram and a Frequency Polygon for the following distribution.

| Degree of Cloudiness: | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency: | 580 | 150 | 196 | 75 | 55 | 40 | 45 | 68 | 75 | 130 | 220 |

(P.U., B.A. (Part-I); 1962)

2.34 a) Draw a histogram illustrating the following data:

| Age nearest birth day | 20–24 | 25–29 | 30–39 | 40–44 | 45–49 | 50–54 | 55–64 |
|---|---|---|---|---|---|---|---|
| Number of men | 1 | 2 | 26 | 22 | 20 | 15 | 14 |

(P.U., B.A./B.Sc. 1973, 75, 78; M.A. Econ. 1980, P.C.S., 1971)

b) Given the following frequency table of the heights in centimeters of 1,000 students, draw its histogram showing the class marks:

x 155–157, 158–160, etc.

f 4 8 26 53 89 146 188 181 125 92 60 22 4 1 1

(P.U., B.A./B.Sc., 1971)

2.35 a) What is a cumulative frequency curve? How does it differ from the ordinary curve? (P.U., M.A. Econ. 1985)

b) Describe the common types of frequency curves. Indicates their shapes.

2.36 Construct an Ogive from the following table:

| Weights | Frequency |
|---|---|
| 118 – 126 | 3 |
| 127 – 135 | 5 |
| 136 – 144 | 9 |
| 145 – 153 | 12 |
| 154 – 162 | 5 |
| 163 – 171 | 4 |
| 172 – 180 | 2 |

(P.U., B.A./B.Sc., 1967)

2.37 Pupils were asked how long it took them to walk to school on a particular morn cumulative frequency distribution was formed.

| Time taken (minutes) | < 5, | < 10, | < 15, | < 20, | < 25, | < 30, | < 35, | < 40, | < 45 |
|---|---|---|---|---|---|---|---|---|---|
| Cum. Frequency ($F$) | 28 | 45 | 81 | 143 | 280 | 349 | 374 | 395 | 400 |

a) Draw a cumulative frequency curve and estimate how many pupils took less th minutes.

b) 6% of the pupils took $x$ minutes or longer. Find $x$.

c) Take equal class-intervals of 0–, 5–, 10–, etc., construct a frequency distribution and a histogram.

2.38 Describe in detail the method of drawing Ratio Charts and explain their uses in econo statistics. (P.U., M.A. Econ., 19

2.39 Explain what is meant by a ratio chart and discuss its advantages over the natural diagram. Describe and illustrate two practical applications of a ratio chart.

2.40 Toss five coins together and note the number of heads ($x$). Do this 64 times and coun number of times that $x$ = 0, 1, 2, 3, 4, 5. Construct a frequency polygon and an Og represent these results.

2.41 a) Distinguish between primary and secondary data. Describe briefly the method collecting primary data.

b) Find the missing entries in the following frequency distribution table?

| Class Limits | Frequency | Relative Frequency | Cumulative Frequency | Cumulative Percentage |
|---|---|---|---|---|
| 8 to – | – | – | – | 25 |
| – to – | – | 0.05 | – | – |
| – to – | – | – | 9 | – |
| – to – | – | 0.30 | 15 | – |
| – to 32 | | – | | |
| | – | – | | |

(P.U., B.A./B.Sc., 2

2.42 The following data give the annual earnings (rounded to thousands of dollars) o households.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 30 | 5 | 29 | 7 | 20 | 31 | 28 | 10 |
| 13 | 24 | 25 | 7 | 22 | 15 | 15 | 22 |
| 78 | 79 | 99 | 30 | 80 | 80 | 75 | 63 |
| 24 | 35 | 25 | 90 | 35 | 33 | 70 | 63 |
| 35 | 17 | 9 | | | | | |

i) Prepare a stem-and-leaf display for these data.

ii) Condense the stem-and-leaf display by grouping the stems as 0–2, 3–5 and 6–9.

(P.U., B.A./B.Sc., 2

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# CHAPTER 3

# MEASURES OF CENTRAL TENDENCY OR AVERAGES

# MEASURES OF CENTRAL TENDENCY OR AVERAGES

## 3.1 INTRODUCTION

For practical purposes, the condensation of data set into a frequency distribution and the visual presentation are not enough, particularly, when two or more different data sets are to be compared. A data set can be summarized in a single value. Such a value, usually somewhere in the centre and representing the entire data set, is a value at which the data have a tendency to concentrate. The tendency of the observations to cluster in the central part of the data set is called *Central Tendency* and the summary value as a *measure of central tendency*. Since a measure of central tendency indicates the *location* or general position of the distribution or the data set in the range of observations, it is also known as a *measure of location or position*. The measures of central tendency or location are generally known as *Averages*. But in everyday language, 'the average' is often understood to refer to the arithmetic mean (a form of average to be discussed in section 3.4), it is for this reason that when anyone speaks of 'the average' (without qualification) of a set of observations, it may, as a rule, be assumed that the arithmetic mean is meant. The use of the term average has been traced to the time of Pythagoras (570–500 B.C.). Two points should be noted. First, a measure of central tendency should be somewhere within the range of the data, and secondly, it should remain unchanged by a rearrangement of the observations in a different order.

Since the late nineteenth century, the practice has been to make a distinction between a sample and a population from which the sample is drawn, by using Latin letters for numerical quantities describing the sample and Greek letters for corresponding quantities characterizing the population. It should be noted that population parameters are rarely calculated directly as all observations from the population are not usually available. The measures corresponding to population parameters are generally calculated from sample data and are regarded as the *estimates* of population parameters.

## 3.2 CRITERIA OF A SATISFACTORY AVERAGE

Several types of averages are defined to measure the representative or "typical" value of a set of data or distribution. It is therefore desirable that an average should be

i) rigorously defined,

ii) based on all the observations made,

iii) simple to understand and easy to interpret,

iv) quickly and easily calculated,

v) amenable to mathematical treatment,

vi) relatively stable in repeated sampling experiments, and

vii) not unduly influenced by abnormally large or small observations.

An average that possesses all or most of the conditions stated above, is considered a *satisfactory average*.

## 3.3 TYPES OF AVERAGES

The most common types of averages are (i) the arithmetic mean or simply the mean, (ii) the geometric mean, (iii) the harmonic mean, (iv) the median and (v) the mode. The first three types are mathematical in character and give an indication of the magnitude of the observed values. The fourth type indicates the middle position while the last provides information about the most frequent value in the distribution or the data set.

## 3.4 THE ARITHMETIC MEAN

The *arithmetic mean* or simply the *mean* is the most familiar average. It is defined as a value obtained by dividing the sum of all the observations by their number, that is

$$\text{Mean} = \frac{\text{Sum of all the observations}}{\text{Number of the observations}}$$

The mean may correspond to either a population or a sample from the population. If the given set of observations represents a population, the mean is called the *population mean* which is traditionally denoted by $\mu$ (the Greek letter *mu*). Thus the population mean of a set of N observations $x_1, x_2, \ldots\ldots, x_N$ is given as

$$\mu = \frac{x_1 + x_2 + \ldots + x_N}{N} = \frac{\sum x_i}{N}, \qquad (i = 1, 2, \ldots\ldots, N)$$

where $\Sigma$, the Greek capital *sigma*, is a convenient symbol for summation.

If, instead, the given set of observations represents a sample, the mean is called the *sample mean*, usually denoted by placing a bar over the symbol used to represent the observations or the variable. Thus the mean of a set of $n$ observations $x_1, x_2, \ldots\ldots, x_n$ is defined as

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{\sum x_i}{n}, \qquad (i = 1, 2, \ldots\ldots, n)$$

where $\bar{x}$ is the mean of a sample of size $n$.

It is worthwhile to note that the population mean is a fixed quantity, whereas $\bar{x}$, the sample mean, is a variable because different samples from the same population tend to have different means.

In order to interpret the meaning of arithmetic mean, let $x_i$ denote the marks obtained by the *ith* student in a class. Then $\sum x_i$ stands for the total marks obtained by all students and $\bar{x}$, the mean, represents the number of marks that *would have been obtained by each student if everyone in the class had obtained the same number of marks*. Geometrically the mean represents a point at which the distribution or the set of observations would balance.

**Example 3.1** The marks obtained by 9 students are given below:

45, 32, 37, 46, 39, 36, 41, 48, 36.

Calculate the arithmetic mean.

The mean is given by

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{45 + 32 + 37 + 46 + 39 + 36 + 41 + 48 + 36}{9}$$

$$= \frac{360}{9} = 40 \text{ marks}$$

It is relevant to note that, if these marks represent the entire set of observations for the population, the above calculation gives the population mean, i.e. $\mu$ would equal to 40 marks.

**3.4.1 The Weighted Arithmetic Mean.** The multipliers or a set of numbers which express more or less adequately the relative importance of various observations in a set of data are technically called the *weights*. We assign weights $w_1, w_2,..., w_n$ to the observations in a set of data according to their relative importance, when the observations are not of equal importance. The *weighted mean*, denoted by $\bar{x}_w$, of a set of $n$ values $x_1, x_2,....., x_n$ with corresponding weights $w_1, w_2,..., w_n$ is then defined as

$$\bar{x}_w = \frac{x_1 w_1 + x_2 w_2 + ... + x_n w_n}{w_1 + w_2 + ... + w_n}$$

$$= \frac{\sum x_i w_i}{\sum w_i} \qquad (i = 1, 2,...., n)$$

A weighted average is generally employed in the calculation of index numbers, birth and death rates, etc.

**Example 3.2** Calculate the weighted mean from the following data:

| Item | Expenditure (Rs.) | Weights |
|---|---|---|
| Food | 290 | 7.5 |
| Rent | 54 | 2.0 |
| Clothing | 98 | 1.5 |
| Fuel and Light | 75 | 1.0 |
| Other items | 75 | 0.5 |

(P.U., B.A. (Optional), 1969, 94)

We calculate the weighted mean as below:

| Item | Expenditure ($x_i$) | Weights ($w_i$) | $x_i w_i$ |
|---|---|---|---|
| Food | 290 | 7.5 | 2175.0 |
| Rent | 54 | 2.0 | 108.0 |
| Clothing | 98 | 1.5 | 147.0 |
| Fuel and Light | 75 | 1.0 | 75.0 |
| Other items | 75 | 0.5 | 37.5 |
| Total | -- | 12.5 | 2542.5 |

Hence $\bar{x}_w = \frac{\sum x_i w_i}{\sum w_i} = \frac{2542.5}{12.5} = \text{Rs. } 203.4$

**3.4.2 Properties of the Arithmetic Mean.** The arithmetic mean has the following four properties:

i) For a set of data, the sum of the deviations of the observations $x_i$'s from their mean, $\bar{x}$, taken with their proper signs, is equal to zero.

The sum of the deviations $= \sum (x_i - \bar{x})$, $(i = 1, 2, \ldots, n)$

$$= \sum x_i - n\bar{x} \qquad (\because \bar{x} \text{ is constant})$$

$$= \sum x_i - \sum x_i = 0 \qquad (\because \bar{x} = \sum x/n)$$

ii) The sum of squared deviations of the $x_i$'s from the mean, $\bar{x}$, is a minimum. In other words, $\sum (x_i - \bar{x})^2 \le \sum (x_i - a)^2$, where $a$ is an arbitrary value other than the mean.

Now $\quad \Sigma(x_i - a)^2 = \Sigma(x_i - \bar{x} + \bar{x} - a)^2$

$$= \Sigma[(x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - a) + (\bar{x} - a)^2]$$

$$= \Sigma(x_i - \bar{x})^2 + 2(\bar{x} - a)\Sigma(x_i - \bar{x}) + n(\bar{x} - a)^2$$

$$= \Sigma(x_i - \bar{x})^2 + n(\bar{x} - a)^2 \quad [\because \Sigma(x_i - \bar{x}) = 0]$$

It is obvious that $\Sigma(x_i - a)^2 > \Sigma(x_i - \bar{x})^2$ by $n(\bar{x} - a)^2$. The equality sign holds only when $\bar{x} = a$.

Hence $\Sigma(x_i - \bar{x})^2$ is always less than $\Sigma(x_i - a)^2$ if $a \ne \bar{x}$.

This property is usually called the *minimal* property of the mean.

iii) If $k$ subgroups of data consisting of $n_1, n_2, \ldots, n_k$, $(\Sigma n_i = n)$ observations have respective means $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$, then $\bar{x}$, the mean for all the data, is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots + n_k\bar{x}_k}{n_1 + n_2 + \ldots + n_k}$$

$$= \frac{\sum n_i\bar{x}_i}{n}, \qquad (i = 1, 2, \ldots, k)$$

*i.e.* a weighted mean of all the subgroup means.

iv) If $y_i = ax_i + b \, (i = 1, 2, \ldots, n)$, where $a$ and $b$ are any two numbers and $a \ne 0$, then $\bar{y} = a\bar{x} + b$.

Now summing over all values of $i$, we obtain

$$\Sigma y_i = a\Sigma x_i + nb$$

Dividing both sides by $n$, we get

$$\bar{y} = a\bar{x} + b$$

As the equation $y = ax + b$ represents a linear transformation from $x$ to $y$, this property is usually called the *invariance* of the mean under a *linear transformation* and it provides the basis for so-called *coding*

which refers to the operation of subtracting (or adding) a constant from each observation and then dividing (or multiplying) by another constant for computational convenience.

**Example 3.3** The mean heights and the number of students in three sections of a statistics class are given below:

| Section | Number of boys | Mean height |
|---|---|---|
| A | 40 | 62" |
| B | 37 | 58" |
| C | 43 | 61" |

Find the overall mean height of 120 boys.

Here $n_1 = 40;\ n_2 = 37;\ n_3 = 43$, and

$\bar{x}_1 = 62",\ \bar{x}_2 = 58",\ \bar{x}_3 = 61"$

The mean height of the combined class is given as

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3}$$

$$= \frac{(40 \times 62) + (37 \times 58) + (43 \times 61)}{40 + 37 + 43} = \frac{7249}{120} = 60.41$$



**3.4.3 Mean From Grouped Data.** When the number of observations is very large, the data are grouped into a frequency distribution, which is used to calculate the *approximate* values of descriptive measures as the identity of the observations is lost. To calculate the approximate value of the mean, the observations in each class are assumed to be identical with the class midpoint so that the product of the midpoint by the number of observations, i.e. frequency, would be approximately equal to the sum of observations for each class. Thus, if a frequency distribution has $k$ classes with midpoints $x_1, x_2, ..., x_k$ and the corresponding frequencies $f_1, f_2, ..., f_k$ $(\Sigma f_i = n)$, the mean is then given by the formula

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + ... + f_kx_k}{f_1 + f_2 + ... + f_k}$$

$$= \frac{\Sigma f_i x_i}{n}, \qquad (i = 1, 2, ...., k)$$

As a *weight* indicates the number of times an observation is to be counted, the mean calculated from a frequency distribution may also be regarded as the *weighted mean* where each class midpoint $x_i$, taken as the average value of the observations in that class, is weighted by the respective frequency $f_i$ and the sum of the weighted products is divided by the sum of frequencies, i.e. weights.

Sometimes, there may be a slight difference in the values of $\bar{x}$ on account of errors caused by the assumption that all observations in any class may be treated as approximately the midpoint of that class, but experience tells us that this error is usually small and never serious. The following example illustrates the situation.

**Example 3.4** Calculate the mean weight of apples from the data given in Example 2.2 d from the observed values and from the data grouped into a frequency distribution.

The calculations are outlined below:

| Weight (grams) | Sum of actual observations | $f_i$ | Mean Weight of each class ($\bar{x}_i$) | $f_i\bar{x}_i$ | Midpoints ($x_i$) | $f_i x_i$ |
|---|---|---|---|---|---|---|
| 65 – 84 | 695 | 9 | 77.2 | 694.8 | 74.5 | 670.5 |
| 85 – 104 | 947 | 10 | 94.7 | 947.0 | 94.5 | 945.0 |
| 105 – 124 | 1919 | 17 | 112.9 | 1919.3 | 114.5 | 1946.5 |
| 125 – 144 | 1325 | 10 | 132.5 | 1325.0 | 134.5 | 1345.0 |
| 145 – 164 | 766 | 5 | 153.2 | 766.0 | 154.5 | 722.5 |
| 165 – 184 | 716 | 4 | 179.0 | 716.0 | 174.5 | 698.0 |
| 185 – 204 | 956 | 5 | 191.2 | 956.0 | 194.5 | 972.5 |
| Total | 7324 | 60 | -- | 7324.1 | -- | 7350.0 |

*Calculation based on Ungrouped data*

We calculate the mean weight, $\bar{x}$, directly from all the observed values, which add to 7324 second column consists of subtotal of actual observations in any class).

$$\therefore \qquad \bar{x} = \frac{\sum x_i}{n}, \qquad (i = 1, 2, \ldots, 60)$$

$$= \frac{7324}{60} = 122.07 \text{ grams}$$

This is the exact mean of the given data.

*Next,* we find the mean weight, $\bar{x}$, by multiplying the actual mean of the observations in any c by the corresponding frequency, adding the products and then dividing by $n$ (column 5).

$$\text{Thus} \qquad \bar{x} = \frac{\sum f_i \bar{x}_i}{n}, \qquad (i = 1, 2, \ldots, 7)$$

$$= \frac{7324.1}{60} = 122.07 \text{ grams}$$

*Calculation based on Grouped data*

Here we calculate the mean weight from grouped data, assuming that all observations in any c are identical with the midpoint of that class. The sixth column consists of class midpoints, $x_i$ and products are given in column 7.

$$\text{Then} \qquad \bar{x} = \frac{\sum f_i x_i}{n}, \qquad (i = 1, 2, \ldots, 7)$$

$$= \frac{7350.0}{60} = 122.5 \text{ grams}$$

It should be noted that the numerical value of $\bar{x}$, calculated from the frequency distribution is slightly different from the value obtained directly from the ungrouped data.

**3.4.4 Change of Origin and Scale.** To reduce the computational labour and to save time, a change of the origin and scale can be made. If $x_i$ denotes an observed value, $a$ and $b$ are two constants with $b \neq 0$, then the operations $x_i + a$, $bx_i$ and $bx_i + a$ are known respectively as the *change of origin*, the *change of scale* and both *change of origin and scale.*

Let $a$ be an arbitrary origin, sometimes called *assumed mean*, and let $x_i = a + hu_i$ where $h$ denotes the unit of measurement. Then its corresponding *coded* value is $u_i = \dfrac{x_i - a}{h}$.

Now
$$\bar{x} = \frac{1}{n}\sum x_i = \frac{1}{n}\sum(a + hu_i)$$
$$= \frac{na}{n} + \frac{h\sum u_i}{n} = a + h\bar{u}$$

Thus the arithmetic mean can be calculated from any *origin* we may choose and using any *scale* we desire. This transformation is particularly useful for calculations based on grouped data, where $h$ is the width of class interval and $a$ is usually chosen the class midpoint lying in the region of the higher frequencies so that the larger frequencies may be multiplied by smaller values of $u$. This procedure gives us a *Short method* for hand calculations.

**Example 3.5** Given the following frequency distribution of weights, calculate the mean weight by the Short Method.

| Weight (grams) | 65–84 | 85–104 | 105–124 | 125–144 | 145–164 | 165–184 | 185–204 |
|---|---|---|---|---|---|---|---|
| $f$ | 9 | 10 | 17 | 10 | 5 | 4 | 5 |



To calculate the mean weight, let us take $u_i = \dfrac{x_i - 114.5}{20}$, where $a$ = 114.5 is the midpoint corresponding to the largest frequency and $h$ = 20 is the width of the uniform class interval. The necessary calculations are shown in the table below:

| Weight (grams) | Midpoints ($x_i$) | $f_i$ | $u_i$ | $f_i u_i$ |
|---|---|---|---|---|
| 65 – 84 | 74.5 | 9 | –2 | –18 |
| 85 – 104 | 94.5 | 10 | –1 | –10 |
| 105 – 124 | 114.5 | 17 | 0 | –28 |
| 125 – 144 | 134.5 | 10 | 1 | 10 |
| 145 – 164 | 154.5 | 5 | 2 | 10 |
| 165 – 184 | 174.5 | 4 | 3 | 12 |
| 185 – 204 | 194.5 | 5 | 4 | 20 |
| Total | -- | 60 | -- | +52<br>24 |

Thus
$$\bar{x} = a + h\bar{u}, \text{ where } \bar{u} = \frac{\sum f_i u_i}{n}$$
$$= 114.5 + \frac{(24)(20)}{60} = 114.5 + 8.0 = 122.5 \text{ grams.}$$

**Example 3.6** Compute the mean for the following frequency distribution of annual death rates:

| Death Rate | Frequency |
|---|---|
| 3.5 – 4.4 | 1 |
| 4.5 – 5.4 | 4 |
| 5.5 – 6.4 | 5 |
| 6.5 – 7.4 | 13 |
| 7.5 – 8.4 | 12 |
| 8.5 – 9.4 | 19 |
| 9.5 – 10.4 | 13 |
| 10.5 – 11.4 | 10 |
| 11.5 – 12.4 | 6 |
| 12.5 – 13.4 | 4 |
| 13.5 – 14.4 | 1 |
| **Total** | **88** |

The necessary calculations are given below:

| Death Rate | Midpoints $(x_i)$ | $f_i$ | $u_i(=x_i-8.95)$ | $f_iu_i$ |
|---|---|---|---|---|
| 3.5 – 4.4 | 3.95 | 1 | –5 | –5 |
| 4.5 – 5.4 | 4.95 | 4 | –4 | –16 |
| 5.5 – 6.4 | 5.95 | 5 | –3 | –15 |
| 6.5 – 7.4 | 6.95 | 13 | –2 | –26 |
| 7.5 – 8.4 | 7.95 | 12 | –1 | –12 |
| 8.5 – 9.4 | 8.95 | 19 | 0 | –74 |
| 9.5 – 10.4 | 9.95 | 13 | 1 | 13 |
| 10.5 – 11.4 | 10.95 | 10 | 2 | 20 |
| 11.5 – 12.4 | 11.95 | 6 | 3 | 18 |
| 12.5 – 13.4 | 12.95 | 4 | 4 | 16 |
| 13.5 – 14.4 | 13.95 | 1 | 5 | 5 |
| **Total** | -- | **88** | -- | +72 / –2 |

Hence $\bar{x} = a + \frac{\sum f_i u_i}{n}$, where $a$ is assumed mean and $h = 1$.

$$= 8.95 + \frac{(-2)}{88} = 8.95 - 0.02 = 8.93$$

## 3.5 THE GEOMETRIC MEAN

The *geometric mean, G,* of a set of $n$ positive values $x_1, x_2, ..., x_n$ is defined as the positive *nth* root of their product, *i.e.*

$$G = \sqrt[n]{x_1\, x_2 \ldots x_n}, \qquad \text{where } x > 0$$

When $n$ is large, the computation of the geometric mean becomes laborious, as we have to multiply all the values and then extract the *nth* root. The arithmetic is simplified by using logarithms to the base 10. Thus, taking logarithms, we get

$$\log G = \frac{1}{n}[\log x_1 + \log x_2 + ... + \log x_n]$$

$$= \frac{1}{n}\sum \log x_i$$

Hence $$G = \text{antilog}\left[\frac{1}{n}\sum \log x_i\right]$$

It means that geometric mean is the anti-logarithm of the arithmetic mean of the logarithms of the values themselves.

For data organized into a grouped frequency distribution, having $k$ classes with classmarks $x_1, x_2, ..., x_k$ and the corresponding frequencies $f_1, f_2, ..., f_k (\sum f_i = n)$, the formula for the geometric mean is given by

$$G = [x_1^{f_1} . x_2^{f_2} ... x_k^{f_k}]^{1/n},$$

In terms of logarithms, the formula becomes

$$\log G = \frac{1}{n}[f_1 \log x_1 + f_2 \log x_2 + ... + f_k \log x_k]$$

$$= \frac{1}{n}\sum f_i \log x_i$$

or $$G = \text{antilog}\left[\frac{1}{n}\sum f_i \log x_i\right]$$

The *weighted* geometric mean of the values $x_1, x_2, ..., x_k$ with corresponding weights $w_1, w_2, ..., w_k$ is given by

$$\log G_w = \frac{1}{\sum w_i}[\sum w_i \log x_i]$$

The geometric mean is appropriate to average ratios and rates of change.

**Example 3.7** Find the geometric mean of 45, 32, 37, 46, 39, 36, 41, 48 and 36.

The geometric mean, $G$, is calculated as

$$G = \sqrt[9]{(45 \times 32 \times 37 \times 46 \times 39 \times 36 \times 41 \times 48 \times 36)}$$

Taking logs, we have

$$\log G = \frac{1}{9}[\log 45 + \log 32 + \log 37 + \log 46 + \log 39 + \log 36 + \log 41 + \log 48 + \log 36]$$

$$= \frac{1}{9}[1.65321 + 1.50515 + 1.56820 + 1.66276 + 1.59106 + 1.55630 + 1.61278 + 1.68124 + 1.55630]$$

$$\log G = \frac{1}{9}[14.38700] = 1.59856$$

Hence $G = \text{anti}-\log(1.59856) = 39.68$

**Example 3.8** Given the following frequency distribution of weights, calculate the geometric mean

| Weight (grams) | 65–84 | 85–104 | 105–124 | 125–144 | 145–164 | 165–184 | 185–204 |
|---|---|---|---|---|---|---|---|
| $f$ | 9 | 10 | 17 | 10 | 5 | 4 | 5 |

The computation of the geometric mean is shown below:

| Weight (grams) | $x_i$ | $f_i$ | $\log x_i$ | $f_i \log x_i$ |
|---|---|---|---|---|
| 65 – 84 | 74.5 | 9 | 1.8722 | 16.8498 |
| 85 – 104 | 94.5 | 10 | 1.9754 | 19.7540 |
| 105 – 124 | 114.5 | 17 | 2.0589 | 35.0013 |
| 125 – 144 | 134.5 | 10 | 2.1287 | 21.2870 |
| 145 – 164 | 154.5 | 5 | 2.1889 | 10.9445 |
| 165 – 184 | 174.5 | 4 | 2.2418 | 8.9672 |
| 185 – 204 | 194.5 | 5 | 2.2889 | 11.4445 |
| $\Sigma$ | -- | 60 | -- | 124.2483 |

$$\therefore \quad \log G = \frac{1}{n}\Sigma f_i \log x_i = \frac{124.2483}{60} = 2.0708$$

Hence $G = \text{Anti-log}\,(2.0708) = 117.7$ grams



## 3.6 THE HARMONIC MEAN

The *harmonic mean, H,* of a set of $n$ values $x_1, x_2, \ldots, x_n$ is defined as the reciprocal of the arithmetic mean of the reciprocals of the values. In symbols,

$$H = \text{Reciprocal of } \frac{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}{n}$$

$$= \frac{n}{\Sigma \frac{1}{x_i}}, \qquad \text{where } x \neq 0$$

The harmonic mean is an appropriate type to be used in averaging certain kinds of ratios or rates of change. To illustrate this formula, let us take an example. Suppose a car is running at the rate of 15 *km/hr* during the first 30 *km*; at 20 *km/hr* during the second 30 *km*; and at 25 *km/hr* during the third 30 *km*. The

distance is constant but the times are variable. Therefore, the harmonic mean is the correct average. In this case, the harmonic mean is

$$H = \text{Reciprocal of } \frac{\frac{1}{15}+\frac{1}{20}+\frac{1}{25}}{3}$$

$$= \frac{3}{0.06667+0.5000+0.04000}$$

$$= \frac{3}{0.15667} = 19.15\, km/hr \text{ approximately.}$$

Care should be exercised to apply the harmonic mean. The following rule will help determine the application of the harmonic mean.

*"When rates are expressed as x per y, and x is constant, the harmonic mean is required; but if y is constant, the arithmetic mean is required."*

For data organised into a frequency distribution having $k$ classes with classmarks $x_1, x_2, \ldots, x_k$ and the corresponding frequencies $f_1, f_2, \ldots, f_k$ $(\Sigma f_i = n)$ the harmonic mean of the distribution is given by

$$\text{H} = \text{Reciprocal of } \frac{\frac{f_1}{x_1}+\frac{f_2}{x_2}+\ldots+\frac{f_k}{x_k}}{f_1+f_2+\ldots+f_k}$$

$$= \frac{n}{\Sigma f_i \frac{1}{x_i}}$$

Similarly, the weighted harmonic mean is defined as

$$H_w = \frac{w_1+w_2+\ldots+w_n}{w_1\left(\frac{1}{x_1}\right)+w_2\left(\frac{1}{x_2}\right)+\ldots+w_n\left(\frac{1}{x_n}\right)}$$

$$= \frac{\Sigma w_i}{\Sigma w_1\left(\frac{1}{x_i}\right)}$$

**Example 3.9** Find the harmonic mean from the following frequency distribution of weights:

| Weight (grams) | 65–84 | 85–104 | 105–124 | 125–144 | 145–164 | 165–184 | 185–204 |
|---|---|---|---|---|---|---|---|
| $f$ | 9 | 10 | 17 | 10 | 5 | 4 | 5 |

We calculate the harmonic mean as below:

| Weight (grams) | $x_i$ | $f_i$ | $f_i\left(\frac{1}{x_i}\right)$ |
|---|---|---|---|
| 65 – 84 | 74.5 | 9 | 0.12081 |
| 85 – 104 | 94.5 | 10 | 0.10582 |
| 105 – 124 | 114.5 | 17 | 0.14847 |
| 125 – 144 | 134.5 | 10 | 0.07435 |
| 145 – 164 | 154.5 | 5 | 0.03236 |
| 165 – 184 | 174.5 | 4 | 0.02292 |
| 185 – 204 | 194.5 | 5 | 0.02571 |
| $\Sigma$ | -- | 60 | 0.53044 |

Hence $$H = \frac{n}{\Sigma f_i\left(\frac{1}{x_i}\right)} = \frac{60}{0.53044} = 113.11 \text{ grams}$$



**Example 3.10** Compute the Geometric and Harmonic means for the following distributio annual death rates:

| $x_i$ | 3.95 | 4.95 | 5.95 | 6.95 | 7.95 | 8.95 | 9.95 | 10.95 | 11.95 | 12.95 | 13.95 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_i$ | 1 | 4 | 5 | 13 | 12 | 19 | 13 | 10 | 6 | 4 | 1 |

(B.I.S.E. Lahore 1

We can construct the following table to compute the geometric and harmonic means:

| $x_i$ | $f_i$ | $\log x_i$ | $f_i \log x_i$ | $\frac{1}{x_i}$ | $f_i\left(\frac{1}{x_i}\right)$ |
|---|---|---|---|---|---|
| 3.95 | 1 | 0.59660 | 0.59660 | 0.25316 | 0.25316 |
| 4.95 | 4 | 0.69461 | 2.77844 | 0.20202 | 0.80808 |
| 5.95 | 5 | 0.77452 | 3.87260 | 0.16807 | 0.84035 |
| 6.95 | 13 | 0.84198 | 10.94574 | 0.14388 | 1.87044 |
| 7.95 | 12 | 0.90037 | 10.80444 | 0.12579 | 1.50948 |
| 8.95 | 19 | 0.95182 | 18.08458 | 0.11173 | 2.12287 |
| 9.95 | 13 | 0.99782 | 12.97166 | 0.10050 | 1.30650 |
| 10.95 | 10 | 1.03945 | 10.39450 | 0.09132 | 0.91320 |
| 11.95 | 6 | 1.07740 | 6.46440 | 0.08368 | 0.50208 |
| 12.95 | 4 | 1.11229 | 4.44916 | 0.07722 | 0.30888 |
| 13.95 | 1 | 1.14459 | 1.14459 | 0.07168 | 0.07168 |
| Total | 88 | -- | 82.50671 | -- | 10.50672 |

Now $$\log G = \frac{1}{n}\Sigma f_i \log x_i = \frac{82.50671}{88} = 0.93758$$

Hence $G = \text{anti-log } (0.93758) = 8.66$, and

$$\text{Harmonic mean} = \frac{n}{\Sigma f_i\left(\frac{1}{x_i}\right)} = \frac{88}{10.50672} = 8.38$$

## 3.7 THE MEDIAN

The *median* is defined as a value which divides a data set that have been ordered, into two equal parts, one part comprising of observations greater than and the other part smaller than it. Or more precisely, the median is a value at or below which 50% of the ordered data lie.

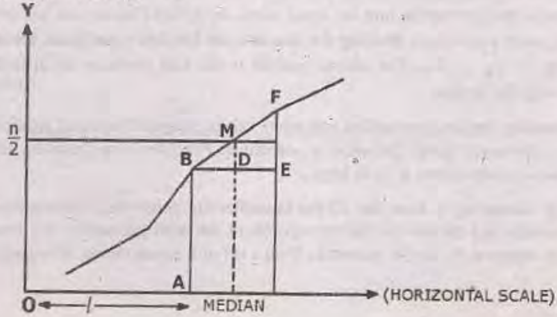
Thus the sample median of the $n$ observations $x_1, x_2, ..., x_n$ when arranged in *order* from smallest to largest, is the middle value if $n$ is odd, and the average of two middle values if $n$ is even. Stated differently, when $\frac{n}{2}$ is not an integer, the median is $\left(\frac{n+1}{2}\right)th$ observation, and when $\frac{n}{2}$ is an integer, the median is the average of $\frac{n}{2}th$ and $\left(\frac{n}{2}+1\right)th$ observations.

The median in case of a discrete or ungrouped frequency distribution can be found as above by forming a cumulative frequency distribution.

For data grouped into a frequency distribution, the *median* is a value or a point on the horizontal scale through which a vertical line divides the histogram of the distribution into two parts of equal area. In other words, the median is that value on the horizontal scale which corresponds to a cumulative frequency $\frac{n}{2}$. This value would lie in a certain group, called the *median group*, but a single value for the median is often desirable. To obtain this single value, we *assume that the observations are evenly distributed within the median group*. Then it may be obtained as follows:

Let us consider a relevant portion of the cumulative frequency polygon as drawn below. Then the median is the abscissa of the point M. That is

Median = OA + BD (see figure below).



Since BDM and BEF are similar triangles, therefore $\dfrac{BD}{BE} = \dfrac{DM}{EF}$

or $\quad BD = BE.\dfrac{DM}{EF}$

Now evidently BE is the width of the class-interval containing median and hence is equal to $h$.

$DM = \dfrac{n}{2} - AB$, where AB represents the cumulative frequency corresponding to the gro

preceding the median group. Let it be equal to C. Then $DM = \dfrac{n}{2} - C$.

EF is the difference between two cumulative frequencies, which is clearly the frequ corresponding to the median group and is denoted by $f$. OA = $l$ (which is the lower boundary of me group), then substituting these values, we get the following formula

$$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - C\right)$$

This process of determining the median is called *linear interpolation* and it does not requ uniform class interval. If this arithmetical process is not used but the value of median (on the X-ax corresponding to a cumulative frequency $\dfrac{n}{2}$ is read directly from the graph of Ogive curve, the proc called the *location of the median graphically*. The median is an average of position. It is also know partition value. The population median may be found in the same way from all the observations population.



**3.7.1 Quantiles.** When the number of observations is quite large, the principle according to wh a distribution or an ordered data set is divided into two equal parts, may be extended to any numbe divisions. The three values which divide the distribution into four equal parts, are called the *Qua* These values are denoted by $Q_1$, $Q_2$ and $Q_3$ respectively. $Q_1$ is called the *first* or *lower quartile* and known as the *third or upper quartile*. In other words, the quartiles $Q_1$, $Q_2$ and $Q_3$ are the values below which lie respectively, the lowest 25, 50 and 75 per cent of the data. Similarly, the nine which divide the distribution into ten equal parts, are called *Deciles* and are denoted by $D_1, D_2, \ldots$ while the ninety nine values dividing the data into one hundred equal parts, are called *Percentiles* an denoted by $P_1, P_2, \ldots, P_{99}$. The second quartile or the fifth decile or the fiftieth percentile is obv identical with the median.

Quartiles, deciles, percentiles and other values obtained by equal subdivision of the given se data, are collectively called *Quantiles* or sometimes *Fractiles*. The Quantiles should be calculated the number of observations is quite large.

It is interesting to note that all the Quantiles are percentiles. For example, the $3^{rd}$ quartile *75th* percentile and the *6th* decile corresponds to the *60th* percentile. We therefore use the follo formula to compute $P_j$, the *jth* percentile from a set of $n$ observations, arranged in *order* from smalle largest.

i) When $\frac{jn}{100}$ is not an integer, the *jth* percentile is given as

$P_j$ = Observation with ordinal number $[\frac{jn}{100}]+1$; and

ii) When $\frac{jn}{100}$ is an integer, the *jth* percentile is

$P_j$ = Average of two observation with ordinal numbers

$$\left(\frac{jn}{100}\right) \text{ and } \left(\frac{jn}{100}\right)+1,$$

where [x] stands for the largest integer in $x$.

In case of grouped data, Quantiles are calculated in the same way as the median.

Sir Francis Galton (1822--1911) is considered the originator of the terms *median* and *percentiles*, although it was Gauss (1777--1855) who actually suggested *median*.

**Example 3.11** Given below are the marks obtained by 9 students:

45, 32, 37, 46, 39, 36, 41, 48 and 36.

Find the median and the quartiles.

To find the median and the quartiles, we first arrange the marks in *order* from lowest to highest. The ordered marks are;

32, 36, 36, 37, 39, 41, 45, 46, 48.

Hence $n = 9$ and $\frac{n}{2}$, i.e. $\frac{9}{2}$ is not an integer, therefore

Median = Marks obtained by $\left([\frac{n}{2}]+1\right)th$ student

= Marks obtained by (4 + 1), *i.e. 5th* student in ordered data,

= 39 marks

Now $Q_1$ = Marks obtained by $\left([\frac{n}{4}]+1\right)th$ student as $\frac{n}{4}$ is not an integer.

= Marks obtained by $\left([\frac{9}{4}]+1\right)th$, *i.e.* 3*rd* student.

= 36 marks.

Similarly,

$$Q_3 = \text{Marks obtained by } \left([\frac{3n}{4}]+1\right)th \text{ student}$$

$$= \text{Marks obtained by } \left([\frac{27}{4}]+1\right)th, \text{ i.e. } 7th \text{ student.}$$

$$= 45 \text{ marks.}$$

**Example 3.12** The following distribution relates to the number of assistants in 50 establishments.

| No. of assistants | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f$ | 3 | 4 | 6 | 7 | 10 | 6 | 5 | 5 | 3 | 1 |

Find the median number of assistants. Also calculate the quartiles and the *7th* decile.

This is an example of ungrouped frequency distribution with unit class interval. To locate median, the quartiles and the *7th* decile for such a distribution, we cumulate the frequencies as shown the table.

| No. of assistants ($x$) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 3 | 4 | 6 | 7 | 10 | 6 | 5 | 5 | 3 | 1 |
| Cumulative Frequency | 3 | 7 | 13 | 20 | 30 | 36 | 41 | 46 | 49 | 50 |



Since $\frac{n}{2}$, *i.e.* $\frac{50}{2}$ is an integer, therefore the median is the average number of assistants $\left(\frac{n}{2}\right)th$ and $\left(\frac{n}{2}+1\right)th$, *i.e., 25th* and *26th* retail establishments. Looking at the cumulative frequency we find that these two values corresponding to the same value of $x$, *i.e.* 4.

Hence *median* number of assistants = 4.

For $Q_1$, we see that $\frac{n}{4}$, *i.e.* $\frac{50}{4}$ is not an integer, therefore

$$Q_1 = \text{No. of assistants in } \left([\frac{50}{4}]+1\right)th \text{ establishment.}$$

= No. of assistants in (12 + 1), *i.e. 13th* establishment

= 2 assistants.

Similarly,

$$Q_3 = \text{No. of assistants in } \left([\frac{3 \times 50}{4}]+1\right)th \text{ establishment as } \frac{3n}{4} \text{ is also not an integer.}$$

= No. of assistants in 38*th* establishment.

= 6 assistants.

Again $D_7$ = Average number of assistants $\left(\frac{7 \times 50}{10}\right)th$ and $\left(\frac{7 \times 50}{10}+1\right)th$ establishment as $\frac{7n}{10}$ is an integer.

= Average number of assistants in 35*th* and 36*th* establishments

= 5 assistants (since both values correspond to 5)

**Example 3.13** Find the median, the quartiles and the 8*th* decile for the distribution of examination marks given below:

| Marks | 30–39 | 40–49 | 50–59 | 60–69 | 70–79 | 80–89 | 90–99 |
|---|---|---|---|---|---|---|---|
| Number of students | 8 | 87 | 190 | 304 | 211 | 85 | 20 |

(P.U., B.A/B.Sc. 1970)

To find the median marks, quartiles, etc. by the process of *linear interpolation*, the data are assumed to be continuous. Thus we need the class boundaries as the cumulative frequencies correspond to upper class boundaries, *i.e.* to 39.5, 49.5, etc., and not to 39, 49, etc., the upper class limits. Hence the class boundaries are shown in the first column and the last column of the table below consists of cumulative frequencies.

| Class-boundaries (marks) | Midpoints ($x_i$) | Frequency ($f_i$) | Cumulative frequency ($F$) |
|---|---|---|---|
| 29.5 – 39.5 | 34.5 | 8 | 8 |
| 39.5 – 49.5 | 44.5 | 87 | 95 |
| 49.5 – 59.5 | 54.5 | 190 | 285 |
| 59.5 – 69.5 | 64.5 | 304 | 589 |
| 69.5 – 79.5 | 74.5 | 211 | 800 |
| 79.5 – 89.5 | 84.5 | 85 | 885 |
| 89.5 – 99.5 | 94.5 | 20 | 905 |



Median = Marks obtained by $\left(\frac{n}{2}\right)th$ student

= Marks obtained by $\frac{905}{2}$, *i.e.* 452.5*th* student which corresponds to marks in the class 59.5 – 69.5. Therefore

Median $= l+\frac{h}{f}\left(\frac{n}{2}-C\right)$, where the letters have their usual significance.

$= 59.5 + \frac{10}{304}(452.5 - 285)$

$= 59.5 + \frac{1675}{304} = 59.5 + 5.5 = 65$ marks.

And $\quad Q_1 = \text{Marks of } \left(\frac{n}{4}\right)th \text{ student}$

$= \text{Marks of } \frac{905}{4}$, *i.e.* 226.25*th* student which corresponds to a value in the cla
49.5 – 59.5. Therefore

$$Q_1 = l + \frac{h}{f}\left(\frac{n}{4} - C\right) = 49.5 + \frac{10}{190}(226.25 - 95)$$

$$= 49.5 + 6.9 = 56.4 = 56 \text{ marks}$$

Again $\quad Q_3 = \text{Marks of } \left(\frac{3n}{4}\right)th \text{ student}$

$= \text{Marks of } \frac{3 \times 905}{4}$, *i.e.* 678.75*th* student which lies in the class 69. 5 – 79.5

$$\therefore \quad Q_3 = l + \frac{h}{f}\left(\frac{3n}{4} - C\right)$$

$$= 69.5 + \frac{10}{211}(678.75 - 589) = 69.5 + 4.2 = 73.7 = 74 \text{ marks}$$

And $\quad D_8 = \text{Marks of } \left(\frac{8n}{10}\right)th \text{ student}$

$= \text{Marks of } \frac{8 \times 905}{10}$, *i.e.* 724*th* student which also lies in the class 69.5 – 79.5

Hence $\quad D_8 = l + \frac{h}{f}\left(\frac{8n}{10} - C\right)$

$$= 69.5 + \frac{10}{211}(724 - 589) = 69.5 + 6.4 = 76 \text{ marks}$$



## 3.8 THE MODE

The French word *mode* meaning fashion, has been adopted to convey the idea of "most frequen
The *mode* is defined as a value which occurs most frequently in a set of data, that is it indicates the mo
common result. A set of data may have more than one mode or no mode at all when each observati
occurs the same number of times.

In an ungrouped frequency distribution with classes consisting of single values, the mode can b
immediately located by examining the distribution. For example, in the distribution relating to the numbe
of assistants in 50 retail establishments (Example 3.12) the mode is 4, as the frequency for $x = 4$ is grea
than for any other value of $x$.

When the data are organised into a grouped frequency distribution, the mode would lie in the class that carries the highest frequency. This class is called the *modal class*. For most practical purposes, it is sufficient to take the midpoint of the modal class as the mode but generally it is a poor approximation. It therefore becomes desirable to decide at what point of the modal class, the mode should be located. To meet this requirement a method based on three adjacent rectangles of the histogram, with the tallest in the middle, has been developed. The method is

$$\text{Mode} = l + \frac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h,$$

where $l$ = lower class boundary of the modal class,

$f_m$ = frequency of the modal class,

$f_1$ = frequency associated with the class *preceding* the modal class,

$f_2$ = frequency associated with the class *following* the modal class, and

$h$ = width of class interval.

The mode can also be calculated by the following formula:

$$\text{Mode} = l + \frac{f_2}{f_1 + f_2} \times h,$$

where the letters have their usual meaning. It should be noted that the first formula is more accurate and should be generally used in calculating the mode.

When a frequency distribution is displayed as a smooth curve, the mode is the abscissa of the highest ordinate. A distribution having a single mode,, is called a *unimodal* distribution, while a distribution with two or more modes, is called a *bimodal* or *multimodal* distribution. It has no meaning for flat-topped distributions. It should be remembered that, when a frequency distribution has classes of unequal widths, the modal class is the class with maximum frequency per unit. The mode should be calculated if the frequency distribution has a class that carries more frequencies than the others and this class should not be at the extremity of the distribution.

**Example 3.14** Calculate the mode for the distribution of examination marks given in Example [illegible]

The class that caries the highest frequency is 59.5 – 69.5, which is thus the modal class.

Also $l = 59.5$, $f_1 = 190$, $f_2 = 211$, $f_m = 304$ and $h = 10$.

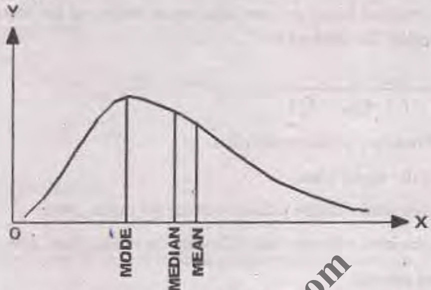
Hence
$$\text{Mode} = l + \frac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h,$$

$$= 59.5 + \frac{304 - 190}{(304 - 190) + (304 - 211)} \times 10$$

$$= 59.5 + 5.8 = 65.3 = 65 \text{ marks.}$$

## 3.9 EMPIRICAL RELATION BETWEEN MEAN, MEDIAN AND MODE

In a single-peaked frequency distribution, the values of the mean, median and mode coincide if the frequency distribution is absolutely symmetrical. But if these values differ, the frequency distribution is said to be skewed or *asymmetrical*.




Experience tells us that in a unimodal curve of moderate skewness, the median is usually sandwiched between the mean and the mode and between them the following approximate relation holds good.

$$\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$$

or $$\text{Mode} = 3\ \text{Median} - 2\ \text{Mean}.$$

This empirical relation does not hold in case of a J-shaped or an extremely skewed distribution.

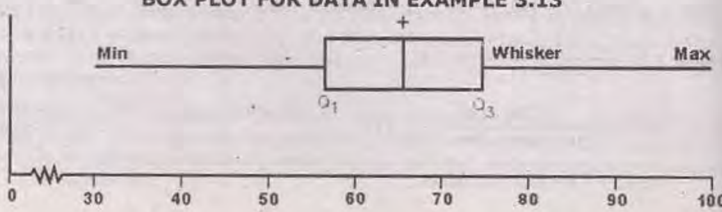


## 3.10 THE BOX PLOTS

*The Box plots*, which are graphically very simple, are based on the Median, a measure of location and the Interquartile Range (*IQR*), a measure of data's variability. They are informative and effective for comparing two or more data sets' distributions.

A box plot is constructed by drawing a rectangle (the *box*) with the ends (called the *hinges*) drawn at the lower and upper quartiles ($Q_1$ and $Q_3$). The median of the data is shown in the box usually by a "+" sign. The straight lines (called the *whiskers*) are drawn from each hinge to the most extreme observations. The entire graph is called a *Box and Whiskers plot*. If one whisker is longer, the distribution of data is skewed in the direction of the longer whisker. The box plot given below represents the distribution of examination marks given in Example 3.13.

**BOX PLOT FOR DATA IN EXAMPLE 3.13**



When two or more distributions are to be compared by drawing box plots, the scale of measurement is usually plotted vertically. Sometimes, two sets of limits, called *inner fences* and *outer fences* are also used.

## 3.11 RELATIVE MERITS AND DEMERITS OF VARIOUS AVERAGES

It is necessary to understand the merits and demerits of each one of the averages in order that it can be appropriately employed.

**3.11.1 The Arithmetic Mean.** The advantages of the mean are:

(i) It is rigorously defined by a mathematical formula.

(ii) It is based on all the observations in the data.

(iii) It is easy to calculate and simple to comprehend. understand

(iv) It is determined for almost every kind of data.

(v) It is a relatively stable statistic with the fluctuations of sampling. That is why it is universally used.

(vi) It is amenable to mathematical treatment.

The disadvantages of the mean are:

(i) It is greatly affected by extreme values in the data.

(ii) It gives sometimes fallacious conclusions.

(iii) In a highly skewed distribution, the mean is not an appropriate measure of average.

(iv) If the grouped data have "open-end" classes, mean cannot be calculated without assuming the limits.

**3.11.2 The Geometric Mean.** The advantages of the geometric mean are:

(i) It is rigorously defined by a mathematical formula.

(ii) It is based on all observed values.

(iii) It is amenable to mathematical treatment in certain cases.

(iv) It gives equal weightage to all the observations.

(v) It is not much affected by sampling variability.

(vi) It is an appropriate type of average to be used in case rates of change or ratios are to be averaged.

Its disadvantages are:

(i) It is neither easy to calculate nor to understand.

(ii) It vanishes if any observation is zero.

(iii) In case of negative values, it cannot be computed at all.

**3.11.3 The Harmonic Mean.** The advantages of the harmonic mean are:

i) It is rigorously defined by a mathematical formula.

ii) It is based on all the observations in the data.

iii) It is amenable to mathematical treatment.

iv) It is not much affected by sampling variability.

v) It is an appropriate type for averaging rates and ratios.

The disadvantages of the harmonic mean are:

i) It is not readily understood.

ii) It cannot be calculated, if any one of the observations is zero.

iii) It gives too much weightage to the smaller observations.

**3.11.4 The Median.** The advantages of the median are:

i) It is easily calculated and understood.

ii) It is located even when the values are not capable of quantitative measurement.

iii) It is not affected by extreme values. It can be computed even when a frequency dis involves "open-end" classes like those of income and prices.

iv) In a highly skewed distribution, median is an appropriate average to use.

The median has the following disadvantages:

i) It is not rigorously defined.

ii) It is not capable of lending itself to further statistical treatment.

iii) It necessitates the arrangement of data into an array which can be tedious and time co for a large body of data.

**3.11.5 The Mode.** The advantages of the mode are:

i) It is simply defined and easily calculated. In many cases, it is extremely easy to lo mode.

ii) It is not affected by abnormally large or small observations.

iii) It can be determined for both the quantitative and the qualitative data.

The disadvantages of the mode are:

i) It is not rigorously defined.

ii) It is often indeterminate and indefinite.

iii) It is not based on all the observations made.

iv) It is not capable of lending itself to further statistical treatment.

v) When the distribution consists of a small number of values, the mode may not exist.

## EXERCISES

### OBJECTIVE

Answer 'True' and 'False'. If the statement is not true then replace the underlined words with words that make the statement true:

i) A measure of central tendency is a quantitative value that tends to locate in some sense the middle of a set of data.

ii) The mean of a sample always divides the data into two equal halves – half larger and half smaller.

iii) For any distribution, the sum of the deviations from the mean equals zero.

iv) The arithmetic mean is not affected by the extreme values.

v) A distribution always has exactly one median score, but it can have more than one mode score.

vi) The value that occurs most frequently is known as median.

vii) The mean may not exist for some sets of data.

viii) Another name for the median is the score at the third quartile.

ix) The Harmonic mean, the geometric mean and the arithmetic mean are equal only if all the numbers $x_1, x_2, \ldots, x_n$, are identical.

x) The Harmonic mean is the $N^{th}$ root of the product of the numbers.

xi) The first quartile is also referred to as the $25^{th}$ percentile.

xii) If a distribution of scores is symmetric, then the median and the mode will be the same.

larger

xiii) If a distribution is positively skewed, then the mean is smaller than the median.

xiv) If a distribution is skewed to left, generally mean > median > mode. F

xv) If the mean, median, and mode of a distribution are 5, 6, and 8, respectively, the distribution is positively skewed.

MULTIPLE CHOICE QUESTIONS.

i) Half the observations are always larger than the

a) Mean b) Total c) Median d) Mode

ii) The value that occurs most often in a set of data is called the

a) Mean b) Mode c) Geometric mean d) Harmonic mean

iii) In case of an open-end class,

a) A median cannot be computed.

b) The arithmetic mean and the median will always be exactly equal.

c) A mean cannot be computed.

d) The distribution is always positively skewed.

iv) Which of the following is a true statement about the median?

a) It is always one of the data values.

b) It is influenced by extreme values.

c) Fifty percent of the observations are larger than the median.

d) It is the middle value of the data values.

v) Which of the following is not a characteristic of the arithmetic mean?

a) It is influenced by extreme values.

b) The sum of the observations from the mean is zero.

c) Fifty percent of the observations are larger than the mean.

d) The sum of the squared directions from mean is always minimum.

vi) Find the mean of the following sample of distances of stars from the earth: 18.2, 56.9, 24.6, 13.5

a) $\bar{X} = 28.30$ b) $\bar{X} = 43.40$ c) $\mu = 28.30$ d) $\mu = 43.40$

vii) In a positively skewed distribution, the mean is always

a) Smaller than the median b) Equal to the median

✓c) Larger then the median d) Equal to the mode

viii) The median is larger than the arithmetic mean when

a) The distribution is positively skewed.

√b) The distribution is negatively skewed.

c) The data is organized into a frequency distribution.

d) The distribution is symmetrical.

ix) The geometric mean of the numbers 2, 4 and 8 is

a) 3.67 b) 4 c) 3.43 d) 5

x) Which of the following statement is not true for Harmonic Mean?

a) Harmonic mean is smaller than the mean.

b) It is based on all the values.

c) It is an appropriate type for averaging rates and ratios.

d) It gives equal weightage to all the values.

## SUBJECTIVE

3.1 What is a statistical average? Name the important types of averages. Discuss the advantages and disadvantages of each average. (P.U., B.A. (Hons.) 196

3.2 What is a measure of "central tendency"? What is the purpose served by it? What are its desirable qualities?

3.3 What are the principal criteria for a satisfactory average? State giving reasons the circumstances in which it would be preferable to use (i) the mean, (ii) the median (iii) the mode, (iv) geometric mean and (v) harmonic mean.

3.4 What criteria do you apply to judge the merits of an average? Discuss the merits and demerits of the different averages in common use with special reference to these criteria.

3.5 In what circumstances would you consider the Arithmetic mean, the Geometric mean and the Harmonic mean respectively, the most suitable statistic to describe the central tendency of distributions? (P.U., B.A./B.Sc. 1989)

3.6 What are the different measures of central tendency? Describe the manner of computation of any *three* of them with suitable illustrations. (P.U., M.A. Econ. 1967)

3.7 Define weighted average and explain how it differs from simple mean. Give the method of its computation and discuss the use of weighted mean in Statistics. (P.U., M.A. Econ. 1974)

3.8 What is the median? What are its advantages and disadvantages? Give reasons why the statistician usually prefers the arithmetic mean to the median. (P.U., M.A. Econ. 1981)

3.9 Define, and explain how to compute, the following quantities for a grouped distribution: $\bar{x}, Q_1, Q_3, D_7$ and Mode.

3.10 Define the arithmetic mean, the mode and the median. Discuss the relationship of these three measures of location in a skewed distribution. State the chief advantages of the arithmetic mean as a form of average.

3.11 Define Mean, Median and Mode. What are their advantages and limitations in the analysis of data? Give various methods of calculating Arithmetic mean, with illustrations. (P.U., B.A./B.Sc. 1958, 1960)

3.12 a) Define Mean, Median and Mode. Give an empirical relation between them. Does this relation give correct value for the mode?

b) Criticise the following statements:

i) An average does not reveal all the information about the data.

ii) The median is described as *the value of the average* rather than the *average value*.

3.13 Comment on the following statements:

i) The depth of a river at four different points is 2, 7, 5, 6 feet respectively. The average depth is 5 feet. Therefore all the people with heights above 5 feet can cross it.

ii) The average marks of one class of students are 30. Therefore every student is hopeless.

iii) The average income of a king and his household servants is £20,000 per month, therefore all the household servants must be fabulously paid.

iv) On an average, the number of accidents occurring in the middle of the road are 5 per thousand. The number of deaths at other places is 30 per thousand. Therefore, it is safer to walk in the middle of the road.

v) In a country, 2,000 vaccinated persons died. Therefore vaccination is useless.

3.14 Define the mean, the median and the mode of a frequency distribution.
It is commonly true that the median lies between the other two measures and is approximately twice as far from the mode as from the mean. State with reasons, whether you expect this relationship to hold, and which of the three statistics is likely to be the most useful single statistic for each of the following distributions:

i) The annual earnings of employed males in Pakistan.
ii) The percentage of sky, to the nearest 10 per cent, covered by cloud at Karachi at mid-day.
iii) The exact length of rods cut to a standard size by machine.

(P.C.S., 1971)

3.15 a) Define arithmetic mean and describe its properties.

b) If the arithmetic mean of $n$ numbers $x_1, x_2, \ldots, x_n$ is $M$ and $A$ is any arbitrary number, then show that

$\Sigma(x_i - A)^2 = \Sigma(x_i - M)^2 + n(M - A)^2$ (P.U., B.A./B.Sc. 1977, 82)

c) A distribution consists of three components with frequencies 3, 4 and 5, and having means 2, 5.5 and 10. Find the mean of the combined distribution. (P.U., B.A./B.Sc. 1977)

3.16 a) State the properties of the arithmetic mean.

b) Show that $\Sigma(x_i - a)^2 = \Sigma(x_i - \bar{x})^2 + n(\bar{x} - a)^2$. In other words, show that the sum of squares about $a = \bar{x}$ is smallest.

c) A distribution consists of 3 components with frequencies 45, 40 and 65, having their means 2, 2.5 and 2 respectively. Show that the mean of the combined distribution is 2.13 approximately.

3.17 a) The number of cars crossing a certain bridge in a big city in 10 intervals of five minutes each was recorded as follows:

25, 15, 18, 30, 30, 20, 12, 9, 16, 15

Calculate (i) the arithmetic mean, (ii) the median and (iii) the geometric mean.

b) Explain why the mean calculated for a set of ungrouped data might differ from the mean if the same data were grouped into a frequency distribution.

3.18 a) Define Arithmetic mean, Geometric mean and Harmonic mean; and prove that for any two positive numbers $a$ and $b$,

A.M. $\geq$ G.M. $\geq$ H.M. (P.U., B.A./B.Sc. 1991)

b) The monthly incomes of ten families in rupees in a certain locality are given below:

| Family: | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Income (Rs.): | 85, | 70, | 10, | 75, | 500, | 8, | 42, | 250, | 40, | 36. |

Calculate the arithmetic mean, the geometric mean and the harmonic mean of the above incomes. Which one of the above three averages represents the above figures best?

3.19 Calculate the arithmetic mean, the geometric mean and the harmonic mean of the annual incomes of fifteen families as given below:

Rs. 60, 80, 90, 96, 120, 150, 200, 360, 480, 520, 1060, 1200, 1450, 2500, 7200.

3.20 a) In a company having 80 employees, 60 earn Rs.3.00 per hour and 20 earn Rs.2.00 per hour. (i) Determine the mean earnings per hour. (ii) Do you consider this mean hourly wage to be typical? (P.U., B.A./B.Sc. 1980-S)

b) An examination candidate's percentages are: English, 73; French, 82; Mathematics, 57; Science, 62; History, 60. Find the candidate's weighted mean if weights of 4, 3, 3, 1, 1 respectively are allotted to the subjects.

Find (i) the simple average of prices in column 2 and (ii) the weighted average, using the quantities in column 3 as weights, and explain the difference between the two results.

| (1) Piece goods | (2) Price per metre (Rs.) | (3) Quantity (millions metres) |
|---|---|---|
| Unbleached | 8.37 | 286 |
| Bleached | 9.50 | 255 |
| Printed flags | 9.16 | 64 |
| Other sorts | 9.84 | 172 |
| Dyed in piece | 13.65 | 165 |
| Of dyed yarn | 11.95 | 80 |

The following are the monthly salaries in rupees of 30 employees of a firm:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 139 | 126 | 114 | 100 | 88 | 62 | 77 | 99 | 103 | 108 |
| 144 | 129 | 148 | 63 | 69 | 148 | 132 | 118 | 142 | 116 |
| 123 | 104 | 95 | 80 | 85 | 106 | 123 | 133 | 140 | 134 |

The firm gave bonuses of Rs. 10, 15, 20, 25, 30 and 35 for individuals in the respective salary groups: exceeding 60 but not exceeding 75, exceeding 75 but not exceeding 90 and so on upto exceeding 135 but not exceeding 150. Find the average bonus paid per employee.

(P.U., M.A. Econ. 1974; B.Z.U. M.A., Econ. 1991)

The following table shows the age distribution of 1,143 horses.

| Age (years) | Number of horses ($f_i$) | Average age ($\bar{x}_i$) |
|---|---|---|
| 1 – 4 | 12 | 2.7 |
| 5 – 9 | 223 | 7.6 |
| 10 – 14 | 435 | 12.0 |
| 15 – 19 | 272 | 16.3 |
| 20 – 24 | 161 | 20.8 |
| 25 – 29 | 34 | 25.8 |
| 30 and over | 6 | 31.8 |

Compute the average age of these horses (a) from the first two columns of the table by the usual short method, (b) from the last two columns by weighting the group averages by the number of horses in the groups. Compare the two results. Which one is more nearly the real average age?

Find the arithmetic and geometric means of the series 1, 2, 4, 8, 16, …, $2^n$. Find also the harmonic mean. (P.U. D. St. 1960)

Find (i) arithmetic mean, (ii) geometric mean, and (iii) harmonic mean of the series 1, 3, 9, 27, 81, …, $3^n$. (P.U., B.A./B.Sc. 1973, 82)

3.26 a) Define Geometric mean and describe its advantages and disadvantage.

b) Given two sets, each of n positive values, $x_{11}, x_{12}, \ldots, x_{1n}; x_{21}, x_{22}, \ldots, x_{2n}$; prove the geometric mean of the ratios of corresponding values in the two sets is equal the ratio of the geometric means of the two sets. (P.U., B.A./B.Sc.

*Hint.* Let a ratio be defined as $X = \dfrac{X_1}{X_2}$.

Then $\log X = \log X_1 - \log X_2$

Sum for all pairs of $X_1$'s and $X_2$'s.

Hence $G = \dfrac{G_1}{G_2}$.

3.27 A man gets a rise of 10% in salary at the end of his first year of service, and further 20% and 25% at the end of the second and third years respectively, the rise in each case calculated on his salary at the beginning of the year. To what annual percentage incr this equivalent?

3.28 a) Define Harmonic mean. How does it differ from arithmetic mean? What advantages and disadvantages?

b) A man travels from A to B at average speed of 30 miles per hour and returns from along the same route at an average speed of 60 miles per hour. Find the average s the entire journey. (P.U., B.A./B.Sc

c) Find out the average speed of person who rides the first mile at the rate of 8 m hour, the next mile at the rate of 7.5 miles an hour and the third mile at the rate miles an hour. (P.U., B.A./B.Sc

3.29 a) A bus traveling 200 kilometers has 10 stages at equal intervals. The speed of the the various stages was observed to be 10, 15, 20, 25, 20, 30, 40, 50, 30, 40 kilome hour. Find the average speed at which the bust travels.

b) Find our the average rate of (i) motion in the case of a person who rides the first the rate of 10 miles an hour, the next mile at the rate of 8 miles per hour, and the mile at the rate of 6 miles per hour; (ii) increase in population, which in the first has increased 20%, in the next 25% and in the third 44%. (P.U., B.A. (Part-I) 19

3.30 Find the geometric mean and the harmonic mean of the following frequency distributio

| Weekly Income (Rs.) | 35–39 | 40–44 | 45–49 | 50–54 | 55–59 | 60–64 | 65–69 |
|---|---|---|---|---|---|---|---|
| No. of workers | 15 | 13 | 17 | 29 | 11 | 10 | 5 |

(P.U., B.A. (Hons. in Econ.)

3.31 Calculate the geometric and the harmonic means for the distribution given below:

| Variable | 0–5 | 5–10 | 10–15 | 15–20 | 20–25 | 25–30 | 30–35 | 35–40 |
|---|---|---|---|---|---|---|---|---|
| Frequency | 2 | 5 | 7 | 13 | 21 | 16 | 8 | 3 |

(P.U., M.A. Econ.

3.32 Find the mean or median, whichever you think more suitable, in each of the following

i) Salaries of 5 men in an industrial concern:
Rs.950, Rs.2100, Rs.1500, Rs.100, Rs.10,000.

ii) Heights of 6 boys: 64", 65", 65", 66", 66", 67".

iii) Handicaps of four golfers: 4, 18, 18, 20.

3.33 The following data relate to sizes of shoes sold at a store during a given week. Find the median of the shoes. Also calculate the quartiles, the 7th decile and the 64th percentile.

| Size of Shoes | 5 | 5½ | 6 | 6½ | 7 | 7½ | 8 | 8½ | 9 | 9½ |
|---|---|---|---|---|---|---|---|---|---|---|
| No. of Pairs | 2 | 5 | 15 | 30 | 60 | 40 | 23 | 11 | 4 | 1 |

(P.U., B.A. (Hons.) 1962)

3.34 Calculate the Mean, Median and Modal numbers of persons per house from the data:

| No. of persons per house | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| No. of houses | 26 | 113 | 120 | 95 | 60 | 42 | 21 | 14 | 5 | 4 |

(P.U., B.A. (Hons.) 1969)

3.35 Draw (i) a Histogram and (ii) an Ogive from the following data:

| Daily wages (Rs.) | 4–6 | 6–8 | 8–10 | 10–12 | 12–14 | 14–16 |
|---|---|---|---|---|---|---|
| No. of employees | 13 | 111 | 182 | 105 | 19 | 7 |

Find approximate value of the median from the Ogive and check your answer by calculation.
(B.I.S.E. Sargodha, 1969-S)

3.36 Estimate graphically and by formula the median and quartile ages of head of household from the following distribution:

| Age of head (yrs) | Number of households |
|---|---|
| under 25 | 44 |
| 25 and under 30 | 79 |
| 30 and under 40 | 152 |
| 40 and under 50 | 122 |
| 50 and under 60 | 141 |
| 60 and under 65 | 100 |
| 65 and under 70 | 58 |
| 70 and under 75 | 32 |
| 75 and under 85 | 28 |



3.37 Compute the median and quartiles of the following distribution of heights and check the results on a graph.

| Heights (inches) | 57.5–, | 60.0–, | 62.5–, | 65.0–, | 67.5–, | 70.0–, | 72.5– |
|---|---|---|---|---|---|---|---|
| Number | 6 | 26 | 190 | 281 | 412 | 127 | 38 |

(P.U., M.A. Econ., 1969)

(Practical)

3.38 Explain when median is more representative than mean. Calculate the median of the following distribution.

| Class | Number | Class | Number | Class | Number |
|---|---|---|---|---|---|
| 100–104 | 4 | 125–129 | 298 | 150–154 | 260 |
| 105–109 | 14 | 130–134 | 380 | 155–159 | 128 |
| 110–114 | 60 | 135–139 | 450 | 160–164 | 66 |
| 115–119 | 138 | 140–144 | 500 | 165–169 | 28 |
| 120–124 | 236 | 145–149 | 430 | 170–174 | 12 |

(P.U., B.A./B.Sc. 1960)

3.39 The frequency distribution of a group of persons according to age is given below:

| Age in years | < 1 | 1–4 | 5–9 | 10–19 | 20–29 | 30–39 | 40–59 | 60–79 |
|---|---|---|---|---|---|---|---|---|
| No. of persons | 5 | 10 | 11 | 12 | 22 | 18 | 8 | 7 |

Calculate the Mean and the Median ages of the distribution.

3.40 a) Describe the merits and demerits of mean and median.

b) Calculate the median, the upper and lower quartiles from the following data: Also draw a box plot.

| Class-Interval | Frequency |
|---|---|
| under 25 | 222 |
| 25 – 29 | 405 |
| 30 – 34 | 508 |
| 35 – 39 | 520 |
| 40 – 44 | 525 |
| 45 – 49 | 490 |
| 50 – 54 | 457 |
| 55 – 59 | 416 |
| 60 and over | 166 |

(P.U., B.A./B.Sc. 1968)

3.41 The following distribution shows Kilowatt-Hours of Electricity used in one month by 75 residential consumers in a certain locality of Lahore.

| Consumption in kilowatt hours | 5–24 | 25–44 | 45–64 | 65–84 | 85–104 | 105–124 | 125–144 | 145–164 |
|---|---|---|---|---|---|---|---|---|
| No. of consumers | 4 | 6 | 14 | 22 | 14 | 5 | 7 | 3 |

Estimate the mean, the median and the two quartiles.

3.42 The yields of grain ( $x$ *lb*) from 500 small plots are grouped in classes with a common class interval (0.2 lb.) in the table below, the values of $x$ given being the midvalues of the classes. Show that the mean of the distribution is 3.95 lb.; the median is 3.95 lb.; and quartiles are 3.6 lb. and 4.28 lb.

| $x$ | $f$ | $x$ | $f$ |
|---|---|---|---|
| 2.8 | 4 | 4.2 | 69 |
| 3.0 | 15 | 4.4 | 59 |
| 3.2 | 20 | 4.6 | 35 |
| 3.4 | 47 | 4.8 | 10 |
| 3.6 | 63 | 5.0 | 8 |
| 3.8 | 78 | 5.2 | 4 |
| 4.0 | 88 | Total | 500 |

3.43 The weights in milligrams of 2538 seeds of the long leef pine were as follows:

| Weight (milligrams) | Number of Seeds | Weight (milligrams) | Number of Seeds |
|---|---|---|---|
| 10 – 24.9 | 16 | 85 – 99.9 | 655 |
| 25 – 39.9 | 68 | 100 – 114.9 | 803 |
| 40 – 54.9 | 204 | 115 – 129.9 | 294 |
| 55 – 69.9 | 233 | 130 – 144.9 | 21 |
| 70 – 84.9 | 240 | 145 – 159.9 | 4 |

a) Find the average weight, the median weight and the most common weight (mode) of the seeds.

b) Find the first and third quartiles. Find the third decile and the 45*th* percentile.

c) Explain your answers as you would to a person who had never studied statistics.

3.44 In a group of 500 wage-earners, the weekly wages of 4% were under Rs.60 and those of 15% were under Rs.62.50. 15% of the workers earned Rs.95 and over, and 5% of them got Rs.100 and over.

The median and quartile wages were Rs.82.25, Rs.72.75 and Rs.90.50; the fourth and sixth decile wages were Rs.78.75 and Rs.85.25 respectively.

Put the above information in the form of a frequency distribution and estimate the mean wage of the 500 wage-earners therefrom.

*Hint.* First put the information in the form of a cumulative frequency table.

3.45 a) Describe the advantages and disadvantages of the mean, the median and the mode. Explain the empirical relation between them.

b) The weight of the 40 male students at a university are given in the following frequency table:

| Weight | 118–126 | 127–135 | 136–144 | 145–153 | 154–162 | 163–171 | 172–180 |
|---|---|---|---|---|---|---|---|
| Frequency | 3 | 5 | 9 | 12 | 5 | 4 | 2 |

Calculate the mean, median and the mode. (P.U., B.A./B.Sc. 1969)

The following table shows the distribution of the maximum loads in short tons supported by certain cables produced by a company.

| [illegible] loads (Short tons) | 9.8–10.2 | 10.3–10.7 | 10.8–11.2 | 11.3–11.7 | 11.8–12.2 | 12.3–12.7 |
|---|---|---|---|---|---|---|
| [illegible] of cables | 7 | 12 | 17 | 14 | 6 | 4 |

Determine the mean, the median and the mode.



The following is the distribution of wages per thousand employees in a certain factory.

| [illegible]ly wages (Rs.) | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| [illegible]ber of employees | 3 | 13 | 43 | 102 | 175 | 220 | 204 | 139 | 69 | 25 | 6 | 1 |

Calculate the Modal and Median wages and explain why there is a difference between the two.

a) Define the mode of a frequency distribution. How does it compare with other types of averages?

b) Write down the empirical relation between mean, median, and mode for unimodal distributions of moderate asymmetry. Illustrate graphically the relative positions of the mean, median and mode for frequency curves which are skewed to the right and to the left. (P.U., B.A./B.Sc. 1972, 80-S)

c) For a certain frequency distribution, the mean was 40.5 and median 36. Find the mode approximately using the formula connecting the three. (P.U., B.A./B.Sc. Optional, 1971-S)

3.49 a) What types of averages would be suitable for the following cases? Give reasons.

i) Size of agricultural holdings.

ii) Heights of students.

iii) Marks obtained in any examination.

iv) Income of workers in a factory.

v) Per capita income in Pakistan.

vi) Comparison of intelligence.

vii) Volumes of sales of ready-made shirts, shoes and collars.

viii) Number of petals of flowers.

b) What measures of central tendency would you recommend for the following cases? G reasons in support of your answer.

i) Symmetrical Distribution.

ii) A J-shaped Distribution.

iii) Distribution having "open-end" classes at the end of the classes.

iv) Frequency distribution of a quantitative variable.

(P.U., M.A. Econ., 19



3.50 a) A distribution $x_1, x_2, ..., x_r, ..., x_k$ with frequencies $f_1, f_2, ..., f_r, ..., f_k$ is transfo into the distribution $X_1, X_2, ..., X_k$, with the same corresponding frequencies b relation $X_r = ax_r + b$, where $a$ and $b$ are constant. Show that the mean, mode median of the new distribution are given in terms of those of the first distribution b same transformation.

b) A distribution has values of the variable $x_1, x_2, ..., x_k$ with corresponding freque $f_1, f_2, ..., f_k$. A new distribution with the same frequencies is formed by $X_r = 2x_r - 3$ for values of $r$ ($r = 1, 2, ..., k$). If the values of the mean, median and of the original distribution are $a$, $b$ and $c$ respectively, what are these values of the distribution?

(P.U., B.A./B.Sc. 1

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Contents

# CHAPTER 4

# MEASURES OF DISPERSION, MOMENTS AND SKEWNESS

# MEASURES OF DISPERSION, MOMENTS AND SKEWNESS

## 4.1 INTRODUCTION

It is quite possible that two or more sets of data may have the same average (mean, median or mode) but their individual observations may differ considerably from the average. Thus a value of central tendency does not adequately describe the data. We therefore need some additional information concerning with how the data are dispersed about the average. This is done by measuring the *dispersion* by which we mean the extent to which the observations in a sample or in a population vary about their mean. A quantity that measures this characteristic, is called a measure of *dispersion, scatter or variability*. It is desirable to have the measure of dispersion (i) in the same units as the observations, (ii) zero when all the observations are equal, (iii) independent of origin, (iv) multiplied or divided by a constant. It is also desirable that it should satisfy the conditions similar to those laid down for an average in previous chapter (see section 3.2).

There are two types of measures of dispersion: *absolute* and *relative*. An *absolute* measure of dispersion is one that measures the dispersion in terms of the same units or in the square of units, as the units of the data. For example, if the units of the data are rupees, metres, kilograms, etc., the units of the measures of dispersion will also be rupees, metres, kilograms, etc. A *relative* measure of dispersion is one that is expressed in the form of a ratio, co-efficient or percentage and is independent of the units of measurement. It is useful for comparison of data of different nature. A measure of central tendency together with a measure of dispersion gives an adequate description of data.

The main measures of dispersion are the following:

i) The Range.
ii) The Semi-Interquartile Range or the Quartile Deviation.
iii) The Mean Deviation or the Average Deviation.
iv) The Variance and the Standard Deviation.



## 4.2 THE RANGE

The *range R*, is defined as the difference between the largest and the smallest observations in a set of data. Symbolically, the range is given by the relation

$$R = x_m - x_0,$$

where $x_m$ stands for the largest observation and $x_0$ denotes the smallest one. When the data are grouped into a frequency distribution, the range is estimated by finding the difference between the upper boundary of the highest class and the lower boundary of the lowest class. The range cannot be computed if there are any open-end classes in the distribution.

The range is a simple concept and is easy to compute. It has, however, two serious disadvantages. *First*, it ignores all the information available from the intermediate observations; and *second*, as its value is based only on the two extreme (unusually large or small) observations, it might give a misleading picture of the spread in the data. It is therefore an unsatisfactory measure of dispersion. However, it is appropriately used in statistical quality control charts of manufactured products, daily temperatures, stock prices, etc. This is an absolute measure of dispersion. Its relative measure known as the *co-efficient of dispersion*, is defined by the following relation:

$$\text{Co-efficient of Dispersion} = \frac{x_m - x_0}{x_m + x_0}.$$

This is a pure (i.e. dimensionless) number and is used for the purposes of comparison.

**Example 4.1** The marks obtained by 9 students are given below:

45, 32, 37, 46, 39, 36, 41, 48, 36.

Find the range and the co-efficient of dispersion.

Here the highest marks, *i.e.* $x_m = 48$,

and the lowest marks, *i.e.*, $x_0 = 32$.

$$\therefore \quad R = x_m - x_0 = 48 - 32 = 16 \text{ marks, and}$$

$$\text{Co-efficient of Dispersion} = \frac{x_m - x_0}{x_m + x_0}$$

$$= \frac{48-32}{48+32} = \frac{16}{80} = 0.2$$

## 4.3 THE SEMI-INTERQUARTILE RANGE OR THE QUARTILE DEVIATION

The *interquartile range* is a measure of dispersion, defined by the difference between the third and first quartiles; and half of this range is called the *semi-interquartile range (S.I.Q.R.)* or *the quartile deviation (Q.D).* Symbolically, we have

$$Q.D. = \frac{Q_3 - Q_1}{2},$$

where $Q_1$ and $Q_3$ are the first and the third quartiles of the data. The quartile deviation has an attractive feature that the range "Median $\pm$ *Q.D.*" contains approximately 50% of the data. The quartile deviation is superior to range as it is not affected by extremely large or small observations. It is simple to understand and easy to calculate. It has certain disadvantages. It gives no information about the position of observations lying outside the two quartiles; is not amenable to mathematical treatment and is greatly affected by sampling variability. The quartile deviation is not as widely used as other measures of dispersion. It is, however, used in situations where extreme observations are thought to be unrepresentative.



The quartile deviation is also an absolute measure of dispersion. Its relative measure called the *Co-efficient of Quartile Deviation* or of *Semi-Interquartile Range,* is defined by the relation

$$\text{Co-efficient of Quartile Deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1},$$

which is a pure number and is used for comparing the variation in two or more sets of data.

**Example 4.2** Find the quartile deviation and the co-efficient of quartile deviation for (i) the data in Example 3.11 and (ii) the frequency distribution in Example 3.13.

i) Using the data of Example 3.11, we find that

$$Q_1 = 36 \text{ marks}, \ Q_3 = 45 \text{ marks, and therefore}$$

$$Q.D. = \frac{45-36}{2} = 4.5 \text{ marks}$$

$$\text{Co-efficient of } Q.D. = \frac{45-36}{45+36} = \frac{9}{81} = 0.11$$

Values of $Q_1$ and $Q_3$ calculated in Example 3.13 are respectively 56 and 74 marks. Therefore

$$Q.D. = \frac{Q_3 - Q_1}{2} = \frac{74-56}{2} = 9 \text{ marks, and}$$

$$\text{Co-efficient of } Q.D. = \frac{74-56}{74+56} = \frac{18}{130} = 0.14$$

## ... THE MEAN (OR AVERAGE) DEVIATION

The *mean deviation (M.D.)* of a set of data is defined as the arithmetic mean of the deviations ...ed either from the mean or from the median, all deviations being counted as positive. The reason ... the deviations as positive, *i.e.* to disregard the algebraic signs (+ and –) is to avoid the difficulty ... from the property that the sum of deviations of the observations from their mean is zero. The ... definition of the mean deviation from the mean is

$$M.D. = \frac{\sum |x_i - \bar{x}|}{n}, \quad \text{for sample data,}$$

$$M.D. = \frac{\sum |x_i - \mu|}{N}, \quad \text{for population data,}$$

... $|x_i - \bar{x}|$ and $|x_i - \mu|$ (pronounced "mod. deviations") indicate the *absolute* deviations of the ... from the mean of a sample and population respectively. It is more appropriate to call it the ... *absolute deviation (M.A.D.)*.

... the data organised into a grouped frequency distribution having $k$ classes with midpoints ... $x_k$ and the corresponding frequencies $f_1, f_2 \ldots, f_k (\Sigma f_i = n)$, the mean deviation of the ... is given by

$$M.D. = \frac{\Sigma f_i |x_i - \bar{x}|}{n}$$

... deviation is also defined in terms of absolute deviations from the median in a similar way. ... tells us that the mean deviation is *least* when the deviations are measured from the median. But in ... it is generally calculated from the arithmetic mean. The mean deviation gives more information ... *range* or the *quartile* deviation as it is based on all the observed values. It is easily calculated and ... understood. As it is not amenable to mathematical treatment, its usefulness is limited. We ... artificiality in its calculation by ignoring the algebraic signs of the deviations and this step is ... mathematically defensible. As the mean deviation does not give undue weight to occasional large ..., so it is used in situations where such deviations are likely to occur. It is unsatisfactory for ... inference.

... deviation is an absolute measure of dispersion. Its relative measure, known as the *co-efficient ... deviation*, is obtained by dividing the mean deviation by the average used in the calculation of ... Thus

$$\text{Co-efficient of } M.D. = \frac{M.D.}{\text{Mean}} \text{ or } \frac{M.D.}{\text{Median}}$$

**Example 4.3** Calculate the mean deviation from (i) the mean, (ii) the median, of the following of examination marks:

45, 32, 37, 46, 39, 36, 41, 48 and 36.

Also calculate the co-efficient of mean deviation.

We first arrange the given marks in an increasing sequence to find the median. The ordered marks are

32, 36, 36, 37, 39, 41, 45, 46, 48.

$\therefore$ Median = Marks obtained by $\left(\left[\frac{n}{2}\right]+1\right)th$ student in ordered data as $\frac{n}{2}$ is not an integer.

= Marks obtained by $\left(\left[\frac{9}{2}\right]+1\right)th$, *i.e.* 5th student

= 39 marks

and $\bar{x} = \frac{\sum x}{n} = \frac{360}{9} = 40$ marks

The necessary calculations are given below:



| | $x_i$ | $x_i - \bar{x}$ | $\lvert x_i - \bar{x}\rvert$ | $\lvert x_i - \text{median}\rvert$ |
|---|---|---|---|---|
| | 32 | –8 | 8 | 7 |
| | 36 | –4 | 4 | 3 |
| | 36 | –4 | 4 | 3 |
| | 37 | –3 | 3 | 2 |
| | 39 | –1 | 1 | 0 |
| | 41 | 1 | 1 | 2 |
| | 45 | 5 | 5 | 6 |
| | 46 | 6 | 6 | 7 |
| | 48 | 8 | 8 | 9 |
| $\Sigma$ | 360 | 0 | 40 | 39 |

$\therefore$ $M.D.$ (from mean) $= \frac{\sum\lvert x_i - \bar{x}\rvert}{n} = \frac{40}{9} = 4.4$ marks

and $M.D.$ (from median) $= \frac{\sum\lvert x_i - \text{median}\rvert}{n} = \frac{39}{9} = 4.3$ marks

Co-efficient of $M.D. = \frac{M.D.}{\bar{x}}$ or $\frac{M.D.}{\text{median}}$

$= \frac{4.4}{40}$ or $\frac{4.3}{39} = 0.11$ or $0.11$

**Example 4.4** Calculate the mean deviation of the following frequency distribution showing the weights of apples:

| Weight (grams) | 65–84 | 85–104 | 105–124 | 125–144 | 145–164 | 165–184 | 185–204 |
|---|---|---|---|---|---|---|---|
| $f$ | 9 | 10 | 17 | 10 | 5 | 4 | 5 |

The calculation of the mean deviation (*M.D.*) from the mean is illustrated below:

| Weight | $x_i$ | $f_i$ | $f_i x_i$ | $x_i - \bar{x}$ | $f_i \lvert x_i - \bar{x} \rvert$ |
|---|---|---|---|---|---|
| 65 – 84 | 74.5 | 9 | 670.5 | –48.0 | 432.0 |
| 85 – 104 | 94.5 | 10 | 945.0 | –28.0 | 280.0 |
| 105 – 124 | 114.5 | 17 | 1946.5 | –8.0 | 136.0 |
| 125 – 144 | 134.5 | 10 | 1345.0 | +12.0 | 120.0 |
| 145 – 164 | 154.5 | 5 | 772.5 | 32.0 | 160.0 |
| 165 – 184 | 174.5 | 4 | 698.0 | 52.0 | 208.0 |
| 185 – 204 | 194.5 | 5 | 972.5 | 72.0 | 360.0 |
| Total | -- | 60 | 7350.0 | -- | 1696.0 |

Here $\bar{x} = \dfrac{\Sigma f_i x_i}{n} = \dfrac{7350.0}{60} = 122.5$ grams

Thus $M.D. = \dfrac{\Sigma f_i \lvert x_i - \bar{x} \rvert}{n} = \dfrac{1696.0}{60} = 28.27$ grams.

## THE VARIANCE AND STANDARD DEVIATION

The *variance* of a set of observations is defined as the mean of the squares of deviations of all the observations from their mean. When it is calculated from the entire population, the variance is called the *population variance*, traditionally denoted by $\sigma^2$ ($\sigma$ is the Greek lowercase "sigma"). If, instead, the data from the sample are used to calculate the variance, it is referred to as the *sample variance* and is denoted by $S^2$ in order to distinguish between the two. The symbolic definition for variance is

$$\sigma^2 = \frac{\Sigma(x_i - \mu)^2}{N}, \quad \text{for population data,}$$

$$S^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}, \quad \text{for sample data,}$$

The variance is also denoted by Var($X$). The term *variance* was introduced in 1918 by R.A. Fisher (1890–1962).

It should be noted that the variance is in square of units in which the observations are expressed and the variance is a large number compared to observations themselves. The variance because of its nice mathematical properties, assumes an extremely important role in statistical theory.

**Standard Deviation.** The positive square root of the variance is called *standard deviation*. Symbolically,

$$\sigma = \sqrt{\frac{\Sigma(x_i - \mu)^2}{N}}, \quad \text{for population data,}$$

$$S=\sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n}},\quad \text{for sample data,}$$

The standard deviation is expressed in the same units as the observations themselves and is a mea the average spread around the mean. Karl Pearson (1857–1936), "founder of the science of Stati credited with the name standard deviation, the most useful measure of dispersion. The *sample var* some texts is defined as

$$s^2=\frac{\Sigma(x_i-\bar{x})^2}{n-1},$$

where $n$ is replaced by $n-1$ on the basis of the argument that *knowledge of any $n-1$ dev automatically determines the remaining deviation as the sum of n deviations must be zero.* This is an *unbiased estimator* of the population variance $\sigma^2$, the explanation for which is deferred to cha *estimation* where we shall learn that sample variance $S^2=\frac{\Sigma(x_i-\bar{x})^2}{n}$, for small sam *underestimates* the population variance $\sigma^2$.

When the data are grouped into a frequency distribution having $k$ classes with mid $x_1, x_2, \ldots, x_k$ and the corresponding frequencies $f_1, f_2, \ldots, f_k (\Sigma f_i = n)$, the sample variance and sta deviation are given by

$$s^2=\frac{\Sigma f_i(x_i-\bar{x})^2}{n}, \text{ and}$$

$$s=\sqrt{\frac{\Sigma f_i(x_i-\bar{x})^2}{n}}$$



It should be noted that for a frequency distribution, as the number of observations or the frequency $n$ is usually large, dividing the sum of squared deviations by $n-1$ is practically equ dividing it by $n$.

The standard deviation has a definite mathematical meaning, utilizes all the observed valu amenable to mathematical treatment but is affected by extreme values. The standard deviati absolute measure of dispersion. Its relative measure called *coefficient of standard deviation*, is def

$$\text{Coefficient of } S.D. = \frac{\text{Standard Deviation}}{\text{Mean}}$$

The quantity $\sqrt{\frac{\Sigma(x_i-a)^2}{n}}$, where $a$ is some arbitrary origin, is called the *root-mean- deviation* which becomes the standard deviation when this arbitrary origin coincides with the mea

To calculate the variance and standard deviation on an *electronic calculator*, the alt formulas for use are obtained by showing that $\Sigma(x_i-\mu)^2=\Sigma x_i^2-(\Sigma x_i)^2/N$.

$$\text{Now} \quad \Sigma(x_i - \mu)^2 = \Sigma(x_i^2 - 2x_i\mu + \mu^2)$$

$$= \Sigma x_i^2 - 2\mu \Sigma x_i + N\mu^2$$

$$= \Sigma x_i^2 - 2N\mu^2 + N\mu^2 \qquad (\because \mu = \frac{\Sigma x_i}{N})$$

$$= \Sigma x_i^2 - N\mu^2 = \Sigma x_i^2 - \frac{(\Sigma x)^2}{N}$$

Thus the sum of squares of the deviations from the mean is equal to the sum of the squares of all minus a *correction factor* which is the (1/N)*th* of the square of the sum of all $x_i$'s.

$$\therefore \qquad \sigma^2 = \frac{\Sigma(x_i - \mu)^2}{N} = \frac{\Sigma x_i^2}{N} - \left(\frac{\Sigma x_i}{N}\right)^2$$

*the variance is the mean of the squares minus the square of the mean.* The corresponding formula for sample variance is

$$S^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

The alternative formulas for standard deviations are

$$\sigma = \sqrt{\frac{\Sigma x^2}{N} - \left(\frac{\Sigma x_i}{N}\right)^2}, \text{ and}$$

$$S = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x_i}{N}\right)^2}$$

The following alternative formulas for the sample variance and standard deviation of a frequency distribution are obtained in a similar way.

$$s^2 = \frac{\Sigma fx^2}{n} - \left(\frac{\Sigma fx}{n}\right)^2, \text{ and}$$

$$s = \sqrt{\frac{\Sigma fx^2}{n} - \left(\frac{\Sigma fx}{n}\right)^2}.$$

**Example 4.5** A population of N = 10 has the observations 7, 8, 10, 13, 14, 19, 20, 25, 26 and 28. Find its variance and standard deviation.

Calculations appear in the following table:

| | $x_i$ | $x_i - \mu$ | $(x_i - \mu)^2$ | $x_i^2$ |
|---|---|---|---|---|
| | 7 | −10 | 100 | 49 |
| | 8 | −9 | 81 | 64 |
| | 10 | −7 | 49 | 100 |
| | 13 | −4 | 16 | 169 |
| | 14 | −3 | 9 | 196 |
| | 19 | +2 | 4 | 361 |
| | 20 | 3 | 9 | 400 |
| | 25 | 8 | 64 | 625 |
| | 26 | 9 | 81 | 676 |
| | 28 | 11 | 121 | 784 |
| $\Sigma$ | 170 | 0 | 534 | 3424 |

Now $\mu = \dfrac{\sum x_i}{N} = \dfrac{170}{10} = 17.$

Therefore $\sigma^2 = \dfrac{\Sigma(x_i - \mu)^2}{N} = \dfrac{534}{10} = 53.4,$

and $\sigma = \sqrt{\dfrac{\Sigma(x_i - \mu)^2}{N}} = \sqrt{53.4} = 7.31$

*Using the alternative method:*

$$\sigma^2 = \frac{\sum x_i^2}{N} - \left(\frac{\sum x_i}{N}\right)^2$$

$$= \frac{3424}{10} - \left(\frac{170}{10}\right)^2 = 342.4 - 289 = 53.4$$

and $\sigma = \sqrt{\dfrac{\sum x_i^2}{N} - \left(\dfrac{\sum x_i}{N}\right)^2} = \sqrt{53.4} = 7.31$

**Example 4.6** Calculate the variance and standard deviation from the following marks obta 9 students.

45, 32, 37, 46, 39, 36, 41, 48, 36

The variance $S^2$ and the standard deviation $S$ for the sample are calculated as below:

| | $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $x_i^2$ |
|---|---|---|---|---|
| | 45 | 5 | 25 | 2025 |
| | 32 | –8 | 64 | 1024 |
| | 37 | –3 | 9 | 1369 |
| | 46 | 6 | 36 | 2116 |
| | 39 | –1 | 1 | 1521 |
| | 36 | –4 | 16 | 1296 |
| | 41 | 1 | 1 | 1681 |
| | 48 | 8 | 64 | 2304 |
| | 36 | –4 | 16 | 1296 |
| $\Sigma$ | 360 | 0 | 232 | 14632 |

Here $\bar{x} = \dfrac{\sum x_i}{n} = \dfrac{360}{9} = 40$ marks

Therefore $S^2 = \dfrac{\sum(x_i - \bar{x})^2}{n} = \dfrac{232}{9} = 25.78 \text{ (marks)}^2$

and $S = \sqrt{\dfrac{\sum(x_i - \bar{x})^2}{n}} = \sqrt{25.78} = 5.08$ marks

Using the alternative method.

$$S^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

$$= \frac{14632}{9} - \left(\frac{360}{9}\right)^2 = 1625.78 - 1600 = 25.78 \text{ (marks)}^2$$

and $S = \sqrt{\dfrac{\sum x_i^2}{n} - \left(\dfrac{\sum x_i}{n}\right)^2} = \sqrt{25.78} = 5.08$ marks



**Example 4.7** Calculate the variance and standard deviation from the data of Example 4.4.

The necessary calculations may be carried out on an *electronic calculator* as below:

| $x_i$ | $f_i$ | $f_i x_i$ | $f_i x_i^2$ |
|---|---|---|---|
| 74.5 | 9 | 670.5 | 49 952.25 |
| 94.5 | 10 | 945.0 | 89 302.50 |
| 114.5 | 17 | 1946.5 | 222 874.25 |
| 134.5 | 10 | 1345.0 | 180 902.50 |
| 154.5 | 5 | 772.5 | 119 351.25 |
| 174.5 | 4 | 698.0 | 121 801.00 |
| 194.5 | 5 | 972.5 | 189 151.25 |
| $\Sigma$ | 60 | 7350.0 | 973 335.00 |

Thus we find $$s^2 = \frac{\sum fx^2}{n} - \left(\frac{\sum fx}{n}\right)^2$$

$$= \frac{973335}{60} - \left(\frac{7350}{60}\right)^2 = 16222.25 - 15006.25$$

$$= 1216\,(\text{grams})^2$$

and $$s = \sqrt{\frac{\sum fx^2}{n} - \left(\frac{\sum fx}{n}\right)^2} = \sqrt{1216} = 34.87 \text{ grams}$$

**4.5.1 Change of Origin and Scale.** The computational labour can be reduced by using the transformation as was used for computing the arithmetic mean.

Let $u_i = \frac{x_i - a}{h}$. Then $x_i = a + hu$ and $\bar{x} = a + h\bar{u}$.

Therefore $$\sum(x_i - \bar{x})^2 = \sum[(a + hu_i) - (a + h\bar{u})]^2$$

$$= h^2 \sum(u_i - \bar{u})^2$$

Further $$\sum(u_i - \bar{u})^2 = \sum u_i^2 - \frac{(\sum u_i)^2}{n}$$

Hence $$S^2 = \frac{\sum(x_i - \bar{x})^2}{n} = \frac{h^2}{n}[\sum u_i^2 - \frac{(\sum u_i)^2}{n}]$$

$$= h^2[\frac{\sum u_i^2}{n} - (\frac{\sum u_i}{n})^2]$$

and $$S = h\sqrt{\frac{\sum u^2}{n} - \left(\frac{\sum u}{n}\right)^2}$$

This gives us a *short method* for hand calculations.

When the data are grouped into a frequency distribution, the corresponding *short method* fo calculations, is

$$s = h\sqrt{\frac{\sum f_i u_i^2}{n} - \left(\frac{\sum f u_i}{n}\right)^2}$$

where h is the width of the class-interval, $f_i$ is the frequency of the *ith* class and $u_i$ is the deviation from an assumed mean in terms of class intervals. This method is also known as the *step-de method* or *coding method*.

**Example 4.8** Find the standard deviation by the short method from the data of Example 4.4.

Let $u_i = \dfrac{x_i - 114.5}{20}$, where $a = 114.5$, value corresponding to the highest frequency, and $h = 20$, the class-interval. Then $u_i = -2, -1, 0, 1, 2, 3, 4$. Other calculations appear below:

| $x_i$ | $f_i$ | $u_i$ | $f_i u_i$ | $f_i u_i^2$ |
|---|---|---|---|---|
| 74.5 | 9 | −2 | −18 | 36 |
| 94.5 | 10 | −1 | −10 | 10 |
| 114.5 | 17 | 0 | −28 | 0 |
| 134.5 | 10 | 1 | 10 | 10 |
| 154.5 | 5 | 2 | 10 | 20 |
| 174.5 | 4 | 3 | 12 | 36 |
| 194.5 | 5 | 4 | 20 | 80 |
| $\Sigma$ | 60 | | +52 / +24 | 192 |

Therefore $\quad s = h \times \sqrt{\dfrac{\Sigma fu^2}{n} - \left(\dfrac{\Sigma fu}{n}\right)^2}$

$$= 20 \times \sqrt{\frac{192}{60} - \left(\frac{24}{60}\right)^2} = 20 \times \sqrt{3.20 - 0.16}$$

$$= 20 \times \sqrt{3.04} = 20 \times (1.7436) = 34.87 \text{ grams.}$$



**4.5.2 Interpretation of the Standard Deviation.** The standard deviation ($\sigma$ or $s$) has not a simple interpretation like the arithmetic mean ($\mu$ or $\bar{x}$) that is interpreted as the balancing point for the distribution. The standard deviation is a very important concept that serves as a basic measure of variability. A smaller value of the standard deviation indicates that most of the observations in a data set are close to the mean while a large value implies that the observations are scattered widely about the mean. However, a connection between the standard deviation and fraction of data included in intervals constructed around the mean, was discovered by the Russian mathematician P.L. Chebyshev (1821–1894). This result, generally known as *Chebyshev's rule*, is stated below:

"For any set of data, the interval $\bar{x} - ks$ to $\bar{x} + ks$, where $k$ is any number greater than 1, contains *at least* the fraction $\left(1 - \dfrac{1}{k^2}\right)$ of the data." For example, the intervals $\bar{x} \pm 2s$ and $\bar{x} \pm 3s$ will contain respectively *at least* the fractions $\left(1 - \dfrac{1}{2^2}\right)$, *i.e.* $\dfrac{3}{4}$ and $\left(1 - \dfrac{1}{3^2}\right)$, *i.e.* $\dfrac{8}{9}$ of the data.

This rule is applied to any distribution (Population or Sample) and guarantees the inclusion of a minimum fraction of the data in the constructed interval whereas the actual fraction of the included (especially in bell-shaped distributions) will exceed $\left(1-\frac{1}{k^2}\right)$.

**4.5.3 Co-efficient of Variation.** The variability of two or more than two sets of data cannot be compared unless we have a relative measure of dispersion. For this purpose, Karl Pearson (1857–1936) introduced a relative measure of variation, known as the *co-efficient of variation*, abbreviated C.V. which expresses the standard deviation as a percentage of the arithmetic mean of a data set. Symbolically, it is defined as

$$\text{C.V.} = \frac{S}{\bar{x}} \times 100, \text{ for sample data,}$$

$$= \frac{\sigma}{\mu} \times 100, \text{ for population data.}$$

As the coefficient of variation is a pure number without units, it is therefore used to compare the variation in two or more data sets or distributions that are measured in different units, e.g. one may be measured in hours and the other in kilograms or rupees. A large value of C.V. indicates that the variability is great and a small value of C.V. indicates less variability.

The coefficient of variation is also used to compare the performance of two candidates or of two players given their scores in various papers or games. The smaller the coefficient of variation the more consistent is the performance of the candidates or players. Thus it is used as a criterion for the consistent performance of the candidates or the players. It should be noted that this co-efficient is unreliable when the arithmetic mean is very small.

**Example 4.9** Using the co-efficient of variation, determine whether or not there is greater variation among the prices of certain similar commodities given, than among the life in hours under test.

| | |
|---|---|
| Prince in Rupees: | 8, 13, 18, 23, 30 |
| Life in hours: | 130, 150, 180, 250, 345 |

We have to compute the mean and the standard deviation for each set so that the corresponding coefficient of variation can be obtained. The necessary arithmetic is shown below:

| Price in Rupees ($X$) | | Life in hours ($Y$) | |
|---|---|---|---|
| $X$ | $X^2$ | $Y$ | $Y^2$ |
| 8 | 64 | 130 | 16900 |
| 13 | 169 | 150 | 22500 |
| 18 | 324 | 180 | 32400 |
| 23 | 529 | 250 | 62500 |
| 30 | 900 | 345 | 119025 |
| 92 | 1986 | 1055 | 253325 |

**Price of Commodities**

$\bar{X} = \text{Rs.}\frac{92}{5} = \text{Rs.}18.4$

$S_X = \sqrt{\frac{1986}{5} - \left(\frac{92}{5}\right)^2}$

$= \sqrt{397.2 - 338.56}$

$= \sqrt{58.44} = \text{Rs. } 7.66$

$C.V. = \frac{7.66}{18.4} \times 100 = 41.63\%$

**Life in Hours**

$\bar{Y} = \frac{1055}{5} = 211 \text{ hours}$

$S_Y = \sqrt{\frac{253325}{5} - \left(\frac{1055}{5}\right)^2}$

$= \sqrt{50665 - 44521}$

$= \sqrt{6144} = 78.38 \text{ hours}$

$\therefore \ C.V. = \frac{78.38}{211} \times 100 = 37.15\%$

We see that the co-efficient of variation for the prices of commodities ($X$) is larger than that for the life in hours ($Y$). Hence the prices of certain similar commodities are showing greater variation than that among the life in hours under test.

**Example 4.10** Goals scored by two teams A and B in a football season were as follows:

| No. of goals scored in a match ($x_i$) | Number of matches | |
|---|---|---|
| | A | B |
| 0 | 27 | 17 |
| 1 | 9 | 9 |
| 2 | 8 | 6 |
| 3 | 5 | 5 |
| 4 | 4 | 3 |

By calculating the co-efficient of variation in each case, find which team may be considered more consistent.

The necessary arithmetic is shown below:

| No. of goals | Team A | | | Team B | | |
|---|---|---|---|---|---|---|
| ($x_i$) | $f_i$ | $f_i x_i$ | $f_i x_i^2$ | $f_j$ | $f_j x_i$ | $f_j x_i^2$ |
| 0 | 27 | 0 | 0 | 17 | 0 | 0 |
| 1 | 9 | 9 | 9 | 9 | 9 | 9 |
| 2 | 8 | 16 | 32 | 6 | 12 | 24 |
| 3 | 5 | 15 | 45 | 5 | 15 | 45 |
| 4 | 4 | 16 | 64 | 3 | 12 | 48 |
| Total | 53 | 56 | 150 | 40 | 68 | 126 |

**Team A:**

$$\text{Mean} = \frac{\sum f_i x_i}{n} = \frac{56}{53} = 1.06, \text{ and}$$

$$s = \sqrt{\frac{\sum f_i x_i^2}{n} - \left(\frac{\sum f_i x_i}{n}\right)^2}$$

$$= \sqrt{\frac{150}{53} - \left(\frac{56}{53}\right)^2} = \sqrt{1.7138} = 1.308$$

$$\therefore \quad \text{C.V.} = \frac{s}{\overline{x}} \text{x} 100 = \frac{1.308}{1.06} \text{x} 100 = 123.4\%$$

**Team B:**

$$\text{Mean} = \frac{\sum f_j x_i}{n} = \frac{48}{40} = 1.20, \text{ and}$$

$$s = \sqrt{\frac{\sum f_j x_i^2}{n} - \left(\frac{\sum f_j x_i}{n}\right)^2}$$

$$= \sqrt{\frac{126}{40} - \left(\frac{48}{40}\right)^2} = \sqrt{1.71} = 1.308$$

$$\text{Thus} \quad \text{C.V.} = \frac{s}{\overline{x}} \text{x} 100 = \frac{1.308}{1.20} \text{x} 100 = 109.0\%$$

We see that the co-efficient of variation for the team B is smaller than that for the team A. Hence team B is more consistent than team A.



**4.5.4 Properties of Variance and Standard Deviation.** The variance and standard deviation have the following useful and interesting properties:

i) The variance of a constant is equal to zero. If $a$ is any constant, then

$$\text{Var}(a) = \frac{1}{N}\Sigma[a-a]^2 \quad (\because \text{mean of a constant is constant itself})$$

$$= 0$$

ii) The variance is independent of the origin, *i.e.* it remains unchanged when a constant is added to or subtracted from each observation of the variable $X$. Symbolically,

$$\text{Var}(X+a) = \text{Var}(X)$$

Now $$\text{Var}(X+a) = \frac{1}{N}\Sigma[(x_i+a)-(\mu+a)]^2 \quad \left(\because \frac{\Sigma(x_i+a)}{N} = \mu+a\right)$$

$$= \frac{1}{N}\Sigma(x_i - \mu)^2 = \text{Var}(X)$$

Hence Var($X$) is *invariant* to change of the origin.

iii) The variance is multiplied or divided by the square of the constant, when each observation of the variable $X$ is either multiplied or divided by a constant.

$$\text{Var}(aX) = \frac{1}{N}\Sigma(ax_i - a\mu)^2$$

$$= a^2 \frac{\Sigma(x_i - \mu)^2}{N} = a^2 \text{ Var}(X)$$

This may also be interpreted as that the variance increases by $a^2$ when the *scale* of $X$ is changed by $a$.

iv) The variance of the sum or difference of two *independent* variables is equal to the sum of their respective variances.

If $X$ and $Y$ are two *independent* variables, then

$$\text{Var}(X \pm Y) = \frac{1}{\text{N}}\Sigma[(x_i \pm y_i) - (\mu_{x+y})]^2$$

$$= \frac{1}{\text{N}}\Sigma[(x_i - \mu_x) \pm (y_i - \mu_y)]^2$$

$$= \frac{1}{\text{N}}\Sigma(x_i - \mu_x)^2 + \frac{1}{N}\Sigma(y_i - \mu_y)^2 \pm \frac{2}{N}\Sigma(x_i - \mu_x)(y_i - \mu_y)$$

The quantity $\frac{1}{\text{N}}\Sigma(x_i - \mu_x)(y_i - \mu_y)$ is called the *covariance* and is denoted by Cov ($X$, $Y$). We shall show at some later stage that the covariance of two *independent* variables is equal to zero. Thus we are left with



$$\text{Var}(X \pm Y) = \frac{1}{\text{N}}\Sigma(x_i - \mu_x)^2 + \frac{1}{\text{N}}\Sigma(y_i - \mu_y)^2$$

$$= \text{Var}(X) + \text{Var}(Y).$$

v) If $k$ subgroups of data consisting of $N_1, N_2, \ldots\ldots, N_k$ $(\Sigma N_i = N)$ observations have respective means $\mu_1, \mu_2, \ldots, \mu_k$ and variances $\sigma_1^2, \sigma_2^2, \ldots\ldots, \sigma_k^2$, then the variance $\sigma^2$ of the combined observations is given by

$$\sigma^2 = \frac{1}{N}\Sigma N_i(\sigma_i^2 + D_i^2), \quad i = 1, 2, \ldots, k$$

where $D_i = \mu_i - \mu$ and $\mu$ is the mean for all the data.

Let for the *ith* subgroup with mean $\mu_i, \mu$, the general mean, be considered as an arbitrary origin. Then the sum of squares of deviations of the observations in the *ith* subgroup from $\mu$ is given by

$$\sum_1^{N_i}(x_i-\mu)^2=\sum_1^{N_i}(x_i-\mu_i+\mu_i-\mu)^2$$

$$=\sum_1^{N_i}(x_i-\mu_i)^2+N_i(\mu_i-\mu)^2, \quad (\because \text{ product term vanishes})$$

$$=N_i\sigma_i^2+N_iD_i^2$$

$$=N_i(\sigma_i^2+D_i^2)$$

But the variance $\sigma^2$ of the combined observations is the mean of the *sum of the deviations of all observations in k subgroups from the general mean* $\mu$. Hence summing over $k$-subgroups, we get

$$N\sigma^2=\sum N_i(\sigma_i^2+D_i^2)$$

It is relevant to note that all these properties are valid for standard deviation ($S.D$), which is the positive square root of variance. In other words,

i) $S.D.(a)=0$.

ii) $S.D.(X+a)=S.D.(X)$

iii) $S.D.(aX)=|a|\ S.D.(X)$, as $S.D.$ cannot be negative.

iv) $S.D.(X\pm Y)=\sqrt{\text{Var}(X)+\text{Var}(Y)}$

v) $\sigma=\sqrt{\dfrac{1}{N}\sum N_i(\sigma_i^2+D_i^2)}$

For sample data, the corresponding results may be obtained in the same way.

**Example 4.11** Let $\bar{x}_1$ and $S_1^2$ be the mean and variance respectively of $n_1$ observations, $\bar{x}_2$ and $S_2^2$ be the mean and variance respectively of $n_2$ observations. Then, if the variance of all $n_1+n_2$ observations is $S^2$, prove that

$$S^2=\frac{n_1S_1^2+n_2S_2^2}{n_1+n_2}+\frac{n_1n_2}{(n_1+n_2)^2}(\bar{x}_1-\bar{x}_2)^2$$ (P.U., B.A./B.Sc. 1986)

Let $\bar{x}$ denote the general mean and be regarded as an arbitrary origin for the set of $n_1$ observations and set of $n_2$ observations. Then the variance of all $n_1+n_2$ observations, by definition, is given by

$$S^2=\frac{1}{n_1+n_2}\sum_{i=1}^{n_1+n_2}(x_i-\bar{x})^2$$

$$= \frac{1}{n_1+n_2}\left[\sum_{i=1}^{n_1}(x_i-\bar{x})^2 + \sum_{i=n_1+1}^{n_1+n_2}(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n_1+n_2}\left[\sum_{i=1}^{n_1}(x_i-\bar{x}_1+\bar{x}_1-\bar{x})^2 + \sum_{i=n_1+1}^{n_1+n_2}(x_i-\bar{x}_2+\bar{x}_2-\bar{x})^2\right]$$

$$= \frac{1}{n_1+n_2}[n_1\{S_1^2+(\bar{x}_1-\bar{x})^2\} + n_2\{S_2^2+(\bar{x}_2-\bar{x})^2\}]$$

$$= \frac{n_1S_1^2+n_2S_2^2}{n_1+n_2} + \frac{n_1(\bar{x}_1-\bar{x})^2+n_2(\bar{x}_2-\bar{x})^2}{n_1+n_2}$$

Since $\bar{x}$ is the mean of all $n_1+n_2$ observations, *i.e.* $\bar{x} = \frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$,

therefore $\bar{x}_1 - \bar{x} = \bar{x}_1 - \frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2} = \frac{n_2(\bar{x}_1-\bar{x}_2)}{n_1+n_2}$, and

$$\bar{x}_2 - \bar{x} = \bar{x}_2 - \frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2} = \frac{-n_1(\bar{x}_1-\bar{x}_2)}{n_1+n_2}.$$

Hence substituting and simplifying, we get

$$S^2 = \frac{n_1S_1^2+n_2S_2^2}{n_1+n_2} + \frac{n_1n_2}{(n_1+n_2)^2}(\bar{x}_1-\bar{x}_2)^2$$

**4.5.5 Standardized Variables.** A variable is defined to be *standardized* or in *standard units* if it is expressed in terms of deviations from its mean and divided by its standard deviation. It is denoted by $Z$. In symbols, this means that

$$Z_i = \frac{x_i-\mu}{\sigma}, \text{ for population data,}$$

$$z_i = \frac{x_i-\bar{x}}{S}, \text{ for sample data.}$$

This is a very important concept in advanced statistics as the mean of a standardized variable is equal to zero and its variance is equal to one. Thus

$$\bar{Z} = \frac{1}{N}\Sigma\left(\frac{x_i-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{\Sigma(x_i-\mu)}{N} = 0;$$

and $$\text{Var}(Z) = \frac{1}{N}\Sigma\left[\frac{(x_i-\mu)}{\sigma} - 0\right]^2$$

$$= \frac{1}{\sigma^2}\frac{\Sigma(x_i-\mu)^2}{N} = \frac{1}{\sigma^2}.\sigma^2 = 1$$

The $Z$-values, being independent of the units of measurement, provide a basis for compariso between individual values, even though they belong to different distributions. That is why they are ofte used in psychological and education testing, where they are known as *standard scores*. The negativ numbers are avoided by multiplying the $Z$ values by 10, an arbitrary *S.D.*, and adding 50, an arbitrar mean, to them. The values so obtained are called the *standard Z scores*. Thus a standard $Z$ score is give by the relation

$$Z = 50 + 10\left(\frac{x - \bar{x}}{S}\right)$$

## 4.6 TRIMMED AND WINSORIZED MEASURES

Data sets often contain *extreme* (unusually large or small) observations which may be ve different from the main body of the data set and may seem to be incorrect. Such extreme observations a generally called *Wild observations* or *Outliers*. These outliers can cause problems. In the presence outliers, the mean and the standard deviation, being affected by the extreme observations, are theref misleading measures of central tendency and variability. The appropriate measures then may be Median and the Interquartile Range, which are much less sensitive to extreme or wild observatio However, it is important to examine a data set for outliers and if present should be excluded.

For this purpose, we *either* remove a certain percentage of the smallest and largest observations get the so-called *Trimmed data* set *or* replace trimmed values by those next in magnitude to obtain wh is known as *Winsorized data* set (proposed by C.P. Winsor). The mean and the standard deviation of su data sets are known as the *Trimmed Mean* and the *Trimmed Standard Deviation*, and the *Winsori Mean* and the *Winsorized Standard Deviation* respectively.

Generally, the *Trimmed* mean is obtained from the data set after having removed all observat below the first quartile and all observations above the third quartile. The *Winsorized* mean is calcula from the modified data set obtained by replacing each observation below the first quartile with the va of the first quartile and each observation above the third quartile with the value of the third quartile. T *Trimmed* standard deviation and the *Winsorized* standard deviation are computed from the *trimmed* set and *Winsorized* data set as usual. The *trimmed* and *Winsorized* measures have gained importance recent years as they are not disturbed by the presence of a few wild observations and have been fo almost as good as the corresponding measures in symmetric distributions with no unusual observatio



**Example 4.12** Calculate the trimmed and Winsorized means and standard deviations for the d given in Example 3.11.

The data ordered from smallest to largest and the two quartiles were found to be

32, 36, 36, 37, 39, 41, 45, 46, 48

$\uparrow$ $Q_1$ (at 36, the third value) $\quad\quad$ $\uparrow$ $Q_3$ (at 45)

To find the trimmed mean and the trimmed standard deviation, we remove the two observatio and 36 below the first quartile and the two observations 46 and 48 above the third quartile. Thus we h five observations 36, 37, 39, 41, 45 as trimmed data set

$$\therefore \quad \text{Trimmed mean} = \frac{36+37+39+41+45}{5} = \frac{198}{5} = 39.6, \text{ and}$$

$$\text{Trimmed } S.D. = \sqrt{\frac{(36)^2+...+(45)^2}{5}-\left(\frac{198}{5}\right)^2}$$

$$= \sqrt{1598.4-1568.16} = \sqrt{30.24} = 5.5$$

To find the Winsorized mean and standard deviation, we replace the two values 32, 36 below the first quartile with 36, and the two values 46, 48 above $Q_3$ with 45 to get the Winsorized data set as 36, 36, 36, 37, 39, 41, 45, 45, and 45. Thus

$$\text{the Winsorized mean} = \frac{\Sigma X_i}{n} = \frac{360}{9} = 40, \text{ and}$$

$$\text{the Winsorized S.D.} = \sqrt{\frac{\Sigma X^2}{n}-\left(\frac{\Sigma X}{n}\right)^2} = \sqrt{\frac{14534}{9}-\left(\frac{360}{9}\right)^2}$$

$$= \sqrt{1614.89-1600} = \sqrt{14.89} = 3.86$$

Shape of Dist origin where lie

## MOMENTS

A *moment* designates the power to which deviations are raised before averaging them, *e.g.* the quantity $\frac{1}{N}\Sigma(x_i-\mu)$ is called the first population moment and is denoted by $\mu_1$. Similarly, the quantity $\frac{1}{N}\Sigma(x_i-\mu)^2$ is called the second population moment and is denoted by $\mu_2$. The corresponding sample moments are denoted by $m_1$ and $m_2$. In general, the *rth moment about the mean* is the arithmetic mean of the rth power of the deviations of the observations from the mean. In symbols, this means that

$$\mu_r = \frac{1}{N}\Sigma(x_i-\mu)^r, \text{ for population data.}$$

$$m_r = \frac{1}{n}\Sigma(x_i-\bar{x})^r, \text{ for sample data.}$$

These moments are also called the *central moments* or the *mean moments* and are used to describe set of data.

In a similar way, *moments about an arbitrary origin*, say *a*, are defined by the relation

$$\mu'_r = \frac{1}{N}\Sigma(x_i-a)^r, \text{ for population data.}$$

$$m'_r = \frac{1}{n}\Sigma(x_i-a)^r, \text{ for sample data.}$$

Now if we put $r = 0$, we see that

$$\mu_0 = \mu'_0 = 1 \quad \text{and} \quad m_0 = m'_0 = 1$$

For $r = 1$, we have

$$\mu_1 = \frac{1}{N}\Sigma(x_i - \mu) = \frac{\Sigma x_i}{N} - \mu = \mu - \mu = 0, \text{ and}$$

$$\mu_1' = \frac{1}{N}\Sigma(x_i - a) = \frac{\Sigma x_i}{N} - a = \mu - a.$$

The corresponding sample results are $m_1 = 0$ and $m_1' = \bar{x} - a$.

Putting $r = 2$ in the relation for mean moments, we see that

$$\mu_2 = \frac{1}{N}\Sigma(x_i - \mu)^2 = \sigma^2, \text{ which is the population variance,}$$

and $$m_2 = \frac{1}{n}\Sigma(x_i - \bar{x})^2 = S^2, \text{ which is the sample variance,}$$

When $a = 0$, the moment $m_r' = \frac{1}{n}\Sigma x_i^r$ is called the *rth moment about zero.*

The moments about the mean or about the arbitrary origin are also called the *power moments*.

When the sample data are grouped into a frequency distribution having $k$ classes with midpoints $x_1, x_2, \ldots, x_k$ and the corresponding frequencies $f_1, f_2, \ldots, f_k (\Sigma f_i = n)$, the *rth* sample moments are given by

$$m_r = \frac{1}{n}\Sigma f_i(x_i - \bar{x})^r, \text{ and}$$

$$m_r' = \frac{1}{n}\Sigma f_i(x_i - a)^r$$

**4.7.1 Moments about the Mean in terms of Moments about the arbitrary origin, say $a$, and conversely.** It is easier to calculate the moments in the first instance, about an arbitrary origin. They are then transformed to the mean-moments. This is done by using the relationships obtained as follows.

By definition, the *rth* sample moment about the mean is given by

$$m_r = \frac{1}{n}\Sigma f_i(x_i - \bar{x})^r$$

The quantity within brackets may be written as

$$(x_i - \bar{x}) = (x_i - a + a - \bar{x}) = (x_i - a) - (\bar{x} - a)$$

$$= D_i - m_1' \text{ where } D_i = (x_i - a) \text{ and } m_1' = (\bar{x} - a)$$

Thus, we have $m_r = \frac{1}{n}\Sigma f_i(D_i - m_1')^r$

By means of Binomial expansion, we have

$$m_r = \frac{1}{n}\sum f_i[D_i^r - \binom{r}{1}D_i^{r-1}m_1' + \binom{r}{2}D_i^{r-2}(m_1')^2 + \ldots(-1)^r(m_1')^r]$$

where $\binom{r}{j} = \dfrac{r!}{j!(r-j)!}$ and $r! = r(r-1)(r-2)\ldots 3\times 2\times 1.$

or

$$m_r = \frac{1}{n}\sum f_i D_i^r - \binom{r}{1}\frac{1}{n}\sum f_i D_i^{r-1}m_1' + \binom{r}{2}\frac{1}{n}\sum f_i D_i^{r-2}(m_1')^2 - \ldots + (-1)^r(m_1')^r\frac{1}{n}\sum f_i$$

$$= m_r' - \binom{r}{1}m_{r-1}'m_1' + \binom{r}{2}m_{r-2}'(m_1')^2 + \ldots + (-1)^r(m_1')^r$$

Putting $r = 1, 2, 3$ and $4$, we get

$$m_1 = m_1' - m_1' = 0;$$

$$m_2 = m_2' - \binom{2}{1}m_1'.m_1' + \binom{2}{2}(m_1')^2.m_0'$$

$$= m_2' - (m_1')^2;$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3, \text{ and}$$

$$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$$

$m_1 = m_1' - m_1' = 0$
$m_2 = m_2' - (m_1')^2$
$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$
$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$

The corresponding results for population data are:

$$\mu_1 = 0;$$

$$\mu_2 = \mu_2' - (\mu_1')^2;$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3; \text{ and}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

It should be noted that in each of these relations, the sum of the co-efficients of various terms on the right hand side equals zero and each term on the right is of the same dimension as the term on the left.

*Conversely*, the *rth* sample moment about an arbitrary origin, $a$, is given by

$$m_r' = \frac{1}{n}\sum f_i(x_i - a)^r$$

$$= \frac{1}{n}\sum f_i(x_i - \bar{x} + \bar{x} - a)^r$$

$$= \frac{1}{n}\sum f_i(d_i + m_1')^r, \text{ where } d_i = x_i - \bar{x} \text{ and } m_1' = \bar{x} - a$$

$$= \frac{1}{n}\sum f_i d_i^r + \binom{r}{1}m_1'\frac{1}{n}\sum f_i d_i^{r-1} + \binom{r}{2}(m_1')^2 \times \frac{1}{n}\sum f_i d_i^{r-2} + \ldots + (m_1')^r\frac{1}{n}\sum f_i$$

$$= m_r + \binom{r}{1}m_{r-1}(m_1') + \binom{r}{2}m_{r-2}(m_1')^2 + \ldots + (m_1')^r$$

Putting $r = 2$, 3 and 4, we get

$$m_2' = m_2 + (m_1')^2$$
$$m_3' = m_3 + 3m_2(m_1') + (m_1')^3$$
$$m_4' = m_4 + 4m_3(m_1') + 6m_2(m_1')^2 + (m_1')^4.$$

For a population of size N, the corresponding relations are

$$\mu_2' = \mu_2 + \mu_1'^2.$$
$$\mu_3' = \mu_3 + 3\mu_1' \mu_2 + \mu_1'^3;$$
$$\mu_4' = \mu_4 + 4\mu_1' \mu_3 + 6\mu_1'^2 \mu_2 + \mu_1'^4.$$

**4.7.2 Sheppard's Corrections.** In the calculation of moments from a grouped freq distribution, certain errors are introduced by the assumption that the frequencies associated with a are located at the midpoint of the class interval. These errors therefore need corrections. It has shown by W.F. Sheppard that, if the frequency distribution (i) is continuous and (ii) tails off to each end, the corrected moments are as given below:

$$m_2 \text{ (corrected)} = m_2 \text{(uncorrected)} - \frac{h^2}{12};$$
$$m_3 \text{ (corrected)} = m_3 \text{(uncorrected)};$$
$$m_4 \text{ (corrected)} = m_4 \text{(uncorrected)} - \frac{h^2}{2}.m_2 \text{ (uncorrected)} + \frac{7}{240}.h^4;$$

where $h$ denoted the uniform class-interval and m's are the moments about the mean of the g frequency distribution. The important point to note here is that these corrections are not applic highly skewed distributions and distributions having unequal class-intervals.

**4.7.3 Moment-Ratios.** There are certain ratios in which both the numerators and the denom are moments. The most common of these moment-ratios are $\beta_1$ and $\beta_2$ defined by the re

$\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$, $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$. They are independent of origin and units of measurement, *i.e.* they ar members. Actually, $\beta_1$ is the square of the third population moment expressed in *standard* units a is the fourth standardized moment for a population, where a standardized variable has been define

$$Z = (x - \mu)/\sigma.$$

For symmetrical distributions, $\beta_1$ is equal to zero. It is, therefore, used as a measure of skewness used to explain the shape of the curve and is a measure of peakedness. For the normal distribut discussed later, $\beta_2 = 3$.

The moment-ratios (or the standardized moments) for sample data are similarly defined as

$$b_1 = \frac{(m_3)^2}{(m_2)^3} \quad \text{and} \quad b_2 = \frac{m_4}{(m_2)^2}$$

**Example 4.13** Calculate the first four moments about the mean for the following set of ation marks: 45, 32, 37, 46, 39, 36, 41, 48 & 36.

For convenience, the observed values are written in an increasing sequence. The necessary ations appear in the table below:

| $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $(x_i - \bar{x})^3$ | $(x_i - \bar{x})^4$ |
|---|---|---|---|---|
| 32 | –8 | 64 | –512 | 4096 |
| 36 | –4 | 16 | –64 | 256 |
| 36 | –4 | 16 | –64 | 256 |
| 37 | –3 | 9 | –27 | 81 |
| 39 | –1 | 1 | –1 | 1 |
| 41 | 1 | 1 | 1 | 1 |
| 45 | 5 | 25 | 125 | 625 |
| 46 | 6 | 36 | 216 | 1296 |
| 48 | 8 | 64 | 512 | 4096 |
| 360 | 0 | 232 | 186 | 10708 |

a=39

$\beta_1, \beta_2 = ?$



Now $\bar{x} = \dfrac{\sum x_i}{n} = \dfrac{360}{9} = 40$ marks

efore $m_1 = \dfrac{\sum(x_i - \bar{x})}{n} = 0$

$$m_2 = \frac{\sum(x_i - \bar{x})^2}{n} = \frac{232}{9} = 25.78\ (\text{marks})^2$$

$$m_3 = \frac{\sum(x_i - \bar{x})^3}{n} = \frac{186}{9} = 20.67\ (\text{marks})^3$$

$$m_4 = \frac{\sum(x_i - \bar{x})^4}{n} = \frac{10708}{9} = 1189.78\ (\text{marks})^4$$

Practical

**Example 4.14** Compute the first four moments for the following distribution of wages after g Sheppard's corrections.

| eekly Earnings (Rupees) | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| o. of men | 1 | 2 | 5 | 10 | 20 | 51 | 22 | 11 | 5 | 3 | 1 |

(P.U., B.A./B.Sc. (Part-I), 1962)

We first calculate moments about an arbitrary origin. The necessary calculations are shown be
The moments about $x = 10$ are obtained by dividing the column sums by $n$.

| Earnings in Rs. $(x_i)$ | Men $f_i$ | $D_i$ $(x_i - 10)$ | $f_i D_i$ | $f_i D_i^2$ | $f_i D_i^3$ | $f_i D_i^4$ |
|---|---|---|---|---|---|---|
| 5 | 1 | –5 | –5 | 25 | –125 | 625 |
| 6 | 2 | –4 | –8 | 32 | –128 | 512 |
| 7 | 5 | –3 | –15 | 45 | –135 | 405 |
| 8 | 10 | –2 | –20 | 40 | –80 | 160 |
| 9 | 20 | –1 | –20 | 20 | –20 | 20 |
| 10 | 51 | 0 | –68 | 0 | –488 | 0 |
| 11 | 22 | 1 | 22 | 22 | 22 | 22 |
| 12 | 11 | 2 | 22 | 44 | 88 | 176 |
| 13 | 5 | 3 | 15 | 45 | 135 | 405 |
| 14 | 3 | 4 | 12 | 48 | 192 | 768 |
| 15 | 1 | 5 | 5 | 25 | 125 | 625 |
| Sums | 131 | -- | +76 / +8 | 346 | +562 / +74 | 3718 |
| Sums÷n | 1 | -- | 0.06 $= m_1'$ | 2.64 $= m_2'$ | 0.56 $= m_3'$ | 28.38 $= m_4'$ |

*Moments about the mean are:*

$$m_1 = 0$$

$$m_2 = m_2' - (m_1')^2 = 2.64 - (0.06)^2 = 2.64;$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$$

$$= 0.56 - 3(2.64)(0.06) + 2(0.06)^3 = 0.08;$$

$$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$$

$$= 28.38 - 4(0.56)(0.06) + 6(2.64)(0.06)^2 - 3(0.06)^4 = 28.30$$

Applying Sheppard's corrections, we have

$$m_2 \text{ (corrected)} = m_2 \text{(uncorrected)} - \frac{h^2}{12} = 2.64 - 0.08 = 2.56,$$

$$m_3 \text{ (corrected)} = m_3 \text{(uncorrected)} = 0.08,$$

$$m_4 \text{ (corrected)} = m_4 \text{(uncorrected)} - \frac{h^2}{2}.m_2 \text{ (uncorrected)} + \frac{7h^4}{240}$$

$$= 28.30 - 1.32 + 0.03 = 27.01$$

**4.7.4 Change of Origin and Scale.** Let $a$ and $h$ denote the arbitrary origin and the class-
Then we define a new variable $u$ as

$$u_i = \frac{x_i - a}{h}$$

so that $x_i - a = hu_i$; $\bar{x} - a = h\bar{u}$ and hence $x_i - \bar{x} = h(u_i - \bar{u})$.

Substituting these values in the $rth$ sample moments, we get

$$m'_r = \frac{1}{n}\sum f_i(x_i - a)^r = h^r . \frac{1}{n}\sum f_i u_i^r;$$

and $$m_r = \frac{1}{n}\sum f_i(x_i - \bar{x})^r = h^r . \frac{1}{n}\sum f_i(u_i - \bar{u})^r.$$

This shows that the $rth$ moments of the variable $X$ are $h^r$ times the corresponding moments of the variable $u$, and are independent of the origin '$a$'. In other words, the *moments are not affected by a change of origin but are affected by a change of scale.*

**4.7.5 Charlier Check.** We have seen that the computation of the moments depends upon the sum of the products of the frequencies by the corresponding values of the variable. It is, therefore, desirable to check these computations so that arithmetic mistakes, if any, are avoided. For this purpose, L.V. Charlier, a Norwegian statistician, introduced a check known as *Charlier check*. This check actually consists in shifting the assumed origin in the *coded* form by one interval. The relations used for this purpose are given below:

$$\sum f(u+1) = \sum fu + n$$
$$\sum f(u+1)^2 = \sum fu^2 + 2\sum fu + n$$
$$\sum f(u+1)^3 = \sum fu^3 + 3\sum fu^2 + 3\sum fu + n$$
$$\sum f(u+1)^4 = \sum fu^4 + 4\sum fu^3 + 6\sum fu^2 + 4\sum fu + n$$

**Example 4.15** Calculate the first four moments about the mean from the data of Example 4.4.

The necessary calculations by taking $u_i = \frac{x_i - 114.5}{20}$, are set out in the following table. The last column is used for *Charlier's* check and the column sums are divided by $n$ to get $m'_r$.

| Data | | Computations | | | | | |
|---|---|---|---|---|---|---|---|
| $x_i$ | $f_i$ | $u$ | $fu$ | $fu^2$ | $fu^3$ | $fu^4$ | $f(u+1)^4$ |
| 74.5 | 9 | –2 | –18 | 36 | –72 | 144 | 9 |
| 94.5 | 10 | –1 | –10 | 10 | –10 | 10 | 0 |
| 114.5 | 17 | 0 | 0 | 0 | 0 | 0 | 0 |
| 134.5 | 10 | 1 | 10 | 10 | 10 | 10 | 160 |
| 154.5 | 5 | 2 | 10 | 20 | 40 | 80 | 405 |
| 174.5 | 4 | 3 | 12 | 36 | 108 | 324 | 1024 |
| 194.5 | 5 | 4 | 20 | 80 | 320 | 1280 | 3125 |
| Sum | 60 | -- | 24 | 192 | 396 | 1848 | 4740 |
| Sums÷n | 1 | -- | 0.4 $= m'_1$ | 3.2 $= m'_2$ | 6.6 $= m'_3$ | 30.8 $= m'_4$ | For charller's check |

*Charlier's check.*

$$\sum f(u+1)^4 = \sum fu^4 + 4\sum fu^3 + 6\sum fu^2 + 4\sum fu + n$$
$$= 1848 + 4(396) + 6(192) + 4(24) + 60$$
$$= 1848 + 1584 + 1152 + 96 + 60 = 4740,$$ which is the sum in the last column.

Hence the moments about the mean and in *class-interval units* are obtained as below:

$$m_1 = 0$$

$$m_2 = m_2' - (m_1')^2$$

$$= 3.2 - (0.4)^2 = 3.04$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$$

$$= 6.6 - 3(3.2)\,(0.4) + 2(0.4)^3 = 2.89$$

$$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$$

$$= 30.8 - 4(6.6)\,(0.4) + 6(3.2)\,(0.4)^2 - 3(0.4)^4 = 23.24$$

To get the moments about the mean in *ordinary units*, we multiply $m_2$ by $h^2$, *i.e.* 400, $(20)^3$ and $m_4$ by $(20)^4$. Thus $m_2 = 1216$, $m_3 = 23120$ and $m_4 = 3718400$.

**Example 4.16** The first three moments of a distribution about the value 2 of the variable are and –40. Show that the mean is 3, the variance 15 and $m_3 = -86$. Also show that the first three mo about $x = 0$ are 3, 24 and 76.

Here we are given $m_1' = \frac{1}{n}\sum f(x-2) = 1$ ... (1)

$$m_2' = \frac{1}{n}\sum f(x-2)^2 = 16 \quad \ldots (2)$$

$$m_3' = \frac{1}{n}\sum f(x-2)^3 = -40 \quad \ldots (3)$$

We also know that $m_1' = \bar{x} - a$, so that

$$\bar{x} = m_1' + a = 1 + 2 = 3 \qquad (\because a = 2)$$

And variance, $S^2 = m_2$ (second moment about mean)

$$= m_2' - (m_1')^2 = 16 - 1 = 15, \text{ and}$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$$

$$= -40 - 3\,(16)\,(1) + 2(1)3 = -86.$$

To find the moment about $x = 0$, we need the values of $\frac{1}{n}\sum fx$, $\frac{1}{n}\sum fx^2$ and $\frac{1}{n}\sum fx^3$, which obtained from relations (1), (2) and (3).

From (1), $\frac{\sum fx}{n} = 3$

From (2), $\frac{1}{n} = \sum f(x-2)^2 = 16$

or $\quad \frac{1}{n}\sum f(x^2 - 4x + 4) = 16$

or $\quad \frac{1}{n}\sum fx^2 - 4\frac{\sum fx}{n} + 4 = 16$

or $\quad \frac{1}{n}\sum fx^2 = 16 - 4 + 4(3) = 24$

(3) on expansion can be written as

$$\frac{1}{n}\sum fx^3 - 6\frac{1}{n}\sum fx^2 + 12\frac{\sum fx}{n} - 8 = -40$$

or $\quad \frac{1}{n}\sum fx^3 = -40 + 8 - 12(3) + 6(24) = 76$

Hence the moments about $x = 0$ are

$$m_1' = \frac{\sum fx}{n} = 3,$$

$$m_2' = \frac{\sum fx^2}{n} = 24, \text{ and}$$

$$m_3' = \frac{\sum fx^3}{n} = 76.$$



**Example 4.17** Show that for discrete distributions, $\beta_2 > 1$.

(P.U., D. St. 1962)

definition, $\beta_2 = \frac{\mu_4}{\mu_2^2}$.

Now $\beta_2$ will be greater than one if the numerator is greater than the denominator, i.e. if $\mu_4 > \mu_2^2$, or if $\mu_4 - \mu_2^2 > 0$.

Now $\mu_4 - \mu_2^2 = \frac{1}{N}\sum f(x-\mu)^4 - \sigma^4 \quad (\because \mu_2 = \sigma^2)$

$$= \frac{1}{N}\sum f(x-\mu)^4 + \sigma^4 - 2\sigma^4$$

$$= \frac{1}{N}\sum f(x-\mu)^4 + \frac{\sigma^4 \sum f}{N} - 2\sigma^2 . \frac{\sum f(x-\mu)^2}{N}, \quad \left[\sigma^2 = \frac{\sum f(x-\mu)^2}{N}\right]$$

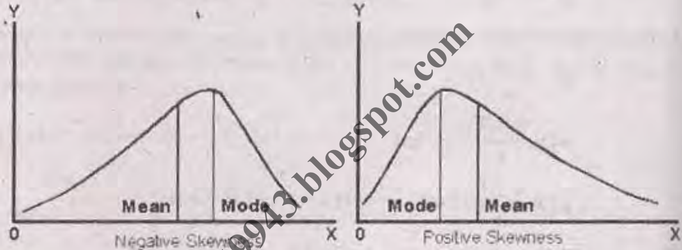
$$= \frac{1}{N}\sum f[(x-\mu)^4 + \sigma^4 - 2\sigma^2(x-\mu)^2] \quad (N\sigma^4 = \sigma^4 \sum f)$$

$$=\frac{1}{N}\sum f[(x-\mu)^2-\sigma^2]^2 \text{ which is essentially positive.}$$

Hence $\beta_2 \geq 1$ because $\mu_4 - \mu_2^2$ is always positive.

## 4.8 SKEWNESS

A distribution in which the values equidistant from the mean have equal frequencies is defined to be *symmetrical* and any departure from symmetry is called *skewness*. It is important to note that in a perfectly symmetrical distribution, the mean, median and mode coincide and that the two tails of the distribution are equal in length from the mean. These values are pulled apart when the distribution departs from symmetry and consequently one tail becomes longer than the other. If the right tail is longer than the left tail, the distribution is said to have *positive skewness*. If the left tail distribution is longer than its right tail, it is said to be *negatively skewed* or to have *negative skewness*. In a positively skewed distribution, the mean is greater than the median and the median is greater than the mode, *i.e.* mean > median > mode and in a negatively skewed distribution, mode > median > mean.






The *difference* between the measures of location, being an indication of the amount of skewness or asymmetry, is used as a measure of skewness. A measure of skewness is defined in such a way that (i) the measure should be zero when the distribution is symmetric and (ii) the measure should be a pure number *i.e.* independent of origin and units of measurements.

According, to measure the degree of skewness of a distribution or curve, Karl Pearson (1857–19..) introduced a coefficient of skewness denoted by $Sk$ and defined by

$$Sk = \frac{\text{Mean} - \text{Mode}}{\text{Standard deviation}}$$

We know that mode is sometimes ill-defined and is difficult to locate by simple methods. It is therefore, replaced by its equivalent from empirical relation holding good in moderately skewed distributions. The *Pearsonian co-efficient of skewness* then becomes

$$Sk = \frac{3(\text{Mean} - \text{Median})}{\text{Standard deviation}}$$

The coefficient usually varies between –3(negative skewness) and +3 (positive skewness) and the sign indicates the direction of skewness. The formula satisfies both the requirements considered essential for a measure of skewness.

Arthur Lyon Bowley (1869–1957), a British statistician, has also proposed a measure of skewness that is based on the median and the two quartiles. In a symmetrical distribution, the two quartiles are equidistant from the median but in an asymmetrical distribution, this will not be the case. The *Bowley's co-efficient of skewness* is

$$Sk = \frac{Q_1 + Q_3 - 2\,\text{Median}}{Q_3 - Q_1}$$

Its values lies between 0 and ±1.

Another measure of skewness that is often used, is the third moment express in standard units (or the moment ratio) and thus is given by

$$Sk = \frac{\mu_3}{\sigma^3}, \quad \text{for population data,}$$

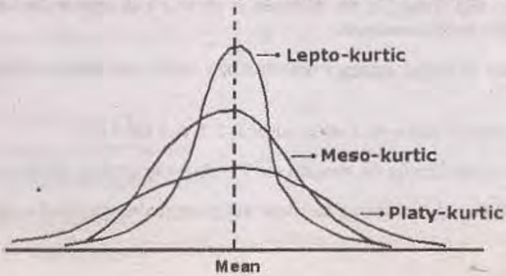
$$= \frac{m_3}{s^3}, \quad \text{for sample data,}$$

This coefficient for most distributions, will be between –2 and +2. Some statisticians denote it by $\alpha_3$ or $\sqrt{\beta_1}$. If the coefficient is greater than zero, the distribution or curve is positively skewed. If $Sk < 0$, there is negative skewness. For symmetrical distributions or curves, the coefficient is zero.

## KURTOSIS

Karl Pearson (1857–1936) introduce the term *Kurtosis* (literally the amount of hump) for the degree of *peakedness* or *flatness* of a unimodal frequency curve. When the values of a variable are closely bunched round the mode in such a way that the curve is *leptokurtic*. If, on the other hand, the curve is flat-topped, we say that the curve is *platykurtic*. Since the *normal curve* (to be described later) is neither very peaked nor very flat-topped, it is taken as basis for comparison. The normal curve itself is called *mesokurtic*.

Kurtosis is usually measured by the fourth standardized moment or the moment-ratio $\beta_2 = (\mu_4 / \mu_2^2)$ whose value for a normal distribution is equal to 3. When $\beta_2$ is greater than 3, the curve is more sharply peaked and has wider tails than the normal curve and is said to be *leptokurtic*. When it is less than 3, the curve has a flatter top and relatively narrower tails than the normal curve and is said to be *platykurtic*.



The corresponding measure of kurtosis for the sample data is $b_2\left(=\frac{m_4}{m_2^2}\right)$. It should be noted that the value of $b_2$ for a large sample from the *normal population* is very nearly 3.

Another measure of Kurtosis not widely used, is given by

$$K=\frac{Q.D.}{P_{90}-P_{10}},$$

where Q.D. is the semi-interquartile range and $P$'s are the *percentiles*. This is known as the *Percentile co-efficient of kurtosis*. It has been shown that $K$ for a normal distribution is 0.263 and that it lies between 0 and 0.50.

## 4.10 DESCRIBING A FREQUENCY DISTRIBUTION

To describe the major characteristics of a frequency distribution, we need the calculations of the following five quantities:

i) The number of observations that describes the *size* of the data.

ii) A measure of central tendency such as the mean or median that provides information about the *centre* or *average* value.

iii) A measure of dispersion such as standard deviation that indicates the *variability* of the data.

iv) A measure of skewness that shows the *lack of symmetry* in the frequency distribution.

v) A measure of kurtosis that gives information about its *peakedness*.



It is interesting to note that all these quantities can be derived from the first four moments. For example, the first moment about $x = 0$ is the arithmetic mean, the second moment about the mean is the variance and its positive square root is the standard deviation. The third mean moment is a measure of skewness while the fourth central moment is used to measure kurtosis. Thus the first four moments play a key role in describing frequency distributions.

## EXERCISES

### OBJECTIVE

a) Answer 'True' and 'False'. If the statement is not true than replace the underlined words with words that make the statement true:

i) A measure of central tendency describes how widely the data are dispersed about a central value.

ii) The standard deviation for a set of values 5, 5, 5, 5, 5 and 5 is 5.

iii) The unit of measure for the standard score is always in standard deviation.

iv) For a bell shaped distribution, the range will be approximately equal to six standard deviations

v) The difference between the largest and the smallest observations is called the Quartile Deviation.

vi) The square of variance gives the standard deviation.

vii) The coefficient of variation is an absolute measure of dispersion.

viii) The coefficient of variation is measured in different units as the data.

ix) The Quartile Deviation is based on only two values in the series.

x) The square root of the variance of a distribution is the absolute deviation.

## MULTIPLE CHOICE QUESTIONS

i) The main disadvantage of the range is that
   a) It does not use all the observations in its calculation.
   b) It can be influenced by an extreme value.
   c) Both a and b are correct.
   d) None of the above.

ii) Which one of the following is not a measure of dispersion?
   a) Range.
   b) Standard deviation.
   c) Second quartile.
   d) Coefficient of variation.

iii) Which of the following is not a measure of dispersion?
   a) Interquartile range.
   b) Difference between the values of the largest and smallest items.
   c) Mean of the values of the largest and smallest items.
   d) Standard deviation.

iv) The standard deviation is
   a) The square of the variance.
   b) Two times the standard deviation.
   c) Half the variance.
   d) The square root of the variance.

v) The coefficient of variation is measured in

a) The same units as the mean and the standard deviation.

b) Percent.

c) Squared units.

d) None of the above.

vi) If the original units are measured in pounds, the variance is

a) Also measured in pounds.

b) Measured in pounds squared.

c) Measured in half pounds.

d) None of the above.

vii) If the tail of a frequency distribution is in positive direction (to the right), the coefficient of skewness is

a) Zero.

b) Positive.

c) Negative.

d) None of the above.

viii) The standard deviation of a frequency distribution is 10, the mean is 250, the median is 2[illegible] and the mode is also 250. The coefficient of skewness is

a) Zero.

b) Positive.

c) Negative.

d) None of the above.



ix) Which of the following statement is true?

a) The standard deviation is less them the range.

b) The range is less than the interquartile range.

c) The arithmetic mean always exceeds the median.

d) The arithmetic man always exceeds the mode.

x) Which of the following is not a property of the standard deviation?

a) It is always negative number.

b) It is affected by extreme values in a data set.

c) It is based on all the values in the data set.

d) It is the most widely used measure of dispersion.

If a distribution has zero standard deviation, then which of the following is true?

a) All observations are negative.

b) All observations are positive.

c) All observations are equal.

d) Number of positive values and negative values are equal.

The empirical rule generally can be applied to

a) Bell shaped distribution.

b) Any distribution.

c) Only continuous distribution.

d) Any skewed distribution.

Symmetrical distribution will always have skewness equal to

a) Negative.

b) Positive.

c) Zero.

d) Approximately zero.

For a normal distribution the measure of Kurtosis equals to

a) Zero.

b) 3.

c) Positive number.

d) Negative number.



For the given sample data set 2, 8, 10, 15, 20, 9, 18, 0, 7, 10, which is the value of coefficient of variation?

a) 70.00 percent.

b) 15.50 percent.

c) 145.00 percent.

d) 61.21 percent.

## BJECTIVE

Explain clearly the meaning of the term Dispersion. What are the most usual methods of measuring dispersion? Indicate the advantages and disadvantages of these methods.
(P.U., B.Com. 1960; B.A. (Hons.), 1960; B.A. (Part-I), 1961)

Discuss the different measures of dispersion. Describe the method of computation of any two of them with suitable examples. (P.U., M.A., Econ. 1969)

4.3 Describe carefully how Mean Deviation, Standard Deviation and Quartile Deviation of any given distribution are obtained. In what problems, should each be used?

(P.U., B.A. (Part-I), 1962-S

4.4 a) What is Range and how is it calculated? What are its uses?

b) Define Quartile Deviation. Find the quartile deviation from the following data (i) graphically, (ii) using an appropriate formula.

| Income per week (Rs.) | 41–50 | 51–60 | 61–70 | 71–80 | 81–90 | 91–100 | Total |
|---|---|---|---|---|---|---|---|
| No. of Earners | 30 | 36 | 43 | 104 | 73 | 14 | 300 |

(P.U., B.A./B.Sc. 196

4.5 The members of a sports club, 60 male adults, had their weights recorded, in pounds. The weights are given below:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 171 | 160 | 144 | 132 | 154 | 160 | 160 | 158 | 148 | 160 | 131 | 153 |
| 131 | 165 | 139 | 163 | 149 | 149 | 140 | 149 | 150 | 161 | 136 | 144 |
| 165 | 174 | 153 | 149 | 157 | 169 | 147 | 156 | 149 | 171 | 149 | 154 |
| 153 | 149 | 147 | 154 | 145 | 158 | 160 | 152 | 156 | 138 | 167 | 142 |
| 165 | 155 | 140 | 155 | 158 | 147 | 149 | 169 | 148 | 174 | 150 | 144 |

Construct a cumulative frequency table for these weights, using classes of width 5 lb, starting at 129.5 lb. Hence draw a cumulative frequency graph, and use this to find the median and semi-interquartile range.

Use the grouped frequency table to calculate the mean and standard deviation, and compare them with the values obtained using the original ungrouped data.

(M.A. Econ. I.U. 1990; B.Z.U. 19



4.6 a) Define Mean Deviation and its co-efficient. Discuss its advantages and uses.

b) Estimate the mean deviation from the arithmetic mean of the following set of examination marks.

| Marks | 0–9 | 10–19 | 20–29 | 30–39 | 40–49 | 50–59 | 60–69 | 70–79 |
|---|---|---|---|---|---|---|---|---|
| No. of students | 2 | 3 | 8 | 24 | 27 | 40 | 11 | 5 |

4.7 When originally collected, the data in the following distribution had been in 5–yearly age groups throughout the whole range 20–64. Using central values of these age-groups, the mean age had been calculated to be 44.5 years and the mean deviation without correction for grouping to be 7.15 years. Reconstruct the original table with 5-yearly age groups from the information given.

| Age nearest birth day: | 20–24 | 25–29 | 30–39 | 40–44 | 45–49 | 50–54 | 55–64 |
|---|---|---|---|---|---|---|---|
| Number of men: | 1 | 2 | 26 | 22 | 20 | 15 | 14 |

(P.C.S., 1971, 93; P.U., B.A./B.Sc.

**Solution:** Let $f_1$ and $f_2$ be the frequencies corresponding to the groups 30–34 and 55–59 in 5–yearly age groups. Then the calculations will proceed as below:

| Age-group | C.V. ($x_i$) | $f$ | $fx$ | $\lvert x-\bar{x}\rvert$ | $f\lvert x-\bar{x}\rvert$ |
|---|---|---|---|---|---|
| 20–24 | 22 | 1 | 22 | 22.5 | 22.5 |
| 25–29 | 27 | 2 | 54 | 17.5 | 35.0 |
| 30–34 | 32 | $f_1$ | $32f_1$ | 12.5 | $12.5f_1$ |
| 35–39 | 37 | $(26-f_1)$ | $962-37f_1$ | 7.5 | $195.0-7.5f_1$ |
| 40–44 | 42 | 22 | 924 | 2.5 | 55.0 |
| 45–49 | 47 | 20 | 940 | 2.5 | 50.0 |
| 50–54 | 52 | 15 | 780 | 7.5 | 112.5 |
| 55–59 | 57 | $f_2$ | $57f_2$ | 12.5 | $12.5f_2$ |
| 60–64 | 62 | $(14-f_2)$ | $868-62f_2$ | 17.5 | $245.0-17.5f_2$ |
| Total | -- | 100 | $4550-5(f_1+f_2)$ | -- | $715+5(f_1-f_2)$ |

Now, Mean $=\frac{1}{n}\sum fx$,

i.e. $44.5=\frac{1}{100}[4550-5(f_1+f_2)]$

or $f_1+f_2=20$ ... (A)

and Mean Deviation $=\frac{1}{n}\sum f\lvert x-\bar{x}\rvert$

i.e. $7.15=\frac{1}{100}[715+5(f_1-f_2)]$

or $f_1-f_2=0$ ... (B)

From (A) and (B) we find that $f_1=10$ and $f_2=10$.

Find the quartile deviation and mean deviation as well as their co-efficients from the following data and comment.

S.D Variance

| Height (inches) | | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 |
|---|---|---|---|---|---|---|---|---|---|
| No. of Persons | Group A | 10 | 18 | 30 | 42 | 35 | 28 | 16 | 8 |
| | Group B | 15 | 20 | 32 | 35 | 33 | 22 | 20 | 10 |

a) Define Variance and Standard Deviation. Describe their properties.

b) For a population of numbers 10, 8, 7, 9, 5, 12, 8, 6, 8, 2, calculate $\sigma^2$ and $\sigma$.

(P.U., M.A. Econ. 1992)

a) Prove that the variance remains unchanged when a constant is added to or subtracted from every value of the variable.

b) The scores obtained by five students on a set of examination papers are 70, 50, 60, 70, 50. These scores are changed by (i) adding 10 points to all scores, (ii) increasing all scores by 10%. What effect will these changes have on the mean and on the standard deviation?

(P.U., B.A./B.Sc., 1971)

4.11 a) Define the mean and standard deviation of a distribution. $\bar{x}$ is the mean and S, the standard deviation. When a provisional mean "$a$" is chosen, the corresponding provisiona standard deviation is found to be $S_1$. Prove that $S_1^2 = S^2 + (\bar{x} - a)^2$. Explain briefly the advantages of this procedure in numerical work.

(P.U., B.A./B.Sc. 197

b) For a set of ungrouped values, the following sums are found: $n$ = $\sum x = 480$, $\sum x^2 = 15735$. Find the mean and the standard deviation.

(B.Z.U., B.A./B.Sc. 19

c) The mean and standard deviation of a sample of 20 observations were found to be 75 an 2.5 respectively. On checking the original figures, it was discovered that one observa which was actually 68, was copied down as 86. Find the correct mean and stand deviation.

4.12 a) Describe the properties of the standard deviation.

b) By multiplying each of the numbers 3, 6, 2, 1, 7, 5 by 2 and then adding 5, we obtain 17, 9, 7, 19, 15. What is the relation between the standard deviations and the means the two sets?

(P.U. B.A./B.Sc. 1972,

c) A child is born to Mrs. $X$ every year for 7 consecutive years. Compute the stan deviation of the children's age when (i) the youngest is 1 year old, and (ii) the young 8 years old. Why do the answers of (i) and (ii) coincide? (P.U., B.A./B.Sc. 1

4.13 Show how to compute the Range and the Standard deviation of a sample of $n$ observa and explain briefly the meaning of these statistics, giving examples of situations in which would be used.

Show that when $n = 2$, the two estimates are simply related.

4.14 It is often stated that in frequency distributions there exists the approximate rela $\frac{\text{Mean Deviation}}{\text{Standard Deviation}} = 0.8$. Test this statement in the following distribution:

| Weight (grams) | 65–84 | 85–104 | 105–124 | 125–144 | 145–164 | 165–184 | 185–204 |
|---|---|---|---|---|---|---|---|
| $f$ | 9 | 10 | 17 | 10 | 5 | 4 | 5 |

4.15 Calculate the mean and standard deviation for the following distribution of lengths of metal bars.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$: | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| $f$: | 4 | 8 | 23 | 35 | 62 | 44 | 18 | 4 | 1 | 1 |

(P.U., D.St. 1963; W.P.C.S.,

4.16 Explain how Chebyshev's rule can be used to answer questions about a data set? Us rule, find and interpret the interval $\bar{x} \pm 2s$ for question 4.15.

The following table gives the frequency distribution of expenditure on food per family per month among working class families in two localities. Find the arithmetic mean and the standard deviation of the expenditure at both places.

| Range of expenditure in rupees per month | No. of Families | |
|---|---|---|
| | Place A | Place B |
| 30–60 | 28 | 39 |
| 60–90 | 292 | 284 |
| 90–120 | 389 | 401 |
| 120–150 | 212 | 202 |
| 150–180 | 59 | 48 |
| 180–210 | 18 | 31 |
| 210–240 | 2 | 5 |

(P.U., B.A., (Part-I), 1961; M.A. Econ. 1970)

What do your understand by Variance? The wages of 1,000 employees range from Rs.4.50 to Rs.19.50. They are grouped in 15 classes with a common class interval of Rs.1, and the class frequencies from the lowest class to the highest class are: 6, 17, 35, 48, 65, 90, 131, 173, 155, 117, 75, 52, 21, 9 and 6. Find the mean wage and its standard deviation.

(P.U., D.St., 1965, P.C.S. 1986)

The breaking strength of 20 test pieces of a certain alloy is given as under:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 95 | 103 | 97 | 130 | 96 | 73 | 78 | 95 | 89 | 68 |
| 82 | 79 | 69 | 67 | 83 | 108 | 94 | 87 | 93 | 117 |

Calculate the average breaking strength of the alloy and the standard deviation. Calculate the percentage of observations lying within the limits: mean $\pm S$; mean $\pm 2S$; mean $\pm 3S$; where $S$ stands for the standard deviation. (P.U., B.Com., 1961, I.U., M.A. Econ. 1989)

A collar manufacturer is considering the production of a new style of collar to attract young men. The following statistics of neck circumferences are available based upon measurements of a typical group of college students.

| Mid-value (inches) | 12.5 | 13.0 | 13.5 | 14.0 | 14.5 | 15.0 | 15.5 | 16.0 | 16.5 |
|---|---|---|---|---|---|---|---|---|---|
| No. of students | 4 | 19 | 30 | 63 | 66 | 29 | 18 | 1 | 1 |

Compute the standard deviation and use the criterion $\bar{x} \pm 3$ (standard deviation) to determine the largest and smallest size of collars he should make in order to meet the needs of practically all his customers, bearing in mind that collars are worn, on average, ¾ inches larger than neck size.

(P.U., B.A./B.Sc. 1960)

The values of the arithmetic mean and standard deviation of the following frequency distribution of a continuous variable derived from the use of working origin and scale are £65.3 and £9.6 respectively. Determine the actual class-intervals.

| $u$ | –4 | –3 | –2 | –1 | 0 | 1 | 2 | 3 | Total |
|---|---|---|---|---|---|---|---|---|---|
| $f$ | 2 | 5 | 8 | 18 | 22 | 13 | 8 | 4 | 80 |

(P.U., D.St., 1964; B.A./B.Sc. 1973)

4.22 In the manufacture of a certain scientific instrument great importance is attached to the a particular critical component. This component is obtained in bulk from two sources B, and in the course of inspection, the lives of 1,000 of the components from each determined. The following frequency tables are obtained:

| c.i | Source A | | Source B | | |
|---|---|---|---|---|---|
| | Life (hours) | No. of components | Life (hours) | | No. of components |
| 1010 | 1,000 – 1,020 | 40 | 1,030 – 1,040 | 1035 | 339 |
| 1030 | 1,020 – 1,040 | 96 | 1,040 – 1,050 | 1045 | 136 |
| 1050 | 1,040 – 1,060 | 364 | 1,050 – 1,060 | 1055 | 25 |
| 1070 | 1,060 – 1,080 | 372 | 1,060 – 1,070 | 1065 | 20 |
| 1090 | 1,080 – 1,100 | 85 | 1,070 – 1,080 | 1075 | 130 |
| 1110 | 1,100 – 1,120 | 43 | 1,080 – 1,090 | 1085 | 350 |

Examine the effectiveness of the measures of dispersion with which you are fam comparing the dispersions of the two distributions.

4.23 Show that, for any discrete distribution, the mean deviation about the mean is not gr the standard deviation. (P.U., B.A./B.Sc.

*Solution.* By definition,

Mean Deviation $= \frac{1}{n}\sum f_i|x_i - \bar{x}|$, and

$$s = \sqrt{\frac{1}{n}\sum f_i(x_i - \bar{x})^2}, \text{ where } n = \sum f_i$$

We are required to show that

$$\frac{1}{n}\sum f_i d_i^2 > \left(\frac{1}{n}\sum f_i d_i\right)^2, \text{ where } d_i = |x_i - \bar{x}|$$

*i.e.* $n[\sum f_i d_i^2] > (\sum f_i d_i)^2$

*i.e.* $(f_1 + f_2 + ... + f_k)(f_1 d_1^2 + f_2 d_2^2 + ... + f_k d_k^2) > (f_1 d_1 + f_2 d_2 + ... + f_k d_k)^2$

*i.e.* $\sum f_i^2 d_i^2 + \sum_{i \neq j} f_i f_j (d_i^2 + d_j^2) > \sum f_i^2 d_i^2 + 2\sum_{i \neq j} f_i f_j d_i d_j)$ .

*i.e.* $\sum_{i \neq j} f_i f_j (d_i - d_j)^2 > 0$

which is true because $(d_i - d_j)^2$ is always positive.

4.24 a) What is the co-efficient of variation? What purpose does it serve?

b) The following data have been obtained from a frequency distribution of a var making the substitution $X = 62 + 5u : \sum f = 120,\ \sum fu = 140,\ \sum fu^2 = 598$, cal co-efficient of variation, using corrected standard deviation.

(P.U., B.A./B.

4.25 a) What do you mean by absolute and relative measures of dispersion? State the u co-efficient of variation in statistical analysis. (P.U., B.A./B.

b) Given below is the distribution of weekly income (to the nearest rupee) of 100 households in a locality A. Calculate the standard deviation.

| Income | 35–39 | 40–44 | 45–49 | 50–54 | 55–59 | 60–64 | 65–69 |
|---|---|---|---|---|---|---|---|
| $f$ | 13 | 15 | 17 | 28 | 12 | 10 | 5 |

37 42 47 52 57 62 67

If 123 households in a different locality B had a mean weekly income of Rs.52.28 and a standard deviation of Rs.4.96, then compare the variability of the weekly income of two localities. (B.Z.U., M.A. Econ. 1986; P.U., B.A./B.Sc. 1974)

a) Define Range, Mean Deviation and Standard Deviation, and compare their merits as descriptive measures of dispersion.

b) Two candidates $X$ and $Y$ at the B.A. (Hons.) Examination obtained the following marks in ten papers. Which of the candidate showed a more consistent performance?

| Paper | I | II | III | IV | V | VI | VII | VII | IX | X |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | 58 | 49 | 76 | 80 | 47 | 72 | 61 | 59 | 77 | 48 |
| $Y$ | 39 | 38 | 86 | 72 | 75 | 69 | 57 | 49 | 83 | 66 |

(B.Z.U., M.A. Econ. 1986; P.U., B.A./B.Sc. 1974)

a) Calculate the corrected co-efficient of variation when mean = 67.45, variance (uncorrected) = 8.5275 and the class-interval is 3. (P.U., B.A./B.Sc. 1968)

b) The following are the scores made by two batsmen A and B in a series of innings:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 12 | 15 | 6 | 73 | 7 | 19 | 199 | 36 | 84 | 29 |
| B | 47 | 12 | 76 | 48 | 4 | 51 | 37 | 48 | 13 | 0 |

Who is better as a run getter? Who is the more consistent player?
(P.U., B.A./B.Sc. 1970; B.Z.U., M.A. Econ. 1984)

a) Explain the difference between absolute dispersion and relative dispersion. Describe the properties of the standard deviation.

b) Find the co-efficient of variation from the following data using both uncorrected and corrected standard deviations.

| Weight (lb) | 118–126 | 127–135 | 136–144 | 145–153 | 154–162 | 163–171 | 172–180 |
|---|---|---|---|---|---|---|---|
| $f$ | 3 | 5 | 9 | 12 | 5 | 4 | 2 |

A manufacturer of television tubes has two types of tubes A and B. The tubes have respective mean life-times $\bar{x}_A = 1495$ hours and $\bar{x}_B = 1895$ hours: and standard deviations $S_A = 280$ hours and $S_B = 310$ hours. Which tube has the greater (i) absolute dispersion, (ii) relative dispersion? (P.U., B.A./B.Sc. 1969, 71)

4.29 Compare the variability of expenditure of families in two towns as given below:

| Expenditure (Rupees) | No. of Families | |
|---|---|---|
| | Town A | Town B |
| 21 – 30 | 3 | 2 |
| 31 – 40 | 61 | 14 |
| 41 – 50 | 132 | 20 |
| 51 – 60 | 153 | 27 |
| 61 – 70 | 140 | 28 |
| 71 – 80 | 51 | 7 |
| 81 – 90 | 2 | 2 |

(B.Z.U., M.A. Econ.

4.30 Compare the variations in the following frequency distributions of weights of bo computing the co-efficient of variation in each case: Also draw a box plot.

| Weight (kilograms) | Classes | | |
|---|---|---|---|
| | A | B | C |
| $20 \le x < 30$ | 7 | 5 | 6 |
| $30 \le x < 40$ | 10 | 9 | 25 |
| $40 \le x < 50$ | 20 | 21 | 24 |
| $50 \le x < 60$ | 18 | 13 | 4 |
| $60 \le x < 70$ | 7 | 6 | 3 |



4.31 If in a series of measurements, we obtain $n_1$ values of magnitude $x_1, n_2$ of magnitu and so on, and if $\bar{x}$ is the mean value of all the measurements, prove that the s deviation is

$$\sqrt{\frac{\sum n_r (k - x_r)^2}{\sum n_r} - \delta^2}, \text{ where } \bar{x} = k + \delta.$$

(B.Z.U., B.A./B.Sc

4.32 a) For a group of 50 boys, the mean score and standard deviation of scores on a test a and 8.38. For a group of 40 girls, the mean and standard deviation are 54.0 and the same test. Find the mean and standard deviation for the combined grou children.

b) A distribution consists of three components with frequencies 200, 250 and 300 means of 25, 10 and 15, and standard deviations of 3, 4 and 5 respectively. Show mean of the combined distribution is 16 and its s.d. is 7.2 approximately. Find coefficient of variation.

a) What is meant by a standardized variable?

b) Show that for any distribution expressed in standard measures, the mean is zero standard deviation is one.

(P.U., B.A./B.Sc

c) The mean of scores for a group of students on a certain test was 63.7 with a standard deviation of 12.3. Find the Z score for the top student, with a score of 98 and the bottom student, with a score of 21.

4.34 Three tests had the values in the following table for mean and standard deviations.

| Test | $\bar{x}$ | S |
|---|---|---|
| 1 | 70 | 5 |
| 2 | 75 | 8 |
| 3 | 60 | 12 |

$\bar{X} = A + m_1'$

Student A received grades of 70, 90, 70 on the three tests, while student B received grades of 90, 70, 70. Assuming that all three tests should carry equal weight, change the grades to Z scores and average each student's scores. (P.U., B.Sc. (Hons.) Part-I, 1971)

4.35 a) Explain what is meant by the trimmed and Winsorized measures of central tendency and dispersion.

b) Calculate the trimmed and the Winsorized means and standard deviations for the following set of 15 scores:

80, 75, 42, 63, 65, 43, 78, 96, 82, 58, 79, 72, 67, 75, 68.

4.36 a) Define Moments about an arbitrary origin and about the mean. Express the moments about the mean in terms of the moments about any point and conversely.

b) Give Sheppard's corrections to moments and explain where they are used.

(P.U., B.A. (Part-I), 1961, 1962-S)

4.37 a) Prove that the second moment about an arbitrary origin equals the second moment about the mean increased by the square of the distance between the arbitrary origin and the mean. (B.Z.U., B.A./B.Sc. 1976)

b) Derive the relations which give the third and fourth moments about the mean in terms of moments about the origin.

4.38 a) What aspects or characteristics of a frequency distribution are measured by the moments? (P.U. B.A./B.Sc. 2008)

b) Prove the general formula connecting the moments about the mean with the moments about the origin

$$\mu_r = \mu_r' - r\mu_r' \mu_{r-1}' + \frac{r(r-1)}{2!} \mu_1'^2 \mu_{r-2}' - \dots$$

(P.U. B.A./B.Sc. 1985, 89, 08)

c) Explain how changes of origin and of scale affect the following:

Mean, standard deviation, moments and $\beta_2$. (P.U. B.A./B.Sc. 1985, 89)

4.39 a) The first four moments of a distribution about $x = 2$ are 1, 2.5, 5.5 and 16. Calculate the four moments about the mean and about zero. (P.U., B.A./B.Sc. 1973, 76, 08)

b) The following data have been obtained from a frequency distribution of a variable $X$ afte making the substitution $x = 10 + 5u$.

$\sum f = 125$, $\sum fu = -46$, $\sum fu^2 = 806$, $\sum fu^3 = -242$ and $\sum fu^4 = 1962$.

Calculate the Mean, Variance, $b_1$ and $b_2$ (moment-ratios). Would you consider th distribution normal? (P.U., B.A./B.Sc. 1975,

4.40 Calculate the first four moments about the mean for the following data.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $f$ | 1 | 6 | 13 | 25 | 30 | 22 | 9 | 5 | 2 |

4.41 Calculate the first four moments and apply the test of normality to the following data wh pertain to weekly earnings in rupees of 200 labourers.

| Weekly wages | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|
| No. of Labourers | 6 | 19 | 13 | 18 | 20 | 25 | 28 | 34 | 22 | 15 |

4.42 Compute the first four moments about the mean from the following frequency distribu Also calculate $b_1$ and $b_2$.

Practical

| Groups | 2–4 | 4–6 | 6–8 | 8–10 | 10–12 | 12–14 | 14–16 | 16–18 | 18–20 |
|---|---|---|---|---|---|---|---|---|---|
| Frequency | 18 | 24 | 47 | 80 | 102 | 66 | 40 | 21 | 15 |

(P.U., D.St. 19



4.43 a) Define the moment-ratios $\beta_1$ and $\beta_2$. What information do they give?

b) Eight coins were tossed together 256 times and the frequency of the occurrence of 0, 1, …, 8 heads in a throw was:

| Number of heads | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| Frequency | 1 | 7 | 26 | 54 | 74 | 52 | 32 | 9 | 1 |

Calculate the values of moment-ratios $b_1$ and $b_2$.

4.44 Calculate the values of $b_1$ and $b_2$ from the following data. Use Charlier check an Sheppard's corrections.

| $x$ | 10–12 | 12–14 | 14–16 | 16–18 | 18–20 | 20–22 | 22–24 | 24–26 | 26–28 |
|---|---|---|---|---|---|---|---|---|---|
| $f$ | 3 | 30 | 110 | 218 | 275 | 222 | 108 | 32 | 2 |

4.45 a) What is meant by skewness? How would you find it in a non-symmetrical distribu Distinguish between positive and negative skewness.

b) Define the moment-ratios $\beta_1$ and $\beta_2$, and state the purpose for which they are us

(P.U., B.A. (Part-I), 196

4.46 a) State the measures commonly employed to define Skewness and Kurtosis. What aspe of the frequency curve are measured by them?

b) What can you say of the skewness in each of the following cases?

i) The median is 49.21 while the two quartiles are 37.15 and 61.27.

ii) Mean = 1403 and Mode = 1487.

iii) The first three moments about 16 are respectively –0.35, 2.09 and –1.93.

(P.U., B.A./B.Sc. 1974, 78; M.A. Econ. 19

4.47 Calculate skewness by (i) the Pearsonian method, and (ii) by the Bowley's formula from the following frequency distribution and interpret the result.

| Ages (years) | 15–19 | 20–24 | 25–29 | 30–34 | 35–39 | 40–44 | 45–49 | 50–54 |
|---|---|---|---|---|---|---|---|---|
| No. of Men | 29 | 176 | 208 | 173 | 82 | 40 | 15 | 3 |

(P.U., M.A. Econ. 1986)

4.48 Calculate the first four moments about the mean and provide one estimate each of skewness and kurtosis of the following distribution.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Age: | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
| Frequency: | 2 | 8 | 18 | 27 | 25 | 16 | 7 | 2 |

(P.U., M.A. Econ. 1967, I.U., 1992)

4.49 The following values have been obtained from two different frequency distributions of weights (lb) having 125 and 200 observations respectively after making the substitution:

$X = 16 + 5u,\ Y = 20 + 2v,$

a) $\sum fu = -46, \sum fu^2 = 306, \sum fu^3 = -242, \sum fu^4 = 1962$

b) $\sum fv = 21, \sum fv^2 = 1265, \sum fv^3 = -627, \sum fv^4 = 14169$

Find
i) which of the distributions is more consistent,
ii) which of the distributions is negatively skewed;
iii) which of the distributions is Meso-kurtic. (P.U., B.A./B.Sc. 1993)

4.50 a) The fourth mean moment of a symmetrical distribution is 243. What would be the value of the standard deviation in order that the distribution may be mesokurtic? (P.U., B.A./B.Sc. 1981)

b) In a certain distribution, the first four moments about the point 4 are –1.5, 17, –30 and 108. Calculate $b_1$ and $b_2$ and state whether the distribution is leptokurtic or platykurtic. (P.U., D.St., 1962)

4.51 a) The second moments about the mean of two distributions are 9 and 16, while the fourth moments about the mean are 230 and 780 respectively. Which of the distribution is (i) leptokurtic, (ii) mesokurtic, (iii) platykurtic?

b) The second moment about the mean of a symmetrical distribution is 25. What must be the value of the fourth moment about the mean in order that the distribution be (i) leptokurtic, (ii) mesokurtic, (iii) platykurtic? (P.U., B.A./B.Sc. 1971)

4.52 Discuss the various measures or quantities by which the characteristics of frequency distributions are measured and compared. (P.C.S. 1993)

4.53 Explain briefly how averages, measures of dispersion, skewness and kurtosis are complementary to one another in understanding a frequency distributions.

4.54 The mean and standard deviation of a variable X are 60 and 8.944 respectively. Find the mean and standard deviation of a new variable if

i) All the values of X are increased by 20 points.

ii) All the values of X are increased by 25%. (P.U., B.A./B.Sc. 2008-S)

4.55 The mean, mode and standard deviation of the weekly earnings of a random sample of women workers from a locality are 3133.33, 2804.35 and 796.70 respectively.

i) Calculate Skewness of the distribution and interpret the result. Also find coefficient of variation.

ii) What will happen to the values of the mean and standard deviation if every woman has an increase of Rs.500 per week?

iii) What will the mean and standard deviation be if every woman has an increase of 10% of previous earnings?

(P.U., B.A./B.Sc. 2007)

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Contents

CHAPTER 5

# INDEX NUMBERS

# INDEX NUMBERS

## 5.1 INTRODUCTION

An *index number* is a statistical measure of average change in a variable or a group of variables with respect to time or space. The variable may be the enrolment of students in an institution, the cost of education for college students, prices of a particular commodity or a group of commodities, wages of workers, volume of trade, sales, experts and imports, production, unemployment, group health, Government securities, etc. Index numbers are obtained by expressing the data for various periods or places as percentage of some specific period or place selected for the purposes of comparison and technically called the *base*. Index numbers may be computed on weekly or monthly basis but generally they are computed on annual basis.

The classical definition of an index number is provided by the English economist F.Y. Edgeworth (1845–1926) who states that "*an index number is a quantity which shows by its variations, the changes over time or space of a magnitude which is not susceptible either of accurate measurement in itself or of direct valuation in practice.*"

In short, an index number is a device that measures the *changes* occurring in data from time to time or from place to place.

**5.1.1 Simple and Composite Index Numbers.** Index numbers are generally classified into Simple Indices and Composite Indices. An index number is called a *simple index* when it is computed for a *single* variable. Index numbers of enrolment in colleges, index numbers of gold prices, etc. are examples of simple indices. A simple index can be very easily computed. The value of the variable for each period is divided by the value in the base period and the result is multiplied by 100. For example, the wages paid to the workers in a certain institution in 1980 and 1983 were Rs.9,650 and Rs.11,580 respectively. Now taking 1980 as the base year and 1983 as the given year, we have

$$\text{Wage Index for 1983} = \frac{\text{wages paid in 1983}}{\text{wages paid in 1980}} \text{x} 100$$

$$= \frac{\text{Rs.}11,580}{\text{Rs.}9,650} \text{x} 100 = 120$$



This result indicates that if the wage level in 1980 were denoted by 100, it is 120 in 1983. In other words, wages have increased by 20% for 1983 by comparison with 1980.

An index that is computed from two or more variables is referred to as a *composite index*. Examples of composite indices are the wholesale price index numbers, consumer price index numbers, etc. Composite indices are more important as many of the index numbers in common use are composite in nature.

Composite indices may further be classified into *Unweighted* and *Weighted* index numbers. Before we discuss these index numbers and the methods of their computation, let us first consider the problems, one has to face, in the compilation of these index numbers.

**5.1.2 Problems Involved in Index Number Construction.** In the compilation of index numbers, many problems are involved. The first problem is to understand the *purpose* which an index is to serve. The purpose of the index may be to compare the scores of two students, or to measure the changes in the general price level or to measure the changes in the production of scooters or to compare the changes in wages of factory workers over different places, etc. The next problem is to decide what data should be included. The data to be included should relate to purpose for which the index is to be used. This step also

involves the collection of data on scores, production, process, wages or whatever is being compared. Another problem is to decide what period should be chosen as the *base* period, *i.e.* the period with which the other periods are to be compared. In case of composite index numbers, another problem is to decide what method of averaging should be used to arrive at a single index for each period. The method of averaging usually includes the system of weighting but sometimes one faces the problem of assigning some explicit weights to the various items of the data so that their relative importance is taken into account.

These problems are discussed in somewhat detail in section 5.2 with reference to the construction of *price* index numbers as the indices used to measure the change in price level are more important.

## 5.2 MAIN STEPS IN THE CONSTRUCTION OF INDEX NUMBERS OF WHOLESALE PRICES

The usual method of compilation of an index number of wholesale prices involves the following steps:

i) Selection of commodities to be included, their number and price quotations.

ii) Selection of the base period and calculation of price relatives.

iii) Selection of average to be used.

iv) Selection of appropriate weights.



**5.2.1 Selection of Commodities for Inclusion.** The first step is to decide on the number of commodities to be included. There is no hard and fast rule for this purpose. Though in statistical theory there is a well recognised principle that *the larger the number of items included, the greater would be the accuracy*, but we know that a very large number of commodities would involve complications, expense and delay in the construction. Hence a reasonable number of commodities on the basis of their evaluated importance should be used.

As pointed out by Dr. Irving Fisher (1867–1947), *index numbers of prices are seldom of much value unless they consist of more than 20 commodities and 50 is a much better number*. However, it is important to bear in mind that while deciding on their number, the commodities to be selected must be (i) representative of the tastes, habits and requirements of the people concerned, (ii) unlikely to vary in quality or grade and (iii) comparable.

Having decided on the number of commodities, arrangements are made to collect the wholesale prices of the commodities chosen. The prices should be obtained from the various sources, *e.g.* from exchanges and big markets where they are quoted, from price bulletins, trade journals and newspapers and from leading firms. The price quotations obtained should be representative, reliable and comparable as regards the quality of the commodities and the units in which the commodities are expressed.

**5.2.2 Selection of the Base Period.** The next step is the selection of a base period-a period from which the changes are measured. The prices of all other periods are then expressed as percentages of the base period prices. Two methods of selecting the base period are available. They are the Fixed base method and the Chain base method.

**Fixed Base Method.** A *fixed base* method is one in which a particular year is generally chosen as the base period that remains unchanged during the life term of the index. It is relevant to note that the base year should not be too far distant in the past and should be a "normal" year. By a "normal" year, we generally mean a year of economic stability and free from any major financial crisis caused by inflation

depression, wars, labour unrest, lock-outs, famines, etc. In other words, it is a year during which prices may have remained more or less stable. If a single year of normal conditions is not available, an average of the prices of several years is used as the base period price. This average minimizes the influence of abnormal and disrupted economic conditions.

It is customary to denote the base period by the subscript 0, *e.g.* $p_0$ (or $q_0$) will denote the price (or quantity) of a commodity in the base period, while the subscripts, 1, 2, …, $n$ denote the other time periods in chronological order. The average price of the base period chosen is then set equal to 100. Index numbers (or price relatives) for other periods denoted by $P_{01}$, $P_{02}$, …, $P_{0n}$ are then computed as relative to the base period. Thus the *price relative* for the given year $n$, will be

$$\text{Price relative} = \frac{\text{Price of a commodity in the given year}}{\text{Price of the commodity in the base year}} \text{x}100$$

or $$P_{0n} = \frac{p_n}{p_0} \text{x}100$$

A *price relative* expresses the price of a commodity in a given year as a fraction of the price in the base year. It is multiplied by 100 to make it a percentage but is usually expressed without the percent symbol. A price relative is independent of any unit of measurement.

**Chain Base Method.** A *chain base* method is one in which the base period is not fixed but moves with the given year. That is, the relatives are computed with the immediately preceding year as the base and such relatives are called *link relatives*. Thus a

$$\text{Link relative} = \frac{\text{Price of a commodity in the given year}}{\text{Price of the commodity in the preceding year}} \text{ x } 100$$

or $$P_{n-1,n} = \frac{p_n}{p_{n-1}} \text{ x } 100$$



The link relatives are converted back to a fixed base by multiplying together all the link relatives (without the factor 100) involved between the two years. This process of conversion is called the *chaining process* and the indices thus determined are the *chain indices*. In other words, we say that the link relatives are "*chained*" back to a fixed base period by a process of successive multiplication and hence, the name "*chain index*". For example, if $P_{01}$, $P_{12}$, $P_{23}$, …, $P_{n-1,\,n}$ denote the link relatives (or average of link relatives) without the factor 100, then the indices on fixed base are obtained as below:

$$P_{01} = P_{01}$$
$$P_{02} = P_{01} \, x \, P_{12}$$
$$P_{03} = P_{01} \, x \, P_{12} \, x \, P_{23}$$
$$\ldots \quad \ldots$$
$$P_{0n} = P_{01} \, x \, P_{12} \, x \, P_{23} \, x \ldots x \, P_{n-1,n}$$

If link relatives are in percentages, the product considered pair-wise, is divided by 100. It is interesting to note that this chain index was originally suggested in 1887 by Alfred Marshall (1842-1924).

The chain index method has several advantages. They are:

i) The chain method provides a direct comparison between each year and the preceding year
it is in such terms that a businessman often thinks.

ii) The chain base method allows the addition of new commodities, removal of old commodi
or the substitution of one commodity for another.

iii) It is possible to change the geographical coverage or the weight of a commodity to
changing conditions.

iv) It satisfies the so-called *circular* test (to be explained later).

v) An index with a fixed period can be computed by the product of link relatives.

It suffers, however, from the following disadvantages:

i) The computational procedure of chain indices is relatively cumbersome.

ii) If an error is committed during the changing process, it will be carried through the
series.

iii) The changes in two years separated by a long interval cannot be compared.

Since the merits of the chain index outweigh its demerits, it is therefore considered a more re
index.

**5.2.3 Selection of Average.** The next step involves the choice of an appropriate average to
*single* index number for each year. For this purpose, any of the following averages may be used:

(a) The arithmetic mean, (b) the median and (c) the geometric mean.



The advantages and disadvantages of these averages in the construction of index numbers are
below:

**The Arithmetic Mean.** It is readily understood and is easier to compute. It is amenab
mathematical treatment, *i.e.* the mean of subgroups can be averaged to find the mean of all the data.
disadvantages of the mean are that it is greatly affected by extreme values; it gives too much weig
increasing prices and too little to decreasing ones, *i.e.* is biased upward. Moreover, the mean of rel
is not *reversible*, *i.e.* change of base cannot be made without affecting the proportionate change
index number.

**The Median.** It is easy to understand as well as to compute. It is less affected by extreme
than the mean and does not overemphasize increases. The defects of the median are that it
amenable to algebraic treatment, *i.e.* the medians of sub-groups cannot be averaged to obtain the
of all the data, and the median of relatives is not *reversible*.

**The Geometric Mean.** The geometric mean is a suitable average for ratios as it gives
importance to equal ratios of change. It makes possible to replace the commodities which have cea
be representative by those which have become representative. The geometric mean of relati
*reversible* and hence we can change a series of index numbers with any year as base to any other
the series as base by dividing each index number of the series by the index number of the new time
selected for base. But it has two disadvantages. It is an unfamiliar type of average and it in
considerable computational labour.

Theoretically, the geometric mean is the most suitable average but, in practice, the arithmetic
is generally employed.

**5.2.4 Selection of Appropriate Weights.** The last important step in the construction of wholesale price index numbers is to decide how to select weights which would indicate the relative importance of the various commodities in the group. It is evident that all the commodities selected are not equally important. For example, eggs and coffee cannot be given the same importance as wheat and rice. Wheat is much more important than coffee, it is therefore desirable that wheat must be given more importance. To meet this requirement of taking into account the relative importance of each commodity, a sample survey should be conducted. On the basis of this survey, each commodity should be assigned a *multiplier* which expresses more or less adequately its relative importance in the group. Such a multiplier is technically called a *weight*. The weights could be *either* a set of suitable numbers adding to 100, *or* the quantities of the various commodities actually consumed, produced or sold, *or* their money values. It is customary to apply *quantity weights* when dealing with prices themselves and *value weights* when using the price relatives. The quantity or value weights may relate to the base period or to the given period. When the base period quantity ($q_0$) is used as weight, the price index number, $P_{0n}$ for the year $n$ is computed by the formula

$$P_{0n} = \frac{\Sigma p_n q_0}{\Sigma p_0 q_0} \times 100$$

This is called the *base-year weighting*. With the given period quantity ($q_n$) as weight, the corresponding formula becomes

$$P_{0n} = \frac{\Sigma p_n q_n}{\Sigma p_0 q_n} \times 100$$

This is also known as the *given or current year weighting*. It is relevant to note that different systems of weighting would not generally lead to identical results.

## 5.3 UNWEIGHTED INDEX NUMBERS

Unweighted indices are generally classified into *Simple Aggregative* indices and *Simple Average of Relatives*.

**5.3.1 Simple Aggregative Index** is one that indicates the percentage change in the aggregate value of a number of commodities, (say $k$) at different periods. It is obtained by dividing the sum of the given year prices of all commodities by the sum of the base year prices of the same commodities and expressing the result as a percentage. Symbolically, we have

$$P_{0n} = \frac{\Sigma p_n}{\Sigma p_0} \times 100,$$

where $P_{0n}$ denotes the price index for the given year $n$ relative to the base year 0,

$\Sigma p_n$ denotes the sum of prices for the given year, and

$\Sigma p_0$ denotes the sum of prices for the base year.

A simple aggregative index is easy to understand as well as to apply but it has a disadvantage of not taking into account the relative importance of the various commodities. Moreover, the units of prices of different commodities being different, influence the price index, which then becomes an inappropriate measure. That is why it is seldom used in practice.

**5.3.2 Simple Average of Relatives.** A simple average of price relatives is an index obta taking the average of the price relatives of the given commodities for each year and expressing the as a percentage. If we take the arithmetic mean, then we have

$$P_{0n} = \frac{1}{k}\Sigma\left(\frac{p_n}{p_0}\right)\times 100,$$

where $k$ denote the number of commodities whose price relatives are thus combined.

The simple average of relatives index is superior to simple aggregative index. It suffers fr disadvantage that each price relative exerts equal influence and gives no consideration to the eco importance of each commodity. Moreover, the use of arithmetic mean, which is not an app average to use with ratios, results in an *upward bias*. This sort of drawback may be got rid of by us geometric average, in which case the formula becomes

$$P_{0n} = \left[\Pi\left(\frac{p_n}{p_0}\right)\right]^{1/k}\times 100,$$

where $k$ denotes the number of commodities whose price relatives are being averaged.

The median may also be used for averaging the price relatives.

**Example 5.1** Following are the prices of a commodity for the ten years ending with 19 Calculate the index numbers with (i) 1948 as a base; and (ii) average of first five years as a base.

| Year | 1948 | 1949 | 1950 | 1951 | 1952 | 1953 | 1954 | 1955 | 1956 | 1957 |
|---|---|---|---|---|---|---|---|---|---|---|
| Price in Rs. | 5.25 | 5.87 | 6.12 | 5.50 | 6.25 | 6.62 | 6.75 | 7.12 | 6.50 | 7.50 |

(P.U., B.A. (Hons. in Econ.)

Let $p_0$ denote base year's price and $p_n$ the given year's price. Then the prices are conver price relatives or price indices by the formula

Price relative for a given year $= \frac{p_n}{p_0}\times 100$,

| Year | Prices (rupees) | (i) Index No. $\left(=\frac{p_n}{p_0}\times 100\right)$ (1948 as base) | (i) Index No. $\left(=\frac{p_n}{p_0}\times 100\right)$ (average as base) |
|---|---|---|---|
| 1948 | 5.25 | 100 | 91 |
| 1949 | 5.87 | 112 | 101 |
| 1950 | 6.12 | 117 | 106 |
| 1951 | 5.50 | 105 | 95 |
| 1952 | 6.25 | 119 | 108 |
| 1953 | 6.62 | 126 | 114 |
| 1954 | 6.75 | 129 | 116 |
| 1955 | 7.12 | 136 | 123 |
| 1956 | 6.50 | 124 | 112 |
| 1957 | 7.50 | 143 | 129 |

**Example 5.2** The average prices of certain commodities are given below:

| Commodity | Unit | Price (Rs.) in | |
|---|---|---|---|
| | | 1957 | 1958 |
| Wheat | Ton | 351.00 | 335.00 |
| Rice | 40 kg | 35.00 | 32.00 |
| Salt | 40 kg | 10.00 | 11.00 |
| Sugar | kg | 1.25 | 1.40 |
| Cloth | Meter | 2.25 | 2.60 |
| Milk | Litre | 0.75 | 0.85 |
| Oil | Gallon | 1.25 | 1.35 |

Compute a simple aggregative price index number for the year 1958 based on 1957 prices.

(P.U., M.A. Econ. 1969-S)

The simple aggregative price index number is given by

$$P_{0n} = \frac{\sum p_n}{\sum p_0} \times 100$$

$$= \frac{335.00 + 32.00 + ... + 1.35}{351.00 + 35.00 + ... + 1.25} \times 100 = \frac{384.20}{401.50} \times 100 = 95.7$$

The simple aggregative price index for 1958 on the basis of 1957 is 95.7. This means that the prices have decreased by 4.3%.

**Example 5.3** Compute price index for the year 1958 based on 1957 prices, using the simple average of relatives method for the data in Example 5.2.

The computation of the simple average of price relatives index involves the calculation of price relatives for 1958 by dividing the prices in 1958 by the corresponding prices in 1957. The price relatives are shown below:

| Commodity | Price relatives | |
|---|---|---|
| | 1957 | 1958 |
| Wheat | 100 | 95.44 |
| Rice | 100 | 91.43 |
| Salt | 100 | 110.00 |
| Sugar | 100 | 112.00 |
| Cloth | 100 | 115.56 |
| Milk | 100 | 113.33 |
| Oil | 100 | 108.00 |
| Σ | -- | 745.76 |

Thus the simple average of price relatives for 1958 is

$$P_{0n} = \frac{1}{k}\sum\left(\frac{p_n}{p_0}\right) \times 100.$$

$$= \frac{745.76}{7} = 106.5$$

This indicates an increase in prices from 1957 to 1958 by 6.5%.

It should be noted that the index numbers obtained by the two methods show considera difference because the simple aggregative index has been affected by the units of prices of th commodities, and although the influence due to different units has been eliminated by the simple avera of relatives, commodities such as salt, sugar, cloth, etc. have exercised an influence out of all propor of their economic importance.

**Example 5.4** From the data given below, compute the index numbers of prices, taking 198 as base. Use (i) simple average of price relatives and (ii) the median of price relatives.

| Year | Commodity (Prices in Rs.) | | | |
|---|---|---|---|---|
| | A | B | C | D |
| 1980 | 16.25 | 20.00 | 2.40 | 10.50 |
| 1981 | 17.22 | 22.40 | 2.64 | 12.50 |
| 1982 | 19.55 | 16.00 | 3.00 | 12.60 |
| 1983 | 18.70 | 20.00 | 3.80 | 14.65 |

We calculate the price index numbers as in the following table:

| Year | Price Relatives | | | | Total | Index Numbers by | |
|---|---|---|---|---|---|---|---|
| | A | B | C | D | | (i) Mean | (ii) Median |
| 1980 | 100 | 100 | 100 | 100 | 400 | 100 | 100 |
| 1981 | 106 | 112 | 110 | 119 | 447 | 112 | 111 |
| 1982 | 120 | 80 | 125 | 120 | 445 | 111 | 120 |
| 1983 | 115 | 100 | 158 | 140 | 513 | 128 | 128 |

The first entry in row 1981 is $\frac{p_1}{p_0} \times 100 = \frac{\text{Rs.}17.22}{\text{Rs.}16.25} \times 100 = 106$, and the price index for 19

$$\frac{1}{k}\Sigma\left(\frac{p_n}{p_0}\right) \times 100 = \frac{447}{4} = 112.$$

The other entries have been computed in a similar way.

The price index numbers appearing in the last column of the table, have been obtained by fin the medians of the price relatives.

**Example 5.5** Construct *chain indices* for the following years, taking 1940 as the base and u the simple average of relatives.

| Year | Price in Rs. per maund (1 maund = 37.2 kg) | | |
|---|---|---|---|
| | Wheat | Rice | Maize |
| 1940 | 2.80 | 10.50 | 2.70 |
| 1941 | 3.40 | 10.80 | 3.20 |
| 1942 | 3.60 | 10.60 | 3.50 |
| 1943 | 4.00 | 11.00 | 3.80 |
| 1944 | 4.20 | 11.50 | 4.00 |

(P.U., B.A. (Hons. in Econ.), 19

First we calculate the link relatives by the formula

$$P_{n-1,n} = \frac{\text{Price of a commodity in the given year}}{\text{Price of the commodity in the preceding year}} \times 100$$

$$= \frac{p_n}{p_{n-1}} \times 100$$

Next, we multiply the averages of link relatives successively (taking two at a time) and divide the product by 100 to *chain* the relatives back to the base 1940. The calculations appear in the table below:

| Year | Link Relatives | | | Sum of relatives | Simple average of relatives | Chain Indices |
|---|---|---|---|---|---|---|
| | Wheat | Rice | Maize | | | |
| 1940 | 100 | 100 | 100 | 300 | 100 | 100 |
| 1941 | $\frac{3.40}{2.80} \times 100 = 121$ | 103 | 119 | 343 | 114 | $\frac{100 \times 114}{100} = 114$ |
| 1942 | $\frac{3.60}{3.40} \times 100 = 106$ | 98 | 109 | 313 | 104 | $\frac{114 \times 104}{100} = 118.6$ |
| 1943 | $\frac{4.00}{3.60} \times 100 = 111$ | 104 | 109 | 324 | 108 | $\frac{118.6 \times 108}{100} = 128.1$ |
| 1944 | $\frac{4.20}{4.00} \times 100 = 105$ | 105 | 105 | 315 | 105 | $\frac{128.1 \times 105}{100} = 134.5$ |

Hence the Chain Index Numbers are 100, 114, 118.6, 128.1, 134.5.

**Example 5.6** Find the *chain indices* from the following price relatives of four commodities, using the geometric mean of the relatives for each year.

| Year | Commodity | | | |
|---|---|---|---|---|
| | A | B | C | D |
| 1991 | 81 | 77 | 119 | 55 |
| 1992 | 62 | 54 | 128 | 82 |
| 1993 | 104 | 87 | 111 | 100 |
| 1994 | 93 | 75 | 154 | 96 |
| 1995 | 60 | 43 | 165 | 88 |

First we calculate the *link relatives*. The link relatives for commodity A (for instance) are computed below:

$$\text{Link relative for 1992} = \frac{62}{81} \times 100 = 76$$

$$\text{Link relative for 1993} = \frac{104}{62} \times 100 = 168$$

$$\text{Link relative for 1994} = \frac{93}{104} \times 100 = 89$$

$$\text{Link relative for 1995} = \frac{60}{93} \times 100 = 65$$

The link relatives for other commodities are obtained in a similar way. The chain indices, using geometric mean are then computed and shown in the table below:

Calculation of chain indices, using the geometric mean of the relatives

| Year | Link Relatives of Commodity | | | | $\sum \log x$ | $\frac{1}{k}\sum \log x$ | G.M. | Chain Index Numbers |
|---|---|---|---|---|---|---|---|---|
| | A | B | C | D | | | | |
| 1991 | 81 | 77 | 119 | 55 | | | | |
| log | 1.9085 | 1.8865 | 2.0755 | 1.7404 | 7.6109 | 1.9027 | 79.9 | 79.9 |
| 1992 | 76 | 70 | 108 | 149 | | | | $\frac{79.9 \times 96.2}{100}$ |
| log | 1.8808 | 1.8451 | 2.0334 | 2.1732 | 7.9325 | 1.9831 | 96.2 | = 76.9 |
| 1993 | 168 | 161 | 87 | 122 | | | | $\frac{76.9 \times 130.2}{100}$ |
| log | 2.2253 | 2.2068 | 1.9395 | 2.0864 | 8.4580 | 2.1145 | 130.2 | = 100.1 |
| 1994 | 89 | 86 | 139 | 96 | | | | $\frac{100.1 \times 100.6}{100}$ |
| log | 1.9494 | 1.9345 | 2.1430 | 1.9823 | 8.0092 | 2.0023 | 100.6 | = 100.7 |
| 1995 | 65 | 57 | 107 | 93 | | | | $\frac{100.7 \times 77.9}{100}$ |
| log | 1.8129 | 1.7559 | 2.0294 | 1.9685 | 7.5667 | 1.8917 | 77.9 | = 78.4 |

Hence the chain indices by geometric mean are 79.9, 76.9, 100.1, 100.7 and 78.4.



## 5.4 WEIGHTED INDEX NUMBERS

An index number that measures the change in the prices of a group of commodities when the relative importance of the commodities (*i.e.* weight) has been taken into account, is called a weighted *price index number*. Weighted indices are generally divided into Weighted Aggregative indices and Weighted Average of relatives indices.

**5.4.1 Weighted Aggregative Price Index Numbers.** An index is called a *weighted aggregative index* when it is constructed for an aggregate of items (*prices*) that have been weighted in some way (by *corresponding quantities produced, consumed or sold*) so as to reflect their importance. There are various kinds of weighted aggregative index numbers, some of them are discussed below:

(a) **Laspeyres' Price Index,** advocated by the German economist *E'* tienne Laspeyres (1834–1913) in 1864, is defined as

$$P_{0n} = \frac{\sum p_n q_0}{\sum p_0 q_0} \times 100$$

This is the percentage ratio of the aggregate of the given period prices weighted by the quantities produced, consumed or sold in the base period to the aggregate of base period prices weighted by the base period quantities. The index represents the relative cost in different years of purchasing the base year quantities of various commodities at the given year price. The advantage of Laspeyres formula is that the quantity weights remain unchanged for the subsequent periods and only information on latest prices need be obtained. It has, however, a few limitations. The index value obtained by this formula gets somewhat inflated as the series move away from the base period. That is why an index of Laspeyres type is said to have an *upward bias*. It does not satisfy the time-reversal test or the factor-reversal test. (discussed later)

(b) **Paasche's Price Index,** proposed in 1874 by the German economist Herman Paasche (1851–1925) is given by the relation

$$P_{0n}=\frac{\sum p_n q_n}{\sum p_0 q_n}\times 100$$

This is the percentage ratio of the aggregate of given period prices weighed by the quantities produced, consumed or sold in the given period to the aggregate of base period prices weighted by the given period quantities. It represents the relative cost in different years of purchasing the given year quantities of various commodities at the given year price. The computation of this type of index needs up-to-date data on quantities (*i.e.* weights) and to obtain such information on quantities for each given year, fresh enquiry or a statistical survey, which normally involves considerable time and finances, would be needed every year and this is a difficult task. This index has a *downward bias, i.e.* it deflates the index values of distant periods of time. It also does not obey the time-reversal test or the factor-reversal test.

(c) **Marshall-Edgeworth Price Index.** This index was proposed independently by the two English economists Alfred Marshall (1842–1924) and F.Y. Edgeworth (1845–1926). Here the weights are taken as an average of the respective quantities in the base period and in the given period. This is, so to say, a compromise solution, although it is the index which has no general bias in either direction. Since $(q_0+q_n)$ lies between $q_0$ and $q_n$, the Marshall-Edgeworth's index number lies between the Laspeyres' and Paasche's index numbers. The formula for this index is

$$P_{0n}=\frac{\sum p_n(q_0+q_n)}{\sum p_0(q_0+q_n)}\times 100$$

(d) **Fisher's "Ideal" Index.** Fisher's *ideal* index number, named after its inventor, Irving Fisher (1867–1947), is the geometric mean of the Laspeyres and Paasche type of index numbers. Symbolically, it is given by

$$P_{0n}\text{ (Fisher)}=\sqrt{P_{0n}\text{ (Laspeyres)}\times P_{0n}\text{ (Paasche)}}$$

$$=\sqrt{\frac{\sum p_n q_0}{\sum p_0 q_0}\times\frac{\sum p_n q_n}{\sum p_0 q_n}}\times 100.$$

Fisher called it "*ideal*" index because it meets certain theoretical tests of quality which he considered appropriate for a *good* index number. It is sometimes known as *crossed-weight* formula because it is the result of geometrically crossing (averaging) two index numbers with different systems of weighting.

Fisher's *ideal* index has a theoretical advantage over other index numbers as it is the only index that obeys both the time-reversal test and the factor-reversal test. It suffers, however from the following disadvantages:

i) The *Ideal* index number being the geometric mean of the Laspeyres index that has an upward bias and the Paasche index which suffers from a downward bias, is considered to give a better result. But we are not sure about the elimination of the biases as "*the average of two wrong answers does not necessarily give one right answer.*"

ii) It is difficult to say specifically what the *ideal* index number measures because of its being a hybrid of two index numbers.

iii) The computation of the *ideal* index is relatively difficult and laborious.

iv) Its computation needs information on quantities (consumed, produced or sold) for each given period. To get such information, a fresh survey which may be too costly or time-consuming would be needed every year.

(e) **Walsh Index.** Walsh advocated that the weights should be the geometric mean of the base and given period quantities instead of taking their simple average. Symbolically, it is given as

$$P_{0n} = \frac{\sum p_n \sqrt{q_0 q_n}}{\sum p_0 \sqrt{q_0 q_n}} \times 100$$

There is another formula known as the **Lowe** price index number in which average weights are used. It is given as

$$P_{0n} = \frac{\sum p_n q}{\sum p_0 q} \times 100$$

where $q$ is obtained by averaging quantities of several years.

**Example 5.7** From the following table, compute the weighted aggregative price index for the year 1999 on the basis of the year 1995.

| Commodity | Unit | Average monthly consumption per family 1995 | Average prices in rupees 1995 | Average prices in rupees 1999 |
|---|---|---|---|---|
| Milk | Litre | 30 | 3.50 | 4.00 |
| Flour | Kilogram | 25 | 1.25 | 1.75 |
| Cloth | Meter | 12 | 4.00 | 8.00 |
| Tea | Kilogram | 1 | 12.00 | 18.00 |
| Vegetable ghee | Kilogram | 5 | 7.50 | 4.00 |
| Eggs | Dozen | 4 | 5.00 | 7.25 |

To compute the weighted aggregative price index for the year 1999 with 1995 as base, we need $\sum p_0 q_0$ and $\sum p_n q_0$. The calculations appear below:

| Commodity | 1995 ($q_0$) | 1995 ($p_0$) | 1999 ($p_n$) | $p_0q_0$ | $p_nq_0$ |
|---|---|---|---|---|---|
| Milk | 30 | 3.50 | 4.00 | 105.00 | 120.00 |
| Flour | 25 | 1.25 | 1.75 | 31.25 | 43.75 |
| Cloth | 12 | 4.00 | 8.00 | 48.00 | 96.00 |
| Tea | 1 | 12.00 | 18.00 | 12.00 | 18.00 |
| Vegetable ghee | 5 | 7.50 | 4.00 | 37.50 | 20.00 |
| Eggs | 4 | 5.00 | 7.25 | 20.00 | 29.00 |
| $\sum$ | --- | --- | --- | 253.75 | 326.75 |

Since $P_{0n} = \frac{\sum p_n q_0}{\sum p_0 q_0} \times 100 = \frac{326.75}{253.75} \times 100 = 128.77$

The weighted aggregative price index for 1999 on the basis of 1995 is 128.77. This means that the prices have increased by 28.77%.

**Example 5.8** Construct the following weighted aggregative price index numbers for 2000 and 2001 from the given data.

(i) Laspeyres' index, (ii) Paasche's index and (iii) Fisher's "Ideal" index.

| Commodity | Prices (Rs. per 40 kg) | | | Quantities (tons) | | |
|---|---|---|---|---|---|---|
| | 1996 (base) | 2000 | 2001 | 1996 | 2000 | 2001 |
| A | 64 | 75 | 80 | 270 | 276 | 290 |
| B | 40 | 45 | 41 | 124 | 118 | 144 |
| C | 18 | 21 | 20 | 130 | 121 | 137 |
| D | 58 | 68 | 56 | 185 | 267 | 355 |

We calculate the necessary products in the following table:

| [illegible] | Price | | | Quantity | | | $p_0q_0$ | $p_1q_0$ | $p_2q_0$ | $p_0q_1$ | $p_0q_2$ | $p_1q_1$ | $p_2q_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $p_0$ (1996) | $p_1$ (2000) | $p_2$ (2001) | $q_0$ (1996) | $q_1$ (2000) | $q_2$ (2001) | | | | | | | |
| [illegible] | 64 | 75 | 80 | 270 | 276 | 290 | 17280 | 20250 | 21600 | 17664 | 18560 | 20700 | 23200 |
| [illegible] | 40 | 45 | 41 | 124 | 118 | 144 | 4960 | 5580 | 5084 | 4720 | 5760 | 5310 | 5904 |
| [illegible] | 18 | 21 | 20 | 130 | 121 | 137 | 2340 | 2730 | 2600 | 2178 | 2466 | 2541 | 2740 |
| [illegible] | 58 | 68 | 56 | 185 | 267 | 355 | 10730 | 12580 | 10360 | 15486 | 20590 | 18156 | 19880 |
| [illegible] | -- | -- | -- | -- | -- | -- | 35310 | 41140 | 39644 | 40048 | 47376 | 46707 | 51724 |

(i) **Laspeyres' index numbers:**

Index for 2000 $(P_{01}) = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100 = \frac{41140}{35310} \times 100 = 116.5$

Index for 2001 $(P_{02}) = \frac{\sum p_2 q_0}{\sum p_0 q_0} \times 100 = \frac{39644}{35310} \times 100 = 112.3$

(ii) **Paasche's index numbers:**

$$\text{Index for 2000 } (P_{01}) = \frac{\sum p_1q_1}{\sum p_0q_1} \times 100 = \frac{46707}{40048} \times 100 = 116.6$$

$$\text{Index for 2001 } (P_{02}) = \frac{\sum p_2q_2}{\sum p_0q_2} \times 100 = \frac{51724}{47376} \times 100 = 109.2$$

(iii) **Fisher's "Ideal" index numbers:**

$$\text{Index for 2000 } (P_{01}) = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} \times \frac{\sum p_1q_1}{\sum p_0q_1}} \times 100$$

$$= \sqrt{\frac{41140}{35310} \times \frac{46707}{40048}} \times 100$$

$$= \sqrt{1.165 \times 1.166} \times 100 = 116.5$$

$$\text{Index for 2001 } (P_{02}) = \sqrt{\frac{\sum p_2q_0}{\sum p_0q_0} \times \frac{\sum p_2q_2}{\sum p_0q_2}} \times 100$$

$$= \sqrt{\frac{39644}{35310} \times \frac{51724}{47376}} \times 100$$

$$= \sqrt{1.123 \times 1.092} \times 100 = 110.7$$



**Example 5.9** The prices and quantities of three commodities during 1990 and 1994 are given below:

| Commodity | Prices Rs. per maund | | Quantities Produced (maunds) | |
|---|---|---|---|---|
| | 1990 ($p_0$) | 1994 ($p_1$) | 1990 ($q_0$) | 1994 ($q_1$) |
| A | 3.95 | 4.25 | 9,675 | 10,436 |
| B | 34.80 | 38.94 | 78 | 83 |
| C | 61.56 | 59.70 | 118 | 116 |

Compute the Marshall-Edgeworth and the Walsh's price index numbers for 1994, using 1990 as the base period.

First we calculate the necessary products with 1990 as base year. These calculations are shown in the table below:

*Computation of Weighted Aggregative Price Indices*

| Com. | $(q_0+q_1)$ | $p_1(q_0+q_1)$ | $p_0(q_0+q_1)$ | $\sqrt{q_0q_1}$ | $p_1\sqrt{q_0q_1}$ | $p_0\sqrt{q_0q_1}$ |
|---|---|---|---|---|---|---|
| A | 29,111 | 85,471.75 | 79,438.45 | 10,048.3 | 42,705.28 | 39,690.78 |
| B | 161 | 6,269.34 | 5,602.80 | 80.5 | 3,134.67 | 2,801.40 |
| C | 234 | 13,969.80 | 14,405.04 | 117.0 | 6,984.90 | 7,202.52 |
| Total | --- | 105,710.89 | 99,446.29 | --- | 52,824.85 | 49,694.70 |

Thus the Marshall-Edgeworth price index $= \dfrac{\sum p_1(q_0+q_1)}{\sum p_0(q_0+q_1)} \text{x} 100$

$$= \frac{105710.89}{99446.29} \text{x} 100 = 106.3$$

and the Walsh price index $= \dfrac{\sum p_1\sqrt{q_0q_1}}{\sum p_0\sqrt{q_0q_1}} \text{x} 100$

$$= \frac{52824.85}{49694.70} \text{x} 100 = 106.3.$$

**5.4.2 Weighted Average of Relatives Price Index Number.** It is computed by multiplying each price relative by its weight, summing these products and dividing by the sum of the weights. The weights are always the total values of the commodities. The important types of the weighted average of relatives price indices are given below:

(a) **Laspeyres,** index number is

$$P_{0n} = \frac{\sum (p_n/p_0)p_0q_0}{\sum p_0q_0} \text{x} 100$$

where the price relatives are weighted by the *total* value of commodities in the base year. This is equivalent to Laspeyres weighted aggregative price index, *i.e.* $\dfrac{\sum p_nq_0}{\sum p_0q_0} \text{x} 100$. In other words, these methods are alternative ways of getting the same result.

(b) **Paasche's** index number is

$$P_{0n} = \frac{\sum \left(\dfrac{p_n}{p_0}\right) p_0q_n}{\sum p_0q_n} \text{x} 100$$

where the price relatives are weighted by the *total* value of commodities in the given year at base year prices. This is also identical with Paasche's weighted aggregative price index, *i.e.* $\dfrac{\sum p_n q_n}{\sum p_0 q_n} \times 100$.

(c) **Palgrave's** index number is

$$P_{0n} = \frac{\sum (p_n / p_0) p_n q_n}{\sum p_n q_n} \times 100,$$

where the price relatives are weighted by the *total* value of commodities in the given year.

In all these index numbers, the computational labour is reduced if the weights are made to sum to unity. For example, if $w_0 = \dfrac{p_0 q_0}{\sum p_0 q_0}$, then the Laspeyres formula becomes

$$P_{0n} = \sum \left[ \left( \frac{p_n}{p_0} \right) w_0 \right] \times 100, \qquad (\because \sum w_0 = 1)$$

The advantages of this procedure is that it indicates how many points each commodity contributes to the index number each year.

**Example 5.10** Using the data of Example 5.9, compute the weighted average of relatives price index numbers for 1994 by (i) Laspeyres' method, (ii) Paasche's method, and (iii) Palgrave's method.

*Computation of Weighted Average of Relatives Price Indices*

| Com. | Price | | Quantity | | Price relative | Weights | | | Weighted Price Relatives | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $p_0$ (1990) | $p_1$ (1994) | $q_0$ (1990) | $q_1$ (1994) | $\left(\frac{p_1}{p_0}\right)$ | $p_0q_0$ | $p_0q_1$ | $p_1q_1$ | $p_0q_0\left(\frac{p_1}{p_0}\right)$ | $p_0q_1\left(\frac{p_1}{p_0}\right)$ | [illegible] |
| A | 3.95 | 4.25 | 9,675 | 10,436 | 1.076 | 38216.25 | 41222.20 | 44353.00 | 41120.68 | 44355.09 | [illegible] |
| B | 34.80 | 38.94 | 78 | 83 | 1.119 | 2714.40 | 2888.40 | 3232.02 | 3037.41 | 3232.12 | [illegible] |
| C | 61.56 | 59.70 | 118 | 116 | 0.970 | 7264.08 | 7140.96 | 6925.20 | 7046.16 | 6926.73 | [illegible] |
| Total | --- | --- | --- | --- | --- | 48194.73 | 51251.56 | 54510.22 | 51204.25 | 54513.94 | [illegible] |

Thus (i) Laspeyres' index for 1994 $= \dfrac{\sum (p_1 / p_0) p_0 q_0}{\sum p_0 q_0} \times 100$

$$= \frac{51{,}204.25}{48{,}194.73} \times 100 = 106.2,$$

(ii) Paasche's index for 1994 $= \dfrac{\sum (p_1 / p_0) p_0 q_1}{\sum p_0 q_1} \times 100$

$$= \frac{54{,}513.94}{51{,}251.56} \times 100 = 106.4, \text{ and}$$

$$\text{(iii) Palgrave's index for 1994} = \frac{\sum (p_1/p_0)p_1q_1}{\sum p_1q_1} \times 100$$

$$= \frac{58{,}057.90}{54{,}510.22} \times 100 = 106.5.$$

## 4.5 QUANTITY INDEX NUMBERS

They are intended to measure the changes in the physical volume or quantity produced, consumed or sold of certain goods or services with respect to time. Like a price relative, we define a quantity relative by the ratio

$$\text{quantity relative} = \frac{q_n}{q_0} \times 100,$$

where $q_n$ denotes the quantity of a commodity in the given period and $q_0$ denotes the corresponding quantity in the base period and which measures the proportionate change in quantity. The quantity index number formulas are obtained by interchanging $p$'s and $q$'s in the weighted price index number formulas. For instance, the Laspeyres' Weighted Aggregative Quantity Index is given by

$$Q_{0n} = \frac{\sum q_n p_0}{\sum q_0 p_0} \times 100,$$

where prices of the base year are kept fixed as weights. The Paasche's Weighed Aggregative Quantity Index is given as

$$Q_{0n} = \frac{\sum q_n p_n}{\sum q_0 p_n} \times 100,$$

where prices of the given year are kept fixed as weights. Similarly, we define the Fisher's Quantity Index by the relation

$$Q_{0n} = \sqrt{\frac{\sum q_n p_0}{\sum q_0 p_0} \times \frac{\sum q_n p_n}{\sum q_0 p_n}} \times 100.$$



Other formulas are also defined and interpreted in a similar way. As the prices are held constant, the differences are attributed to changes in quantity. These index numbers are also called the *volume index numbers or Quantum index numbers*.

When the price $p$ of a commodity during a period is multiplied by its quantity $q$, produced, purchased or sold during the period, we get the *total value*, by $pq$ or $v$. A *value relative* is then defined as

$$\text{value relative} = \frac{\text{Total value during given year}}{\text{Total value during base year}} \times 100$$

$$\text{or} \quad V_{0n} = \frac{p_n q_n}{p_0 q_0} \times 100 = \frac{v_n}{v_0} \times 100.$$

It is interesting to note that (omitting the factor 100) a value relative is equal to the product of the price relative by the quantity relative. Symbolically, this may be written as

$$\text{value relative} = \frac{p_n q_n}{p_0 q_0}$$
$$= \left(\frac{p_n}{p_0}\right)\left(\frac{q_n}{q_0}\right)$$
$$= \text{Price relative x Quantity relative}$$

In a similar way, a weighted relative quantity-index is defined as

$$Q_{on} = \frac{\sum\left(\frac{q_n}{q_0}\right)w}{\sum w} \times 100,$$

where $(q_n/q_0)$ are the quantity relatives and $w$ denotes the weights.

**Example 5.11** Construct (i) a Laspeyres' type (ii) a Paasche's type and (iii) Fisher's *Ideal* type quantity index numbers from the following data:

| Commodity | Quantity | | | Price | | |
|---|---|---|---|---|---|---|
| | 1958 ($q_0$) | 1959 ($q_1$) | 1960 ($q_2$) | 1958 ($p_0$) | 1959 ($p_1$) | 1960 ($p_2$) |
| Chair | 200 | 350 | 350 | 15 | 16 | 20 |
| Desk | 100 | 220 | 340 | 18 | 20 | 35 |
| Radio | 30 | 45 | 50 | 100 | 120 | 150 |

(I.U., M.A., Econ. 1989, 9...)

The necessary calculations are given below:

| Commodity | $q_0p_0$ | $q_1p_0$ | $q_2p_0$ | $q_0p_1$ | $q_0p_2$ | $q_1p_1$ | $q_2p_2$ |
|---|---|---|---|---|---|---|---|
| Chair | 3000 | 5250 | 5250 | 3200 | 4000 | 5600 | 7000 |
| Desk | 1800 | 3960 | 6120 | 2000 | 2500 | 4400 | 8500 |
| Radio | 3000 | 4500 | 5000 | 3600 | 4500 | 5400 | 7500 |
| Total | 7800 | 13710 | 16370 | 8800 | 11000 | 15400 | 23000 |

(i) **Laspeyres' type:**

$$\text{Quantity index for 1959} = \frac{\sum q_1 p_0}{\sum q_0 p_0} \times 100 = \frac{13710}{7800} \times 100 = 175.8$$

$$\text{Quantity index for 1960} = \frac{\sum q_2 p_0}{\sum q_0 p_0} \times 100 = \frac{16370}{7800} \times 100 = 209.9$$

(ii) **Paasche's type:**

$$\text{Quantity index for 1959} = \frac{\sum q_1 p_1}{\sum q_0 p_1} \times 100 = \frac{15400}{8800} \times 100 = 175.0$$

$$\text{Quantity index for 1960} = \frac{\sum q_2 p_2}{\sum q_0 p_2} \times 100 = \frac{23000}{11000} \times 100 = 209.1$$

(iii) **Fisher's Ideal type:**

$$Q_{01} = \sqrt{\frac{\sum q_1 p_0}{\sum q_0 p_0} \times \frac{\sum q_1 p_1}{\sum q_0 p_1}} \times 100$$

$$= \sqrt{\frac{13710}{7800} \times \frac{15400}{8800}} \times 100 = 175.4$$

$$Q_{02} = \sqrt{\frac{\sum q_2 p_0}{\sum q_0 p_0} \times \frac{\sum q_2 p_2}{\sum q_0 p_2}} \times 100$$

$$= \sqrt{\frac{16370}{7800} \times \frac{23000}{11000}} \times 100 = 209.5$$

**Example 5.12** The following series shows for U.K. total imports (a) the declared value, and (b) the value on the basis of average values in 1930.

*U.K. Total Imports*

| Year | Declared value (£ million) | Value on basis of 1930 values (£ million) |
|---|---|---|
| 1930 | 1044 | 1044 |
| 1931 | 861 | 1069 |
| 1932 | 702 | 939 |
| 1933 | 675 | 946 |
| 1934 | 731 | 991 |
| 1935 | 756 | 1012 |
| 1936 | 848 | 1077 |

https://stat9943.blogspot.com

Taking 1930 as base year, construct index numbers (i) of average values and (ii) of volume for the years 1931–1936. (P.U., M.A., 1961; B.A./B.Sc., 1969; P.C.S. 1971)

It is easy to obtain (i) an index of average values for each year and (ii) a volume index, by writing the items in columns 2 and 3 in symbols first as below:

| Year (1) | Declared value (2) | Value on basis of 1930 values (3) |
|---|---|---|
| 1930 | $\sum p_0 q_0$ | $\sum p_0 q_0$ |
| 1931 | $\sum p_1 q_1$ | $\sum p_0 q_1$ |
| 1932 | $\sum p_2 q_2$ | $\sum p_0 q_2$ |
| 1933 | $\sum p_3 q_3$ | $\sum p_0 q_3$ |
| 1934 | $\sum p_4 q_4$ | $\sum p_0 q_4$ |
| 1935 | $\sum p_5 q_5$ | $\sum p_0 q_5$ |
| 1936 | $\sum p_6 q_6$ | $\sum p_0 q_6$ |

(i) **Index numbers of average values** are thus obtained by dividing the entry in column 2 for year by corresponding entry in column 3, *i.e.* by Paasche's formula. Thus

$$\text{the average value (price) index for 1930} = \frac{\sum p_0q_0}{\sum p_0q_0} \times 100$$

$$= \frac{1044}{1044} \times 100 = 100,$$

$$\text{and average value index for 1931} = \frac{\sum p_1q_1}{\sum p_0q_1} \times 100$$

$$= \frac{861}{1069} \times 100 = 80.7.$$

Similarly, the average value indices for 1932–36 are obtained as 74.8, 71.4, 73.8, 74.7 and 78.7.

(ii) **Index numbers of volumes** are obtained by dividing the values in column 3 by $\sum p_0q_0$, i.e. by Laspeyres' method.

$$\text{Thus the volume index for 1930} = \frac{\sum p_0q_0}{\sum p_0q_0} \times 100$$

$$= \frac{1044}{1044} \times 100 = 100,$$

$$\text{and the volume index for 1931} = \frac{\sum p_0q_1}{\sum p_0q_0} \times 100$$

$$= \frac{1069}{1044} \times 100 = 102.4$$

Proceeding in a similar way, we obtain the volume index numbers for 1932–36 as 89.9, 90.6, 9–, 96.9 and 103.2.



**Example 5.13** Given the following data:

| Commodity | Base Year | | Current Year | |
|---|---|---|---|---|
| | Price ($p_0$) | Quantity | Price ($p_1$) | Quantity |
| A | 80.65 | 276 | 85.00 | 290 |
| B | 155.00 | 18 | 154.75 | 44 |
| C | 32.50 | 121 | 30.50 | 137 |
| D | 75.00 | 267 | 60.95 | 355 |

Calculate (i) the value index number;

(ii) Laspeyres' weighted aggregative price index;

(iii) Paasche's weighted average of relatives volume index.

(B.Z.U., M.A. Econ. 19–)

We first calculate the necessary products in the following table:

| Commodity | $p_0q_0$ | $p_1q_1$ | $p_1q_0$ | $p_0q_1$ | $q_1/q_0$ | $(q_1/q_0) \times q_0p_1$ |
|---|---|---|---|---|---|---|
| A | 22259.4 | 24650.0 | 23460.0 | 23388.5 | 1.0507 | 24649.42 |
| B | 2790.0 | 6809.0 | 2785.5 | 6820.0 | 2.4444 | 6808.88 |
| C | 3932.5 | 4178.5 | 3690.5 | 4452.5 | 1.1323 | 4178.75 |
| D | 20025.0 | 21637.25 | 16273.65 | 26625.0 | 1.3296 | 21637.45 |
| $\sum$ | 49006.9 | 57274.75 | 46209.75 | 61286.0 | -- | 57274.50 |

(i) The value index number is;

$$V_{01} = \frac{\sum p_1q_1}{\sum p_0q_0} \times 100 = \frac{57274.75}{49006.9} \times 100 = 116.87$$

(ii) The Laspeyres' weighted aggregative price index is

$$P_{01} = \frac{\sum p_1q_0}{\sum p_0q_0} \times 100 = \frac{46209.65}{49006.9} \times 100 = 94.29;$$

(iii) The Paasche's weighted average of relatives volume index is

$$P_{01} = \frac{\sum(q_1/q_0)q_0p_1}{\sum q_0p_1} \times 100 = \frac{57274.50}{46209.65} \times 100 = 123.94.$$



## TESTS FOR INDEX NUMBER FORMULAE

From a theoretical view point, a "good" index number formula is required to satisfy the following ...posed by Irving Fisher (1867–1947).

**5.6.1 Time Reversal Test.** This may be stated as follows:

"If the time subscripts 0 and $n$ are interchanged in a price (or quantity) index number formula ...er they appear, then the resulting price (or quantity) formula should be the reciprocal of the ... index formula, ignoring the factor 100." Symbolically, the test requires that

$$P_{0n} = \frac{1}{P_{n0}} \text{ or } P_{0n} \times P_{n0} = 1$$

...ce the index number is designed to measure changing values of prices or production, it is ... to expect that the formula should give the same result regardless of which the two periods is ... as the base. The base period and the given period are only relative terms and hence should be ... An index number satisfying this test gives consistent results. This is a property we shall desire ... to obey.

...r the purposes of illustration, let us consider some of the important formulae and see whether ... do not satisfy this test.

The Laspeyres' price index expressed in ratio rather than percentage form (*i.e.* omitting the factor 100), is given by

$$P_{0n} = \frac{\sum p_nq_0}{\sum p_0q_0}$$

Interchanging the time subscripts, we get

$$P_{n0} = \frac{\sum p_0 q_n}{\sum p_n q_n}.$$

But $P_{0n} \times P_{n0}$, *i.e.* $\frac{\sum p_n q_0}{\sum p_0 q_0} \times \frac{\sum p_0 q_n}{\sum p_n q_n} \neq 1$.

Hence it does not satisfy the time reversal test.

(ii) Again, taking the Paasche's price index formula, *i.e.* $P_{0n} = \frac{\sum p_n q_n}{\sum p_0 q_n}$ and interchanging the time subscripts, we get

$$P_{n0} = \frac{\sum p_0 q_0}{\sum p_n q_0}$$

But again, $P_{0n} \times P_{n0} = \frac{\sum p_n q_n}{\sum p_0 q_n} \times \frac{\sum p_0 q_0}{\sum p_n q_0} \neq 1$, which shows that this formula, too, does not obey the time reversal test.

(iii) Interchanging the time subscripts in the Marshall-Edgeworth price index, *i.e.* in the formula $P_{0n} = \frac{\sum p_n (q_0 + q_n)}{\sum p_0 (q_0 + q_n)}$, we get

$$P_{n0} = \frac{\sum p_0 (q_n + q_0)}{\sum p_n (q_n + q_0)}$$

Multiplying together, we obtain

$$P_{0n} \times P_{n0} = \frac{\sum p_n (q_0 + q_n)}{\sum p_0 (q_0 + q_n)} \times \frac{\sum p_0 (q_n + q_0)}{\sum p_n (q_n + q_0)} = 1.$$

Hence the time reversal test is satisfied.

(iv) Fisher's Ideal index number is

$$P_{0n} = \sqrt{\frac{\sum p_n q_0}{\sum p_0 q_0} \times \frac{\sum p_n q_n}{\sum p_0 q_n}}$$

Interchanging the time subscripts, we have

$$P_{n0} = \sqrt{\frac{\sum p_0 q_n}{\sum p_n q_n} \times \frac{\sum p_0 q_0}{\sum p_n q_0}}$$

Thus $P_{0n} \times P_{n0} = \sqrt{\frac{\sum p_n q_0}{\sum p_0 q_0} \times \frac{\sum p_n q_n}{\sum p_0 q_n}} \times \sqrt{\frac{\sum p_0 q_n}{\sum p_n q_n} \times \frac{\sum p_0 q_0}{\sum p_n q_0}} = 1$

Hence it conforms to the time reversal test.

**5.6.2 Factor Reversal Test.** It may be stated in this way:

"If the factors $p$'s (prices) and $q$'s (quantities) occurring in a price (or quantity) index formula be interchanged (or reversed) so that a quantity (or price) index formula is obtained, then the product of the two index numbers should equal the value index number, *i.e.* $\frac{\sum p_n q_n}{\sum p_0 q_0}$". In other words, the factor reversal test requires that

(Price index) (Quantity index) = Value index,

We give the following illustrations.

(i) Interchanging the factors $p$'s and $q$'s in the formula

$$P_{0n} = \frac{\sum p_n q_0}{\sum p_0 q_0} \text{ (Laspeyres' price index), we obtain}$$

$$Q_{0n} = \frac{\sum q_n p_0}{\sum q_0 p_0} \text{ (Laspeyres' quantity index)}$$

The product of these two is $\frac{\sum p_n q_0}{\sum p_0 q_0} \times \frac{\sum q_n p_0}{\sum q_0 p_0}$, which is not equal to $\frac{\sum p_n q_n}{\sum p_0 q_0}$, the value index.

Hence the factor-reversal test is not obeyed.

(ii) Again interchanging the factors $p$'s and $q$'s, the formula

$$P_{0n} = \frac{\sum p_n q_n}{\sum p_0 q_n} \text{ : (Paasche's price index)}$$

transforms into $Q_{0n} = \frac{\sum q_n p_n}{\sum q_0 p_n}$ (Paasche's quantity index)

Thus $P_{0n} \times Q_{0n} = \frac{\sum p_n q_n}{\sum p_0 q_n} \times \frac{\sum q_n p_n}{\sum q_0 p_n} \neq \frac{\sum p_n q_n}{\sum p_0 q_0}$ (Value index).

Hence Paasche's price index does not satisfy the factor reversal test.

(iii) The Marshall-Edgeworth's price index is

$$P_{0n} = \frac{\sum p_n (q_0 + q_n)}{\sum p_0 (q_0 + q_n)}.$$

Interchanging the factors $p$'s and $q$'s, the quantity index is obtained as

$$Q_{0n} = \frac{\sum q_n (p_0 + p_n)}{\sum q_0 (p_0 + p_n)}.$$

But $$\frac{\sum p_n(q_0+q_n)}{\sum p_0(q_0+q_n)} \times \frac{\sum q_n(p_0+p_n)}{\sum q_0(p_0+p_n)} \neq \frac{\sum p_n q_n}{\sum p_0 q_0},$$

*i.e.* (price index) (quantity index) ≠ value index.

Thus the factor-reversal test is not satisfied.

(iv) By definition, Fisher's price index is given by

$$P_{0n} = \sqrt{\frac{\sum p_n q_0}{\sum p_0 q_0} \times \frac{\sum p_n q_n}{\sum p_0 q_n}}$$

Fisher's quantity index is obtained by interchanging the factors $p$'s and $q$'s as

$$Q_{0n} = \sqrt{\frac{\sum q_n p_0}{\sum q_0 p_0} \times \frac{\sum q_n p_n}{\sum q_0 p_n}}$$

Therefore $$P_{0n} \times Q_{0n} = \sqrt{\frac{\sum p_n q_0}{\sum p_0 q_0} \times \frac{\sum p_n q_n}{\sum p_0 q_n}} \times \sqrt{\frac{\sum q_n p_0}{\sum q_0 p_0} \times \frac{\sum q_n p_n}{\sum q_0 p_n}}$$

$$= \frac{\sum p_n q_n}{\sum p_0 q_0} = \text{value index.}$$

Hence, we see that Fisher's Ideal index satisfies the factor reversal test.

**Example 5.14** Show with the help of the following data that the factor-reversal and time-reversal tests are satisfied by Fisher's Ideal formula for index number construction.

| Commodity | Base Year | | Current Year | |
|---|---|---|---|---|
| | Price (in Rs.) | Quantity (units of 40 kg) | Price (in Rs.) | Quantity (units of 40 kg) |
| A | 6 | 50 | 10 | 56 |
| B | 4 | 100 | 2 | 120 |
| C | 2 | 60 | 6 | 60 |
| D | 10 | 30 | 12 | 24 |
| E | 8 | 40 | 12 | 36 |

(P.U., B.A./B.Sc.

We calculate the necessary products in the following table:

| Commodity | Price | | Quantity | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $p_0$ | $p_1$ | $q_0$ | $q_1$ | $p_0q_0$ | $p_1q_0$ | $p_0q_1$ | $p_1q_1$ |
| A | 6 | 10 | 50 | 56 | 300 | 500 | 336 | 560 |
| B | 4 | 2 | 100 | 120 | 400 | 200 | 480 | 240 |
| C | 2 | 6 | 60 | 60 | 120 | 360 | 120 | 360 |
| D | 10 | 12 | 30 | 24 | 300 | 360 | 240 | 288 |
| E | 8 | 12 | 40 | 36 | 320 | 480 | 228 | 432 |
| Total | -- | -- | -- | -- | 1440 = $\sum p_0q_0$ | 1900 = $\sum p_1q_0$ | 1464 = $\sum p_0q_1$ | 1880 = $\sum p_1q_1$ |

Now Fisher's price index for the current year (omitting the factor 100) is given by

$$P_{01} = \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \times \frac{\sum p_1 q_1}{\sum p_0 q_1}} = \sqrt{\frac{1900}{1440} \times \frac{1880}{1464}},$$

and Fisher's quantity index for the current year (omitting the factor 100) is given by

$$Q_{01} = \sqrt{\frac{\sum q_1 p_0}{\sum q_0 p_0} \times \frac{\sum q_1 p_1}{\sum q_0 p_1}} = \sqrt{\frac{1464}{1440} \times \frac{1880}{1900}},$$

$$\therefore \quad P_{01} \times Q_{01} = \sqrt{\frac{1900}{1440} \times \frac{1880}{1464}} \times \sqrt{\frac{1464}{1440} \times \frac{1880}{1900}}$$

$$= \sqrt{\frac{1880}{1440} \times \frac{1880}{1440}} = \frac{1880}{1440} = \frac{\sum p_1 q_1}{\sum p_0 q_0}$$

= Value index.

Hence we see that Fisher's *Ideal* formula for index numbers satisfies the *factor-reversal test*.

Again interchanging the time subscripts in Fisher's price index (omitting the factor 100), we get

$$P_{10} = \sqrt{\frac{\sum p_0 q_1}{\sum p_1 q_1} \times \frac{\sum p_0 q_0}{\sum p_1 q_0}}$$

$$P_{01} \times P_{10} = \sqrt{\frac{1900}{1440} \times \frac{1880}{1464}} \times \sqrt{\frac{1464}{1880} \times \frac{1440}{1900}} = 1$$

Hence Fisher's *Ideal* formula for index numbers also satisfies the *time-reversal test*.

**.6.3 Circular Test.** The *circular test* is an extension of the time reversal test, which is applicable ... years, *i.e.* the given year and the base year. But in circular test, three or more years are taken into ... This test may be stated thus:

If $P_{ab}$ is the price index for the year '*b*' with year '*a*' as base, $P_{bc}$ is the price index for the year '*c*' ... year '*b*' as base and $P_{ca}$ is the price index for the year '*a*' with year '*c*' as base, then the product of ... numbers should equal 1. In other words, the circular test requires

$$P_{ab} \times P_{bc} \times P_{ca} = 1$$

In general, the circular test is said to be satisfied, if

$$P_{01} \times P_{12} \times P_{23} \times \ldots \times P_{k-1,k} \times P_{k0} = 1.$$

where $P_{ij}$ indicates the price index (without the factor 100) for the year '*j*' with the year '*i*' as the base.

This test is not obeyed by any of the weighted index numbers unless the weights are constant. For purposes of illustration, we consider the weighted aggregate price number with fixed weights, say,

Let there be three years denoted by '*a*', '*b*', and '*c*'.

Then the weighted price index numbers are

$$P_{ab} = \frac{\sum p_b q_0}{\sum p_a q_0},\ P_{bc} = \frac{\sum p_c q_0}{\sum p_b q_0} \text{ and } P_{ca} = \frac{\sum p_a q_0}{\sum p_c q_0}.$$

Multiplying them together, we get

$$P_{ab} \times P_{bc} \times P_{ca} = \frac{\sum p_b q_0}{\sum p_a q_0} \times \frac{\sum p_c q_0}{\sum p_b q_0} \times \frac{\sum p_a q_0}{\sum p_c q_0} = 1,$$

which shows that the circular test is satisfied.

## 5.7 CONSUMER PRICE INDEX NUMBER

**5.7.1 Meaning.** A *consumer price index* (CPI) is designed to measure the changes composite price of a specified "*basket*" of goods and services during the given period as compare the base period. The so-called "basket" would comprise various commodities consumed and se received in the base period or the given period, grouped under the main headings: (i) Food and bev (ii) Clothing and footwear, (iii) Fuel and lighting (iv) Housing, (v) Services, (vi) Miscellaneou customary to exclude the durable goods and non-consumption monetary transactions such contribution to Provident Fund, Savings Certificates, etc. The quantities consumed or the expen incurred on various groups are used as weights for the average retail prices prevailing in the l concerned during the base and given periods.

Some countries still retain the name of the *cost of living* index number as "*it measures the c in the cost of living of a person or of a group of persons, having identical tastes for goods.*" A co price index is also called a *household-budget price index* or a *retail price index*.

**5.7.2 Construction of Consumer Price Index Numbers.** The following steps are involved compilation of the consumer price index numbers:

(i) **Scope.** The first step is to clearly specify the category of people and the locality whe reside as a consumer price index number relates to a particular segment of population low-salaried employees, school teachers, industrial workers, etc. residing in a particul defined area such as a city or an industrial town. As far possible, a homogeneous g person, *i.e.* persons who have identical patterns of living, is considered.

(ii) **Household Budget Inquiry and Allocation of weights.** The next step is to co household budget inquiry of the category of people concerned in order to determine the and services to be included in the construction and to derive weights to be attached This step has many practical problems as no two households have the same income and the same purchasing or consuming patterns. The inquiry or the house hold cons survey should, therefore, include questions on family size, income, number of ear quality and quantities of goods and services consumed and the money spent on the various headings such as food and beverages, clothing and footwear, fuel and l housing, and miscellaneous. The miscellaneous group includes items such as con education, medical care, recreation, gifts, newspaper, barber, laundry and other charges.

An appropriate sampling technique is to be employed to collect consumption data of the ...seholds. The factory payrolls in case of industrial workers or the ration register in other cases may be ...d as *frame*. The households, after having been stratified into different income groups, are drawn into ... sample by the method of either *area sampling or systematic sampling with random start*. The ...mption data collected are then analysed according to the nature, quality and quantities of the ...umed commodities, the proportion which expenditure on each group bears to the total expenditure of ... groups. The weights are then attached to the various consumption groups in proportion to the money-... on them so as to truly reflect their relative importance.

(iii) **Piece Data.** The third step is to collect data on consumer prices of goods and services included in the *basket*. These prices (retail prices) should be obtained both for the base period and the given period from the locality in which the people concerned reside or from which they make their purchases.

(iv) **Computation of the Index.** The last step is the computation of the consumer price index number with the help of an appropriate formula. For this purpose, one of the following two methods is employed with the same result.

**The Aggregate Expenditure Method.** Here the quantities consumed by households in the base ... are taken as the weights. The quantity $\sum p_0q_0$ represents the aggregate expenditure incurred on the ... items in the base year and the quantity $\sum p_nq_0$ that in the given year with base year quantities. ...e Laspeyres' formula, namely

$$P_{0n} = \frac{\sum p_nq_0}{\sum p_0q_0} \text{x} 100$$

...en applied. The given year quantities are not used as weights because they change from year to year ... a fresh sample budget enquiry, which involves considerable expense, labour and time, would be ...ed every year.

**Household Budget Method.** In this method, the price relatives are weighted by either the *money* ... by the households on various items or *fixed weights* derived from the sample household budget ...ry conducted in the category of people concerned. This method is called the *Household Budget* ...d, because the amounts of money spent by the households concerned are obtained from a ...hold consumption survey. This method is also known as the *Weighted Average of Relatives*. The ...la followed is

$$P_{0n} = \frac{\Sigma W\left(\dfrac{p_n}{p_0}\right)}{\Sigma W} \text{x} 100 = \frac{\Sigma W \text{x} 1}{\Sigma W}$$

... W may be $p_0q_0$ and $\mathrm{I} = \dfrac{p_n}{p_0} \text{x} 100.$

...ocedure of computation is illustrated by the following example.

**Example 5.15** The following table gives average annual prices of ten commodities during the years 1990 and 1994. Calculate the consumer price index number for 1994 on the basis of 1990.

| Commodity | Quantity consumed | Unit of Price | Price in 1990 | Price in 1994 |
|---|---|---|---|---|
| Wheat | 20 kgs | Rs. per kg | 1.25 | 1.50 |
| Rice | 10 kgs | Rs. per kg | 3.00 | 3.75 |
| Pulse | 12 kgs | Rs. per kg | 2.50 | 3.00 |
| Sugar | 4 kgs | Rs. per kg | 2.00 | 3.25 |
| Ghee | 3 kgs | Rs. per kg | 3.75 | 4.00 |
| Milk | 30 litres | Rs. per litre | 0.50 | 0.75 |
| Vegetables | 35 kgs | Rs. per kg | 0.25 | 0.40 |
| Fuel | 200 kgs | Rs. per kg | 0.50 | 0.75 |
| Cloth | 22 meters | Rs. per meter | 1.50 | 2.10 |
| House Rent | 1 unit | Rs. per unit | 30.00 | 60.00 |

The necessary calculations are shown in the following tables:

(i) Consumer price index number by Aggregate Expenditure Method.

| Commodity | Unit | Quantity consumed ($q_0$) | Price per unit 1990 $p_0$ | 1994 $p_1$ | Aggregate Exp. $p_0q_0$ | $p_1q_0$ |
|---|---|---|---|---|---|---|
| Wheat | kg | 25 kg | 1.25 | 1.50 | 31.25 | 37.50 |
| Rice | kg | 10 kg | 3.00 | 3.75 | 30.00 | 37.50 |
| Pulse | kg | 12 kg | 2.50 | 3.00 | 30.00 | 36.00 |
| Sugar | kg | 4 kg | 2.00 | 3.25 | 8.00 | 13.00 |
| Ghee | kg | 3 kg | 3.75 | 4.00 | 11.25 | 12.00 |
| Milk | litre | 30 litres | 0.50 | 0.75 | 15.00 | 22.50 |
| Vegetables | kg | 35 kg | 0.25 | 0.40 | 8.75 | 14.00 |
| Fuel | kg | 200 kg | 0.50 | 0.75 | 100.00 | 150.00 |
| Cloth | meter | 22 meters | 1.50 | 2.10 | 33.00 | 46.20 |
| House Rent | unit | 1 unit | 30.00 | 60.00 | 30.00 | 60.00 |
| Total | -- | -- | -- | -- | 297.25 | 428.70 |

Thus the consumer price index number for 1994 is

$$P_{01} = \frac{\sum p_1q_0}{\sum p_0q_0} \times 100 = \frac{428.70}{297.25} \times 100 = 144.2$$

This result indicates that the prices of consumption goods have increased by 44.2% for 1994 in comparison with 1990.

(ii) Consumer price index number by the Household Budget Method.

| Commodity | Unit | Commodity consumed $(q_0)$ | Price per unit | | Price Relatives | Weights | Weights x Price Relative = W x I |
|---|---|---|---|---|---|---|---|
| | | | 1990 $p_0$ | 1994 $p_1$ | $\frac{p_1}{p_0} \times 100 = I$ | $W = p_0 q_0$ | |
| Wheat | kg | 25 kg | 1.25 | 1.50 | 120.00 | 31.25 | 3750.00 |
| Rice | kg | 10 kg | 3.00 | 3.75 | 125.00 | 30.00 | 3750.00 |
| Pulse | kg | 12 kg | 2.50 | 3.00 | 120.00 | 30.00 | 3600.00 |
| Sugar | kg | 4 kg | 2.00 | 3.25 | 162.50 | 8.00 | 1300.00 |
| Ghee | kg | 3 kg | 3.75 | 4.00 | 106.67 | 11.25 | 1200.04 |
| Milk | litre | 30 litre | 0.50 | 0.75 | 150.00 | 15.00 | 2250.00 |
| Vegetables | kg | 35 kg | 0.25 | 0.40 | 160.00 | 8.75 | 1400.00 |
| Fuel | kg | 200 kg | 0.50 | 0.75 | 150.00 | 100.00 | 15000.00 |
| Cloth | meter | 22 meters | 1.50 | 2.10 | 140.00 | 33.00 | 4620.00 |
| House Rent | unit | 1 unit | 30.00 | 60.00 | 200.00 | 30.00 | 6000.00 |
| Total | -- | -- | -- | -- | -- | 297.25 | 42870.04 |

Hence the consumer price index number for 1994 is

$$P_{01} = \frac{\sum W \times I}{\sum W}$$

$$= \frac{42870.04}{297.25} = 144.2$$

We see that the consumer price index numbers constructed by both methods are the same. ...mes, the process of rounding off the figures may result in a small difference between the results.

**5.7.3 Shortcomings or Drawbacks of Consumer Price Index Numbers.** Some of the ...mings of the consumer price index numbers are given below:

(i) It is practically difficult to clearly demarcate one category of people from another.

(ii) As the construction of consumer price indices involves the sampling of goods and services, the sampling errors and biases may affect indices and render them to suspect. Moreover, the *frames* used for household consumption inquiry may be incomplete and outdated.

(iii) In case of certain goods, it is difficult to collect prices actually needed. For example, the prices for clothing usually relate to *cloth* and not to tailored *clothes*.

(iv) It is also difficult to eliminate the effect of changes in quality and grade of goods and services purchased by households.

(v) During the course of household budget inquiry, the price of goods and services and their demand may change; some commodities may change in quality, others may disappear and some new goods may enter the market.

(vi) The consumer price indices cannot be used for comparing the price changes in consumption goods and services in two localities or in two households in the same locality as no two households can be homogeneous, *i.e.* they can neither have precisely the same pattern of consumption nor precisely the same basket of goods and services.

It is therefore relevant to point out that a consumer price index should not be wholly relied upon as it is an imperfect measure.

## 5.8 USES OF INDEX NUMBERS

A few uses of index numbers are given below:

i) The price index numbers are used to measure changes in a particular group of prices and help us in comparing the movement in prices of one commodity with another. They are also designed to measure the changes in the purchasing power of money.

ii) Index numbers of industrial production provide a measure of change in the level of industrial production in a country.

iii) The quantity index numbers show the rise or fall in the volume of production, volume of exports and imports, etc.

iv) The import and export price indices are used to measure the changes in the *terms of trade* of a country. By the terms of trade is meant the ratio of import to export prices.

v) Index numbers are also used to forecast business conditions of a country and to discover seasonal fluctuations and business cycles.

vi) The consumer price indices indicate the movements in retail prices of consumption goods and services. These movements in prices help Government in formulating its policies and in taking appropriate economic measures. They can be used to re-adjust the wages and to adopt measures of relief by granting dearness allowance and bonus to their employees to meet the increased costs by the industrial and commercial establishments as well as mills. They are also used to deflate the gross national product and wages to arrive at the *real* values of the national product and *real* wages.

vii) Index numbers are also used to measure enrolment changes, intelligence quotients and performance of students.

## 5.9 LIMITATIONS OF INDEX NUMBERS

Some of the limitations are described below:

i) It is not practicable to price all the goods and services as well as to take into account changes in quantity or product.

ii) Since the construction of almost all index numbers is based on sampling of some items, therefore the sampling errors creep into their calculations.

iii) In price indices the choice of a *normal* period is difficult as few periods can be regarded normal for all segments of the economy.

iv) The results obtained by different methods of construction may not quite agree.

v) Comparisons of changes in variables over long periods are not reliable.

vi) All index numbers are not suitable for all purposes. The users are therefore strongly advised to essentially understand the purpose for which the index has been constructed.

## EXERCISES

### OBJECTIVE

a) Answer 'True' or 'False'. If the statement is not true then replace the underlined words with words that make the statement true:

i) Weighted aggregate index is the simplest form of an index number.

ii) When the base year values are used as weights, the weighted average of relatives price index is the same as Paasche's.

iii) There are three methods of construction of CPI numbers.

iv) Fisher's ideal index is the mean of Laspeyre's and Paasche's index numbers.

v) The most suitable average for index numbers is median.

vi) A Laspeyres price index is a CPI.

vii) If a price index increased from 150 to 200 over a certain period, the increase in prices was 50% from the beginning to the end of that period.

viii) If the current year and base year index numbers are 210 and 160 respectively, then the value of Fisher's Ideal index number is 185.

ix) Prices are appropriate weights in a weighted aggregates quantity index.

x) If price and quantity of a commodity of a year are multiplied, we get value.

### MULTIPLE CHOICE QUESTIONS

If the price of a kg of meat was Rs.40/- in 2000 and Rs.50/ in 2002, the simple price relative in 2002 is

a) 125

b) 100

c) 80

d) 50

An un-weighted aggregates price index has a limitation that

a) It is difficult to calculate.

b) It is unduly influenced by the price variations of high priced commodities.

c) It is unduly influenced by the price variations of low priced commodities.

d) None of the above.

iii) The best weights to be used in a quantity index calculated by the weighted average of relatives methods are:

a) Base period price weights.

b) Current period price weights.

c) Base period quantity weights.

d) Base period value weights.

iv) The CPI is basically

a) A fixed-weight index.

b) A Laspeyres index.

c) Both of the above.

d) None of the above.

v) The Laspeyres price index is a weighted aggregate index in which the weights are based on

a) Current quantities.

b) Base period quantities.

c) Mean of base and current period quantities.

d) None of the above.

vi) The Paasche's price index is a weighted aggregate index in which the weights are based on

a) Current quantities.

b) Base period quantities.

c) Mean of base and current period quantities.

d) None of the above.

vii) The Laspeyres price index is:

a) Upward biased.

b) Downward biased.

c) No bias.

d) None of the above.

viii) The following is a price index number series: 1995, 100; 1997, 120; 2002, 150; which of the following statement is incorrect?

a) Prices increased by 50% from 1995 to 2002.

b) Prices increased by 30% from 1997 to 2002.

c) Prices in 1995 were $33\frac{1}{3}$% lower than in 2002.

d) Prices increased by 25% from 1997 to 2002.

ix) The following is a price index series for Lahore based on 1990 = 100, 1995 = 120, 2000 = 125. Which of the following statement is correct?

a) Prices have increased by 5% from 1995 to 2000.

b) Prices in 1990 were 25% lower than in 2000.

c) Prices in 2000 were 1.2% higher than in 1995.

d) None of the above.

x) If wages of a group of workers increased from 1995 to 2000 by 10% and a relevant price index increased by 5%; Real wages have increased over this period by:

a) 4.8%.

b) 10%.

c) 6%.

d) None of the above.

xi) Which of the statement is the for Laspeyre's index number.

a) It meets time reversal test.

b) It meets factor reversal test.

c) It meets both time reversal as well as factor reversal tests.

d) None of the above.

xii) Which of the statement is true for Paasche's index?

a) It meets time reversal test.

b) It meets factor reversal test.

c) It meets both time reversal as well as factor reversal tests.

d) None of the above.

xiii) A Laspeyres price index is:

a) A cost of living index.

b) A weighted index.

c) Both of the above.

d) None of the above.

xiv) In 2000 the price for a certain type of fish was Rs.120/- per kg, and 450 tons were consumed. In 2001 the price for this type of fish was Rs.100/- per kg, and 350 tons of fish were consumed. If the simple price relative in 2000 is Rs.100/- then in 2001 simple price relative would be

a) 130

b) 100

c) 90

d) 83

xv) The index number for a base year is always

a) Zero.

b) Greater than 100.

c) Less than 100.

d) None of the above.

## SUBJECTIVE

5.1 Explain the concept of an index number. What is the procedure followed in the preparation of an index number?

5.2 Define an index number. Discuss the main steps involved in the construction of index numbers of wholesale prices. Indicate their uses and limitations.
(C.S.S. 1964; P.U., B.A./B.Sc. 1986, 93

5.3 What is an index number? Describe the important problems involved in the preparation of an index number. What considerations would you weigh while constructing a wholesale price index number, in connection with the selection of commodities and the base year?
(P.U., B.A./B.Sc.,

5.4 It has been stated that the technique of index number construction involves four major factors

(a) Choice of items (b) Base period
(c) Form of averages (d) Weighting system

Do you agree with this? If so, explain these four factors and discuss the problems to which they give rise. If you do not agree, give your views on the main problems involved in construction.
(P.U., M.A., Econ.

5.5 a) What is an index number? Distinguish between fixed base and chain base methods of constructing index numbers. What are their respective merits and demerits?
(P.U., B.A./B.Sc.

b) Describe the various methods of averaging that can be used in constructing an index number, and point out their merits and demerits. (P.U., M.A. Econ.,

5.6 a) What is a weighted index number? Describe the various methods of weighting the index numbers of prices. What are their advantages? (P.U., B.A./B.Sc.

b) Present an interpretation of the (i) Laspeyres and (ii) Paasche price index numbers in terms of the total value of commodities.

5.7 Compare the following concepts:

i) Simple index and Composite index.

ii) Fixed base index and Chain base index.

iii) Laspeyres' price index and Paasche's price index.

iv) Weighted aggregative price index and Weighted average of relatives price index.

(P.U., B.A./B.Sc.

5.8 a) Define Laspeyres, Paasche and Marshall-Edgeworth types of index numbers. Show that the Paasche type is the reciprocal of the Laspeyres type with time subscripts reversed, given that the two have the same value.

b) Define Fisher's *Ideal* index number. Describe its advantages and disadvantages.
(C.S.S., 1960; P.U., B.A./B.Sc. 1973)

5.9 Define and discuss the following index numbers:

i) Quantity Index numbers.
ii) Value Index numbers.

5.10 Explain the time and factor reversal tests.

Which of the following formulae satisfy these tests and which do not?

i) $P_{01} = \dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0}$ ii) $P_{01} = \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}$ iii) $P_{01} = \dfrac{\Sigma p_1 \sqrt{q_0 q_1}}{\Sigma p_0 \sqrt{q_0 q_1}}$

iv) $P_{01} = \dfrac{\Sigma p_1 (q_0 + q_1)}{\Sigma p_0 (q_0 + q_1)}$ v) $P_{01} = \sqrt{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}$

(P.U., B.A./B.Sc. 1994; B.Z.U. M.A., Econ., 1994)

5.11 a) Explain theoretical tests which a good index is expected to satisfy.

b) Show that the Marshall-Edgeworth index $P = \dfrac{\Sigma p_{ra}(q_{ra} + q_{rb})}{\Sigma p_{rb}(q_{ra} + q_{rb})}$ satisfies the time reversal test but not the circle test unless the weights in the three years *a*, *b*, *c*, are equal.
(P.U., B.A./B.Sc. 1977; 1978-S)

5.12 a) Prove that the simple aggregate value index numbers $\left( i.e. \dfrac{\Sigma p_n q_n}{\Sigma p_0 q_0} \right)$ satisfy the time reversal and circular tests but do not satisfy the factor reversal test.

b) Show that weighted aggregate price index numbers with fixed (quantity) weights satisfy the circular test.

5.13 a) Show that the (i) Laspeyres and (ii) Paasche index do not satisfy the time reversal or factor reversal tests.

b) Show that the Marshall-Edgeworth index satisfies the time-reversal test but not the factor-reversal test. (P.U., B.A./B.Sc., 1963; M.A. Econ. 1980)

c) Prove that Fisher's Ideal index satisfies both the time reversal and the factor reversal te but does not conform to the circular test.

5.14 a) Describe the methods for testing the consistency of index numbers. Explain Fisher's formula, giving an example. (C.S.S. 1960, P.U., B.A./B.Sc

b) How would you calculate the Consumer Price Index for factory workers in Pa
(P.U., M.A. Econ.

5.15 Explain the meaning of the consumer price index number. Describe the meth construction adopted. Explain the uses of consumer price index numbers.
(P.U., B.A./B.Sc., 1962, 63, 71, 81, 85; C.S.S., 1960; M.A. Econ.

5.16 You are required to prepare the consumer price index for industrial workers in L Describe how you will proceed. Prepare a short questionnaire for the inquiry.
(P.U., B.A./B.Sc.

5.17 a) Explain the construction of the index of retail prices.

b) Consumer price index for Rawalpindi stood at 137 in July and 140 in August. D same months, the index number at Abbottabad stood at 150 and 151. Does this mea Abbottabad is costlier than Rawalpindi? Give your reasons in detail.
(P.U., B.A./B.Sc.

5.18 a) Discuss the problems which arise in constructing consumer price index numbers.

b) Show that the Fisher's Ideal index satisfies both the time reversal and factor-revers Discuss the other properties of this index number. (P.U., B.A./B.Sc.

5.19 Discuss the nature and method of construction of an index number of wages. Exam possibilities of constructing such an index number for Pakistan.
(C.S.S. 1961, 65; P.U., M.A.

5.20 Find the price relatives for each year from the following average retail prices of whea (i) 1995 as a base (ii) 1998 as a base.

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 |
|---|---|---|---|---|---|---|
| Retail prices | Rs.14.95 | 14.94 | 15.10 | 15.65 | 16.28 | 16.53 |

5.21 The following table gives the production of cotton cloth (in million meters) for Pakist 1954 to 1963:
+chain

| Year | 1954 | 1955 | 1956 | 1957 | 1958 | 1959 | 1960 | 1961 | 1962 | 1963 |
|---|---|---|---|---|---|---|---|---|---|---|
| Production | 282 | 389 | 438 | 470 | 511 | 555 | 564 | 630 | 662 | 681 |

Find out index numbers by taking (i) 1954 as base year, (ii) average of 1958, 59, 60 a period. (P.U., M.A. Econ.

5.22 The following table gives the average wholesale prices in rupees per unit of gold, wh cotton during the years 2001–2006.

| Commodity | Average price in Rs. Per unit | | | | | |
|---|---|---|---|---|---|---|
| | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 |
| Gold | 25.3 | 30.8 | 33.4 | 35.5 | 35.3 | 36.0 |
| Wheat | 17.3 | 14.5 | 4.9 | 5.7 | 17.1 | 11.6 |
| Cotton | 7.8 | 5.4 | 6.7 | 5.6 | 7.2 | 10.2 |

Using 2001 as the base period, compute the simple aggregative price indices and the average of relatives price indices for the years 2002 to 2006.

5.23 The Prices of four commodities quoted at Multan for May 2001, April 2002 and May 2002 are given below:

| Commodity | May 2001 | April 2002 | May 2002 |
|---|---|---|---|
| Wheat | 17.50 | 18.37 | 17.58 |
| Barley | 14.58 | 14.58 | 16.50 |
| Jawar | 14.67 | 13.94 | 15.25 |
| Bajra | 17.50 | 13.75 | 13.42 |

Compute price index numbers for April 2002 and May 2002 with May 2001 as base, using (i) simple aggregative method, (ii) simple average (mean) of price relatives and (iii) geometric mean of price relatives.

5.24 Describe the chain base method used for the construction of index numbers from the following table and calculate such index numbers. Discuss its merits against the fixed base method.

| Commodities | Average Prices in Rs. per 40 kg | | | |
|---|---|---|---|---|
| | 1928 | 1929 | 1930 | 1931 |
| Rice | 7.3 | 7.7 | 5.8 | 4.1 |
| Wheat | 7.5 | 5.5 | 3.6 | 2.7 |
| Linseed | 7.0 | 8.0 | 6.5 | 4.2 |
| Gur | 6.3 | 7.3 | 6.2 | 4.2 |
| Cotton | 34.1 | 29.8 | 27.3 | 13.3 |
| Tobacco | 17.3 | 17.1 | 14.5 | 11.6 |

(P.U., B.A/B.Sc. 1976)

5.25 Construct the Chain Indices for the following data for price relatives for 1941 to 1944.

| Year | Sugar | Gur | Tea | Coffee |
|---|---|---|---|---|
| 1941 | 98 | 75 | 82 | 99 |
| 1942 | 100 | 82 | 74 | 100 |
| 1943 | 114 | 83 | 78 | 104 |
| 1944 | 109 | 84 | 89 | 95 |

(P.U., M.A. Econ., 1975)

5.26 The price relatives of three commodities are given below:

| Year | Price Relatives of Commodities | | |
|---|---|---|---|
| | A | B | C |
| 2001 | 100 | 100 | 100 |
| 2002 | 105 | 97 | 121 |
| 2003 | 110 | 94 | 125 |
| 2004 | 115 | 100 | 130 |
| 2005 | 116 | 99 | 128 |
| 2006 | 120 | 105 | 130 |

a) Compute the link relatives, *i.e.* the price relatives in each year with reference to the previous year as 100, and calculate the index for three commodities for each year.

b) Chain the above indices to a 2001 base.

5.27 The following table gives the price relatives of four commodities for the years 2002–20 inclusive, the price of each commodity in 2001 being stated as 100.

| Commodity | Price Relatives: 2001 = 100 | | | | |
|---|---|---|---|---|---|
| | 2001 | 2002 | 2003 | 2004 | 2005 |
| A | 100 | 125 | 112 | 125 | 131 |
| B | 100 | 120 | 110 | 120 | 127 |
| C | 100 | 87 | 92 | 108 | 122 |
| D | 100 | 75 | 125 | 150 | 140 |

Calculate:

i) An index number for each year, 2002–2005, using the simple A.M. of price relatives.

ii) Index number for each year, using the chain base method.

iii) Explain why, in general, the indices of (i) and the chained indices are not in clo agreement.

5.28 A firm divides its material into four main groups and tries to estimate the overall effe price changes by producing an index number weighted according to the tonnages use 2003. Calculate this index for 2007 from the following information, and comment on result.

| Group | 2003 | | 2007 | |
|---|---|---|---|---|
| | Tons | Price (£) | Tons | Price (£) |
| A | 50 | 85 | 45 | 116 |
| B | 120 | 34 | 185 | 42 |
| C | 35 | 10 | 68 | 15 |
| D | 210 | 48 | 250 | 50 |



5.29 The prices and quantities of three commodities are shown below:

| Commodity | Price (Rs.) | | Quantity | |
|---|---|---|---|---|
| | 1998 | 2005 | 1998 | 2005 |
| Rice | 3.50 | 3.15 | 71 | 80 |
| Barley | 2.00 | 1.80 | 107 | 138 |
| Maize | 2.60 | 1.75 | 62 | 57 |

a) Using 1998 as the base period and the base period quantities as weights, compute weighted-aggregative price index and the weighted average of relatives price index 2005.

b) Using 1998 as the base period and 2005 quantities as weights, compute the weig aggregative price index and the weighted average of relatives price index for 2005.

5.30 The prices and quantities of three commodities during 1997 and 2007 are given below:

| Commodity | Price | | Quantity | |
|---|---|---|---|---|
| | 1997 | 2007 | 1997 | 2007 |
| A | 12 | 10 | 501 | 600 |
| B | 38 | 50 | 100 | 194 |
| C | 40 | 40 | 56 | 76 |

Compute weighted-aggregative price index for 2007 with 1997 = 100 by (i) La methods (ii) Paasche's method.

5.31 Compute the following index numbers for the data in question 5.28 by Paasche's method:

i) Weighted aggregative price index.

ii) Weighted average of relatives price index.

5.32 Construct the following index numbers of prices for 2004 and 2005 from the given data. (i) Base year weighted, (ii) Current year weighted.

| Commodity | Prices (Rs. per 40 kg) | | | Quantities (tons) | | |
|---|---|---|---|---|---|---|
| | 2000 (base) | 2004 | 2005 | 2000 | 2004 | 2005 |
| A | 70.50 | 80.65 | 85.00 | 270 | 276 | 290 |
| B | 146.95 | 155.00 | 154.75 | 24 | 18 | 44 |
| C | 25.50 | 32.50 | 30.50 | 130 | 121 | 137 |
| D | 64.75 | 75.00 | 60.95 | 185 | 267 | 355 |

Compute the index numbers of Marshall-Edgeworth and Fisher's "Ideal" type for the following data.

| Commodity | Base Year | | Current Year | |
|---|---|---|---|---|
| | Price $p_0$ | Quantity $q_0$ | Price $p_1$ | Quantity $q_1$ |
| A | 7 | 70 | [illegible] | 49 |
| B | 5 | 27 | 7 | 28 |
| C | 10 | 35 | 9 | 29 |
| D | 9 | 50 | 4 | 42 |
| E | 3 | 16 | 10 | 25 |

(P.U. B.A./B.Sc., 1973)

Given the following, construct Fisher's Ideal index number for

i) 1964, taking 1957 as base year.

ii) 1957, taking 1964 as base year.

| Commodity | 1957 | | 1964 | |
|---|---|---|---|---|
| | Price | Quantity | Price | Quantity |
| Rice | 9.3 | 100 | 4.5 | 90 |
| Wheat | 6.4 | 11 | 3.7 | 19 |
| Jawar | 5.1 | 5 | 2.7 | 3 |

(P.U., M.A., Econ., 1968)

Using 2003 as base, construct price index numbers from the following foodgrain data.

| Commodity | Prices per 40 kg (Rs.) | | | Production (000 tons) | | |
|---|---|---|---|---|---|---|
| | 2003 | 2004 | 2005 | 2003 | 2004 | 2005 |
| Wheat | 9.35 | 8.12 | 8.78 | 3,974 | 3,862 | 3,930 |
| Rice | 11.25 | 11.73 | 12.08 | 973 | 852 | 722 |
| Gram | 7.00 | 7.68 | 8.23 | 755 | 601 | 744 |

  

5.36 Calculate (i) Laspeyres' index, (ii) Paasche's index, (iii) Fisher's Ideal index, (iv) Mar Edgeworth index, (v) Walsh index and (vi) Palgrave index for the following dairy produc

| Dairy Products | Prices | | | Quantities Produced | | |
|---|---|---|---|---|---|---|
| | 1998 (base) | 1999 | 2007 | 1998 | 1999 | 2007 |
| Milk | 3.95 | 3.89 | 4.13 | 9,675 | 9,717 | 10,436 |
| Butter | 61.50 | 62.20 | 59.70 | 118 | 116 | 116 |
| Cheese | 34.80 | 35.40 | 38.90 | 78 | 78 | 83 |

5.37 a) Obtain (i) a volume index and (ii) an index of average values for each year fro following data:

Retained Imports (£ millions)

| Year | Declared value | Value on basis of 1938 values |
|---|---|---|
| 1938 | 858 | 858 |
| 1939 | 840 | 832 |
| 1940 | 1126 | 807 |
| 1941 | 1132 | 704 |

b) The following table shows the average prices in rupees per quarter.

| Cereals | 1960 | 1961 | 1962 |
|---|---|---|---|
| Wheat | 110.70 | 122.85 | 129.00 |
| Barley | 99.50 | 124.55 | 120.50 |
| Maize | 60.25 | 74.90 | 76.25 |

Taking 1960 as base, construct a price index number for the three cereal together, fo and for 1962, using the weights 12 for wheat, 8 for barley and 5 for maize.

(B.Z.U., M.A., Eco

5.38 Compute Fisher's price index number for 1976 with 1961 as base from the following da

| Commodity | Quantity (units) | | Value (Rs.) | |
|---|---|---|---|---|
| | 1961 | 1976 | 1961 | 1976 |
| A | 100 | 150 | 600 | 1200 |
| B | 80 | 100 | 400 | 700 |
| C | 60 | 72 | 180 | 432 |
| D | 30 | 33 | 450 | 363 |

(P.U., B.A./B.Sc

5.39 Compute the quantity and price index numbers from the following data. Describe ho index number could be computed from data such as those in this question.

| Commodity | Quantity (units) | | Value (£) | |
|---|---|---|---|---|
| | 1997 ($q_0$) | 2007 ($q_1$) | 1997 | 2007 |
| 1 | 100 | 150 | 500 | 900 |
| 2 | 80 | 100 | 320 | 500 |
| 3 | 60 | 72 | 150 | 360 |
| 4 | 30 | 33 | 360 | 297 |

The prices and quantities of three commodities during 1955 and 1965 are given blow:

| Commodity | 1955 | | 1965 | |
|---|---|---|---|---|
| | $P$ | $q$ | $p$ | $q$ |
| A | 10 | 501 | 12 | 600 |
| B | 40 | 100 | 38 | 194 |
| C | 50 | 76 | 40 | 56 |

Using 1955 as the base period and base period quantities as the weights, compute the weighted aggregative price index and the weighted average of relative price index for 1965.

(P.U., B.A./B.Sc. 1988)

Compute a quantity index number for (i) 1943 on 1938 as base, using 1938 values as weights; and another for (ii) 1938 on 1943 as base, using 1943 values as weights.

| Articles | | Export of Cotton Yarns and Manufactures | | | |
|---|---|---|---|---|---|
| | | Quantity | | Values (in 000 £) | |
| | | 1938 | 1943 | 1938 | 1943 |
| Cotton Yarn | A | 10.0 | 5.4 | 397 | 817 |
| | B | 2.9 | 2.8 | 278 | 315 |
| Cotton Manufactures | A | 35.3 | 69.3 | 841 | 2,854 |
| | B | 21.6 | 83.2 | 776 | 3,319 |
| | C | 81.9 | 102.9 | 1,452 | 5,805 |
| | D | 68.8 | 90.2 | 1,028 | 6,381 |

(P.U., B.A., (Hons.), 1961)

An inquiry into the budgets of the middle class families in a city in England gave the following information.

| Expenses on | Food 35% | Rent 15% | Clothing 20% | Fuel 10% | Misc. 20% |
|---|---|---|---|---|---|
| Prices (1928) | £150 | £30 | £75 | £25 | £40 |
| Prices (1929) | £145 | £30 | £65 | £23 | £45 |

What changes in cost of living figures of 1929 as compared with that of 1928 are seen?

(P.U., B.A./B.Sc. 1974, 1981)

Compute the consumer price index number for 1940 on the basis of 1939 from the following data, using (i) Aggregative Expenditure Method and (ii) the Household Budget Method.

| Commodity | Quantity Consumed in 1939 | Unit of Price | Price in 1939 | Price in 1940 |
|---|---|---|---|---|
| Rice | 240 kg | Rs. per kg | 5.75 | 6.00 |
| Wheat | 240 kg | Rs. per kg | 5.00 | 8.00 |
| Gram | 40 kg | Rs. per kg | 6.00 | 9.00 |
| Sugar | 40 kg | Rs. per kg. | 20.00 | 15.00 |
| Arhar | 240 kg | Rs. per kg | 8.00 | 10.00 |
| Ghee | 4 kg | Rs. per kg | 2.00 | 1.50 |

(P.U., B.A./B.Sc. 1973)

5.44 Compute the consumer price index number for the following data for 2007 with 2000 as base. Use as weights (i) the quantities consumed in the base year, (ii) the value in the base year.

| Article | Quantity 2000 | Price (Rs.) in | |
|---|---|---|---|
| | | 2000 | 2007 |
| Food | 5 maunds | 18.00 | 26.50 |
| Cloth | 30 meters | 2.60 | 2.80 |
| Electricity | 75 units | 0.25 | 0.30 |
| Rent | 3 rooms | 30.00 | 27.50 |
| Miscellaneous | 34 units | 0.50 | 0.60 |

5.45 Calculate an index number of retail prices for the following selection of foodstuffs on 1 Sept. 2003 (July 2000 = 100).

| Item | Unit | Purchases (units) | Price (Rs.) | |
|---|---|---|---|---|
| | | | July 2000 | 1 Sept. 2003 |
| Flour | Kilogram | 18 | 1.90 | 2.30 |
| Meat | Kilogram | 2 | 22.00 | 28.00 |
| Bread | 200 gram | 2 | 1.00 | 1.50 |
| Tea | 450 gram | 4 | 8.25 | 10.35 |
| Sugar | Kilogram | 3 | 7.00 | 7.75 |
| Milk | Litre | 2.5 | 3.50 | 4.00 |
| Butter | 450 gram | 2 | 12.30 | 15.00 |
| Eggs | Dozen | 1.5 | 6.50 | 10.50 |
| Potatoes | Kilogram | 10 | 2.60 | 3.20 |

5.46 a) Define an index number. Discuss the main steps involved in the construction of wholesale price index numbers.

b) Given:

| Commodity | Quantity (units) | | Value | |
|---|---|---|---|---|
| | 2001 | 2006 | 2001 | 2006 |
| A | 100 | 150 | 600 | 1200 |
| B | 80 | 100 | 400 | 700 |
| C | 60 | 72 | 180 | 432 |
| D | 30 | 33 | 450 | 363 |

Compute the following:

i) Fisher's quantity index number for 2006.

ii) Simple aggregative value index for 2006. (P.U. B.A./B.Sc.

5.47 Obtain i) A simple aggregative value index

ii) An index of average values, for each year from the following data:

Retained Imports (Billions Rs.)

| Year | Declared value | Value on the basis of 2002 values |
|---|---|---|
| 2002 | 860 | 860 |
| 2003 | 950 | 832 |
| 2004 | 1300 | 807 |
| 2005 | 1450 | 704 |

(P.U. B.A./B.Sc.

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Contents

# CHAPTER 6

# PROBABILITY

# PROBABILITY

## 6.1 INTRODUCTION

The word *probability* has two basic meanings: (i) a quantitative measure of uncertainty and (ii) a measure of degree of belief in a particular statement or problem.

Probability and statistics are fundamentally interrelated. Probability is often called the vehicle of statistics. The area of inferential statistics in which we are mainly concerned with drawing inferences from experiments or situations involving an element of uncertainty, leans heavily upon probability theory. Uncertainty is also an inherent part of statistical inference as inferences are based on a sample, and a sample being a small part of the larger population, contains incomplete information. A similar type of uncertainty occurs when we toss a coin, draw a card or throw dice, etc. The uncertainty in all these cases is measured in terms of probability.

It is always clear what we mean when we make statements of the type that it is very likely to rain today or I have a fair chance of passing the annual examination or $A$ will probably win a prize, etc. In each of these statements, the natural state of uncertainty is expressed, but on the basis of past evidence, we have some degree of personal belief in the truth of each statement.

The foundations of probability were laid by two French mathematicians of the seventeenth century-Blaise Pascal (1623–1662) and Pierre De Fermat (1601–1665)–in connection with gambling problems. Later on it was developed by Jakob Bernoulli (1654–1705), Abraham De Moivre (1667–1754) and Pierre Simon Laplace (1749–1827). The modern treatment of probability theory which consists of using a few axioms and rules resulting from these axioms, was developed during the twenties and thirties of twentieth century.

Today the probability theory has a wider field of application and is used to make intelligent decisions in Economics, Management, Operations Research, Sociology, Psychology, Astronomy, Physics, Engineering and Genetics where risk and uncertainty are involved.

As the probability theory these days is best understood through the application of the modern *set theory*, therefore brief descriptions of the basic concepts, notation and operations of set theory that are relevant to probability, are given here.



## 6.2 AN ASIDE—SETS

A *set* is any *well-defined* collection or list of distinct objects, *e.g.* a group of students, the books in a library, the integers between 1 and 100, all human beings on the Earth, etc. The term *well-defined* here means that any object must be classified as either belonging or not belonging to the set under consideration, and the term *distinct* implies that each object must appear only once. The objects that are in a set are called *members* or *elements* of that set. Sets are usually denoted by capital letters such as $A$, $B$, $C$, etc., while their elements are represented by small letters such as $a$, $b$, $c$, $y$, etc. Elements are enclosed by braces to represent a set, *e.g.*

$$A = \{a, b, x, y\} \text{ or } B = \{1, 2, 3, 7\}$$

If $x$ is an element of a set, $A$, we write $x \in A$, which is read as "$x$ belongs to $A$" or $x$ is in $A$.

If $x$ does not belong to $A$, *i.e.* $x$ is not an element of $A$, we write $x \notin A$. The number of a set A, denoted as $n(A)$, is defined as the number of elements in $A$.

A set that has no elements is called an *empty or a null set* and is denoted by the symbol $\phi$. It ma be noted that {0} is not an empty set as it contains an element 0. If a set contains only one elemen called a *unit set* or a *singleton set*. It is also important to note the difference between an element " x " a a unit set {x}. The elements of a set may be sets themselves.

A set may be specified in two ways. We may either give a list of all the elements of a se "*Roster*" method), *e.g.*

A = {1, 3, 5, 7, 9, 11}; B = {a book, a city, a clock, a teacher};

or we may state a *rule* that enables us to determine whether or not a given object is a member of th (the "*rule*" method or "*set builder*" method), *e.g.*

$$A = \{ x \mid x \text{ is an odd number and } x < 12\}$$

meaning that $A$ is a set of all elements $x$ such that $x$ is an odd number and $x$ is less than 12 vertical line is read as "such that". The repetition or the order in which the elements of a set occur not change the nature of the set. The *size* of a set is given by the number of elements present in number may be finite or infinite. Thus a set is *finite* when it contains a finite number of ele otherwise it is an *infinite set*. The empty set is regarded as a finite set. Examples of finite sets are

i) A = {1, 2, 3, ... , 99, 100};

ii) $B = \{ x \mid x$ is a month of the year};

iii) $C = \{ x \mid x$ is a printing mistake in a book};

iv) $D = \{ x \mid x$ is a living citizen of Pakistan}; etc.

and the sets

i) $A = \{ x \mid x$ is an even integer};

ii) $B = \{ x \mid x$ is a real number between 0 and 1 inclusive}, *i.e.*, $B = \{ x \mid 0 \le x \le 1\}$

iii) $C = \{ x \mid x$ is a point on a line};

iv) $D = \{ x \mid x$ is a sentence in the English language}; etc. are the examples of an infinite se

A set $A$ is said to be in *one-to-one correspondence* with a set $B$ when every element of set $A$ ca made to correspond to one and only one element of set $B$ and conversely. For example, if

$$A = \{1, 2, 3, 4\} \text{ and } B = \{a, b, c, d\}$$

then the sets $A$ and $B$ are in *one-to-one* correspondence.

A set is called *countably infinite* or *denumerable* when its elements can be put into a *one-to-one correspondence* with the sequence of positive integers. A set is said to be *non-denumerable* whe elements cannot be enumerated.

**6.2.1 Subsets.** A set that consists of some elements or members of another set, is called a subs that set. For example, if $B$ is a subset of $A$, then every member of set $B$ is also a member of set $A$ an write

$$B \subset A \text{ or equivalently } A \supset B$$

which is read as '*B is a subset of A*' *or is contained in A*, *or A contains B*. For example, if

$$A = \{1, 2, 3, 4, 5, 10\} \text{ and } B = \{1, 3, 5\}$$

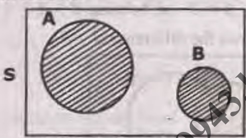
then $B \subset A$, *i.e.* $B$ is contained in $A$.

It should be noted that a set $A$ is always regarded a subset of itself and an empty set $\phi$ is *considered to be* (or accepted as) a subset of every set. Two sets $A$ and $B$ are *equal or identical*, if and only if they contain exactly the same elements. That is $A = B$ if and only if $A \subset B$ and $B \subset A$. If a set $B$ contains *some but not all* of the elements of another set $A$ while $A$ contains each element of $B$, *i.e.* if

$$B \subset A \text{ and } B \neq A$$

then the set $B$ is defined to be a *proper subset* of $A$. The *large* or the *original set* of which all the sets we talk about, are subsets is called the *universal set* or the *space* and is generally denoted by $S$ or $\Omega$. The universal set thus contains all possible elements under consideration. It is also regarded a subset of itself. A set $S$ with $n$ element will produce $2^n$ subsets, including $S$ and $\phi$.

**6.2.2 Venn Diagram.** A diagram that is understood to represent sets by circular regions, parts of circular regions or their complements with respect to a rectangle representing the space $S$ is called a *Venn diagram*, named after the English logician John Venn (1834–1923). The Venn diagrams are used to represent sets and subsets in a pictorial way and to verify the relationship among sets and subsets. An example of a Venn diagram follows:



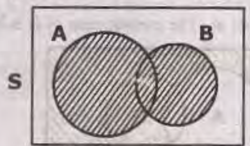

A Simple Venn Diagram

**6.2.3 Operations on Sets.** Let the sets $A$ and $B$ be the subsets of some universal set $S$. Then these sets may be combined and operated on in various ways to form new sets which are also subsets of $S$. The basic operations are *union, intersection, difference* and *complementation*.

(i) The *union or sum* of two sets $A$ and $B$, denoted by $A \cup B$, and read as "$A$ union $B$ or $A$ cup $B$", means the set of all elements that belong to *at least* one of the sets $A$ and $B$, that is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

By means of a Venn diagram, $A \cup B$ is shown by the shaded area as below:

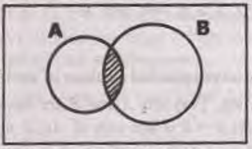



$A \cup B$ is shaded

(ii) The *intersection* of two sets $A$ and $B$, denoted by $A \cap B$ or by $AB$, and read as "$A$ intersection $B$" or "$A$ cap $B$", means the sets of all elements that belong to both $A$ and $B$; that is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Diagrammatically $A \cap B$ is shown by the shaded area as below:




$A \cap B$ is shaded

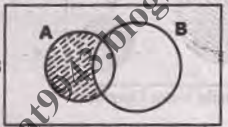
The operations of union and intersection have been defined for two sets only. They may conveniently be extended to any finite number of sets.

Two sets $A$ and $B$ are defined to be *disjoint* or *mutually exclusive* or *non-overlapping* when they have no elements in common, *i.e.* when their intersection is an empty set or $A \cap B = \phi$. On the other hand, two sets $A$ and $B$ are said to be *conjoint* when they have at least one element in common.

iii) The *difference* of two sets $A$ and $B$, denoted by $A - B$ or by $A - (A \cap B)$, is the set of all elements of $A$ which do not belong to $B$. Symbolically,

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

It is to be pointed out that in general $A - B \neq B - A$, because $B - A = \{x \mid x \in B \text{ and } x \notin A\}$. The shaded area of the following Venn diagram shows the difference $A - B$.

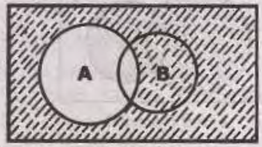



Difference A – B is shaded

It is to be noted that $A - B$ and $B$ are disjoint sets. If $A$ and $B$ are disjoint, then the difference $A - B$ coincides with the set $A$.

iv) *Complementation*: The particular difference $S - A$, that is, the set of all those elements of $S$ which do not belong to $A$, is called the *Complement* of $A$ and is denoted by $\overline{A}$ or by $A^c$.
In symbols $\overline{A} = \{x \mid x \in S \text{ and } x \notin A\}$

The complement of $S$ is the empty set $\phi$. The complement of $A$ is shown by the shaded portion of the following Venn diagram.




$\overline{A}$ is shaded

It should be noted that $A - B$ and $A \cap \overline{B}$, where $\overline{B}$ is the complement of set $B$, are the same

**6.2.4 The Algebra of Sets.** The algebra of sets provides us with laws which can be used to solve many problems in probability calculations. Let $A$, $B$ and $C$ be any subsets of the universal set $S$. Then we have

i) *Commutative laws*

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

ii) *Associative laws*

$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

iii) *Distributive laws*

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv) *Idempotent laws*

$A \cup A = A$ and $A \cap A = A$

v) *Identity laws*

$A \cup S = S, A \cap S = A, A \cup \phi = A,$ and $A \cap \phi = \phi$.

vi) *Complementation laws*

$A \cup \overline{A} = S,\ A \cap \overline{A} = \phi, (\overline{\overline{A}}) = A,$ and $\overline{S} = \phi, \overline{\phi} = S$.

vii) *De Morgan's laws*

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$, and $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

These laws can be visualized by means of the Venn diagrams.

**6.2.5 Partition of Sets.** A *partition* of a set $S$ is a sub-division of the set into non-empty subsets that are disjoint and exhaustive, *i.e.* their union is the set S itself. This implies that

i) $A_i \cap A_j = \phi$, where $i \neq j$;

ii) $A_1 \cup A_2 \cup ... \cup A_n = S$.

The subsets in a partition are called *cells*. Let us consider a set $S = \{a, b, c, d, e\}$. Then $\{a, b\}$ and $\{c, d, e\}$ is a partition of $S$ as each element of $S$ belongs to exactly one cell.

**6.2.6 Class of Sets.** A set of sets is called a *class*, *e.g.* in a set of lines, each line is a set of points. The class of all subset of a set $A$ is called the *power set* of $A$ and is denoted by $\mathcal{P}(A)$. For example, if $A = \{H, T\}$, then $\mathcal{P}(A) = \{\phi, \{H\}, \{T\}, \{H, T\}\}$.

**6.2.7 Cartesian Product Sets.** The *Cartesian product* of sets $A$ and $B$, denoted by $A \times B$, (read as "A cross B"), is a set that contains all *ordered* pairs $(x, y)$, where $x$ belongs to $A$ and $y$ belongs to $B$. Symbolically, we write

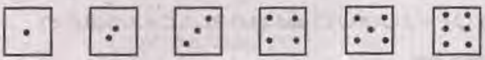
$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

This set is also called the *Cartesian set* of $A$ and $B$, named after the French mathematician Rene' Descartes (1596–1650). The product of a set $A$ by itself is denoted by $A^2$. This concept of product may be extended to any finite number of sets.

Let $A = \{H, T\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Then the *Cartesian product* set is the collection of the following twelve (2 x 6) ordered pairs:

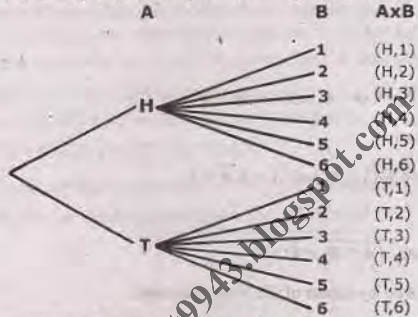
$$A \times B = \{(H, 1); (H, 2); (H, 3); (H, 4); (H; 5); (H, 6); (T, 1); (T, 2); (T, 3); (T, 4); (T, 5); (T, 6)\}$$

Clearly, these twelve elements together make up the universal set $S$ when a coin and a die are tossed together. A die (plural, dice) is a cube of wood or ivory whose six faces are marked with dots as shown below:



It is relevant to note that, in general, $A \times B \neq B \times A$.

The product $A \times B$ may conveniently be found by means of the so-called *tree diagram* shown below:






The "tree" is constructed from the left to the right. A "tree diagram" is a useful device for enumerating all the possible outcomes of two or more sequential events; the possible outcomes being represented by the individual paths or branches of the tree. This diagram is also used when we apply the multiplication rules to compute probabilities.

**6.2.8 Relation and Function.** A *relation* from a set $A$ to a set $B$ is a subset of the Cartesian product of $A \times B$. Such a relation is usually called a *binary relation*. That is, a relation is an association between two or more objects. The set of the first elements of a binary relation is called the *domain* of the relation, while the set of the second elements of the relation is called the *range*. For example, if a relation is $F = \{(1,4), (2,7), (3,12)\}$, then its domain is $D = \{1, 2, 3\}$ and its range is $R = \{4, 7, 12\}$.

A *function* is a rule that assigns values in some manner from a set $A$ to a set $B$ such that for each element $x$ of $A$, there is a unique element $y$ of $B$. Such an assignment is usually written as $f : A \longrightarrow B$ ... is a function from $A$ to $B$. In other words, a *function* is a special kind of binary relation that associates each element of the domain to a unique element of the range. It is to be emphasized that a function is a binary relation in which no first element is repeated. The value of the function f at $x \in A$ is denoted $y = f(x) \in B$.

The variable $x$ that represents the elements of the domain, is called the *independent* variable and the variable $y$ $(= f(x))$ representing elements of the range is referred to as the *dependent* variable. Such functions are also called *single-valued functions*. A function, whose range consists of numbers, is called a *numerical* function. A function whose domain and range consist of sets of real numbers, is said to be

-valued function of a real variable. A function $f(x)$ is defined to be an *even* function, if for every $x$ in a certain range $f(-x) = f(x)$, *e.g.* $f(x) = x^{2n}$. A function $f(x)$ having the property that $f(-x) = -f(x)$, is said to be an *odd* function, *e.g.*, $f(x) = x^{2n+1}$.

## 3 RANDOM EXPERIMENT

The term *experiment* means a planned activity or process whose results yield a set of data. A single performance of an experiment is called a *trial*. The result obtained from an experiment or a trial is called an *outcome*.

An experiment which produces different results even though it is repeated a large number of times under essentially similar conditions, is called a *random experiment*. The tossing of a fair coin, the throwing of a balanced die, drawing of a card from a well-shuffled deck of 52 playing cards, selecting a sample, etc. are examples of random experiments. A random experiment has three properties:

i) The experiment can be repeated, practically or theoretically, any number of times,

ii) The experiment always has two or more possible outcomes. An experiment that has one possible outcome, is not a random experiment.

iii) The outcome of each repetition is unpredictable, *i.e.* it has some degree of uncertainty.

It is to be remembered that an ordinary deck of playing cards contains 52 cards arranged in 4 suits of 13 each. The four suits are called *diamonds*, *hearts*, *clubs* and *spades*; the first two are red and the last two are black. The face values called *denominations*, of the 13 cards in each suit are ace, 2, 3, ... , 10, jack, queen and king. The term *honour card* refers to the denominations ace, 10, jack, queen and king. The *face cards* are king, queen and jack. These cards are used for various games such as whist, bridge, poker, etc.

3.1 **Sample Space.** A set consisting of all possible outcomes that can result from a random experiment (real or conceptual), is defined to be a *sample space* for the experiment and is denoted by the letter $S$. Each possible outcome is a member of the sample space and is called a *sample point* in that space. For instance, the experiment of tossing a coin results in either of the two possible outcomes: a head ($H$) or a tail ($T$); landing on its edge or rolling away is not considered. The sample space for this experiment may be expressed in set notation as $S = \{H, T\}$. The sample space for tossing two coins once (or tossing a coin twice) will contain four possible outcomes denoted by $S = \{HH, HT, TH, TT\}$. Clearly, $S$ is the Cartesian product $A \times A$, where $A = \{H, T\}$. Similarly, the sample space $S$ for the random experiment of throwing two six-sided dice can be described by the Cartesian produce $A \times A$, where $A = \{1, 2, 3, 4, 5, 6\}$. In other words,

$$S = A \times A = \{(x, y) \mid x \in A \text{ and } y \in A\}$$

where $x$ denotes the number of dots on the upper face of the first die and $y$, the number of dots on the upper face of the second die; and $S$ contains 36 outcomes or sample points, which may also be conveniently represented in the following manner:

$$\begin{aligned}
S = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6);\\
&(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6);\\
&(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6);\\
&(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6);\\
&(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6);\\
&(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}
\end{aligned}$$

This sample space may more briefly be expressed as

$$S = \{(i, j) \mid i = 1, 2, 3, 4, 5, 6; \quad j = 1, 2, 3, 4, 5, 6\}$$

A sample space that contains a finite number of sample points is said to be a *finite sample space*. It is defined to be a *discrete sample space* if the sample points can be placed in a one-to-one correspondence with the positive integers or if it is a finite. If it satisfies neither of these criteria, it is called *continuous*. It should be noted that our sample space $S$ will be finite unless otherwise stated.

**6.3.2 Events.** An event is an individual outcome or any number of outcomes (sample points) of a random experiment or a trial. In set terminology, any subset of a sample space $S$ of the experiment is called an *event*. An event that contains exactly one sample point, is defined a *simple event*. A *compound event* contains more than one sample point and is produced by the union of simple events. For instance, the occurrence of a 6 when a die is thrown, is a simple event, while the occurrence of a sum of 10 with a pair of dice, is a compound event, as it can be decomposed into three simple events (4, 6), (5, 5) and (6, 4).

An event $A$ is said to occur if and only if the outcome of the experiment corresponds to some element of $A$. The event "not-$A$" is denoted by $\overline{A}$ or $A^c$ and is called the *negation* (or *complementary event*) of $A$. For example, the *complement* of "heads" is "tails" for tossing of one coin; the complement of "at least one head" on 4 tosses of a coin is "no heads". A sample space consisting of $n$ sample points can produce $2^n$ different subsets or simple and compound events as each sample point can either be included or excluded in forming a subset.

To illustrate, let us consider a sample space $S$ containing three sample point, *i.e.* $S = \{a, b, c\}$.

Then the eight possible subsets are

$$\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$$

Each of these subsets is an *event*. The subset $\{a, b, c\}$ is the sample space itself and is also an event. It always occurs and is known as the *certain* or *sure* event. The empty set $\phi$ is also an event, sometimes known as *impossible* event, because it can never occur.

This class of $2^3 = 8$ subsets (events) can be thought of as a *field* which is denoted by $\mathcal{F}$. These events have the following characteristics:

i) The *union* of any number of events will result in a set that belongs to $\mathcal{F}$.

ii) The *intersection* of any number of events will result in a set that belongs to $\mathcal{F}$.

iii) The *difference* of any two events belongs to $\mathcal{F}$.

iv) The *complement* of any event belongs to $\mathcal{F}$.

**Mutually Exclusive Events.** Two events $A$ and $B$ of a single experiment are said to be *mutually exclusive* or *disjoint* if and only if they cannot both occur at the same time. That is they have no points in common. For instance, when we toss a coin, we get either a head or a tail, but not both, the two events head and tail are therefore mutually exclusive; when a die is rolled, the outcomes are mutually exclusive as we get one and only one of six possible outcomes 1, 2, 3, 4, 5 or 6. Similarly, a student either qualifies or fails, a single birth must be either a boy or a girl, it cannot be both, etc. Three or more events originating from the same experiment are mutually exclusive if pairwise they are mutually exclusive. If the two events can occur at the same time, they are *not mutually exclusive*, e.g., if we draw a card from a

...ary deck of 52 playing cards, it can be both a king and a diamond. Therefore kings and diamonds are ... mutually exclusive. Similarly, inflation and recession are not mutually exclusive events.

...haustive Events. Events are said to be *collectively exhaustive*, when the union of mutually exclusive ... is the entire sample space $S$. Thus, in our coin-tossing experiment, head and tail are collectively ...stive set of events. A group of mutually exclusive and exhaustive events is called a *partition* of the ...e space. For instance, events $A$ and $A^c$ form a partition as they are mutually exclusive and their ... is the entire sample space.

...lly **Likely Events.** Two events $A$ and $B$ are said to be *equally likely*, when one event is as likely to ... as the other. In other words, each event should occur in equal number in repeated trials. For ...ple, when a fair coin is tossed, the head is as likely to appear as the tail, and the proportion of times ... side is expected to appear is ½.

**6.3.3 Events and Symbolic Representations.** For convenience, the verbal statements of some ... and their corresponding symbolic representations in sets are listed below:

| *Verbal statement* | *Set Notation* |
|---|---|
| Event $A$ | $A \subset S$ |
| Event $A$ is impossible | $A = \phi$ |
| Event $A$ is sure (certain) | $A = S$ |
| Not-$A$ (Event $A$ does not occur) | $\overline{A} = S - A$ |
| Event $A$ or event $\overline{A}$ | $A \cup \overline{A} = S$ |
| Event $A$ or event $B$ | $A \cup B$ |
| Event $A$ and event $B$ | $A \cap B$ |
| Events $A$ and $B$ are mutually exclusive | $A \cap B = \phi$ |
| Events $A$ and $B$ are exhaustive | $A \cup B = S$ |
| Event $B$ occurs when $A$ occurs | $A \subseteq B$ |
| Event $A$ occurs but $B$ does not occur | $A \cap \overline{B}$ |
| Event $B$ occurs given that $A$ has occurred | $B \mid A$, |



...4 **Counting Sample Points.** When the number of sample points in a sample space $S$ is *very* ... becomes very inconvenient and difficult to list them all and to count the number of points in the ... space $S$ and in the subsets of $S$. We then need some methods or rules which help us to count the ... of all sample points without actually listing them. A few of the basic rules frequently used in ... are briefly described here.

... **Rule of Multiplication.** If a compound experiment consists of two experiments such that the ... experiment has exactly $m$ distinct outcomes and, if corresponding to each outcome of the first ... there can be $n$ distinct outcomes of the second experiment, then the compound experiment has ... outcomes.

...ample, the compound experiment of tossing a coin *and* throwing a die together consists of ... experiments: the coin-tossing with two distinct outcomes ($H$, $T$), and the die-throwing with six ... outcomes (1, 2, 3, 4, 5, 6). The total number of possible distinct outcomes of the compound

experiment is therefore $2 \times 6 = 12$, as each of the two outcomes of the coin-tossing experiment can ... with each of the six outcomes of die-throwing experiment (see Cartesian product on page 178) ... diagram (page 178) provides a good illustration of this rule.

The rule of multiplication can be readily extended to compound experiments consisting of ... number of experiments performed in a given sequence.

(b) **Rule of Permutation.** A *permutation* is any *ordered* subset from a set of $n$ distinct ... The number of permutations of $r$ objects, selected in a definite order from $n$ distinct objects is denoted ... the symbol $^nP_r$.

To derive the computational formula for $^nP_r (r < n)$, we proceed as below:

The first object may be chosen in $n$ ways, and corresponding to each way of first selection ... second object may be chosen in $(n-1)$ ways. Similarly, once selections have been made for both first ... second objects, there are $(n-2)$ objects left and the third object may be selected in $(n-2)$ ways ... on. The *rth* object may be chosen in $(n-r+1)$ ways. Thus the first $r$ objects may be chosen in ... $(n-2) \ldots (n-r+1)$ ways.

Hence $$^nP_r = n(n-1)(n-2) \ldots (n-r+1)$$

In particular, when $r = n$, $^nP_n = n(n-1)(n-2) \ldots 3 \times 2 \times 1$

$$= n! \text{ (read } n \text{ factorial)}$$

It is relevant to note that $1! = 1$ and that we define $0! = 1$.

The expression for $^nP_r$ on multiplication by $\frac{(n-r)!}{(n-r)!}$, may be written as $^nP_r = \frac{n!}{(n-r)!}$.

The number of permutations of $n$ objects, selected all at a time, when $n$ objects consist of $n_1$ ... kind, $n_2$ of a second kind, ..., $n_k$ of a $k$th kind, $(\sum n_i = n)$ is $P = \frac{n!}{n_1!\, n_2! \ldots n_k!}$.

(c) **Rule of Combination.** A *combination* is any subset of $r$ objects, selected without ... their order, from a set of $n$ distinct objects. The total number of such combinations is denoted ... symbol $^nC_r$ or $\binom{n}{r}$, (read "$n$ above $r$"), where $r \le n$.

The computational formula for $^nC_r$ is derived as below:

Let there be $\binom{n}{r}$ possible combinations of $r$ objects, selected from $n$ distinct objects. ... combination of $r$ objects, if order is not disregarded, can be arranged in $r!$ different orders ... makes $r!$ different permutations. The number of permutations or ordered subsets that correspond ...

combinations, each having $r$ objects, is therefore $r!\binom{n}{r}$. But the total number of permutations of $r$ objects, selected from $n$ objects is $^nP_r$. Thus we conclude that

$$r!\binom{n}{r} = {}^nP_r$$

or

$$\binom{n}{r} = \frac{^nP_r}{r!} = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...3\times2\times1)}$$

$$= \frac{n!}{r!(n-r)!},$$

The quantity $\binom{n}{r}$ or $^nC_r$ is also called a *binomial co-efficient* because of its appearance in the binomial expansion of $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$. The binomial co-efficient has two important properties.

(i) $\binom{n}{r} = \binom{n}{n-r}$, and (ii) $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

Large factorials may be conveniently evaluated by using an approximation known as Stirling's formula:

$$n! \cong n^{n+1/2}e^{-n}\sqrt{2\pi},$$

where $\pi = 3.1416$ and $e = 2.7183$.

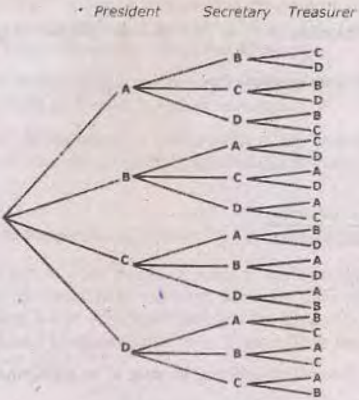


Example 6.1 A club consists of four members. How many sample points are in the sample space when three officers: president, secretary and treasurer, are to be chosen?

It is evident that the *order* in which 3 officers are to be chosen, is of significance. Thus there are 4 choices for the first office, 3 choices for the second office and 2 choices for the third office. Hence the number of sample points is 4 x 3 x 2 = 24. In other words, the number of permutations is

$$^4P_3 = \frac{4!}{(4-3)!} = 4\times3\times2 = 24.$$

Let the four members be A, B, C and D. Then a *tree* diagram which provides an organized way of listing the possible arrangements, for this example, is given on the next page:




Sample Space

ABC
ABD
ACB
ACD
ADB
ADC
BAC
BAD
BCA
BCD
BDA
BDC
CAB
CAD
CBA
CBD
CDA
CDB
DAB
DAC
DBA
DBC
DCA
DCB

**Example 6.2** A three-person committee is to be formed from a list of four persons. How many sample points are associated with the experiment?

Since the order in which the three persons of the committee are chosen, is unimportant, it is therefore an example of a problem involving combinations. Thus the desired number of combinations

$$\binom{n}{r} = \binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$$

In other words, the sample space associated with the experiment contains 4 sample points.

These two examples serve to illustrate the difference between a permutation and a combination.



**Example 6.3** How many sample points are in the sample space when a person draws a hand of 5 cards from a well-shuffled ordinary deck of 52 cards?

The total number of sample points is given by

$$\binom{n}{r} = \binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2{,}598{,}960$$

## 6.4 DEFINITIONS OF PROBABILITY

Probability can be discussed from two points of view: the *objective* and the *subjective*. The objective probability can be classified into the following categories, each of which is briefly described as follows:

**(a) The Classical or *A Priori* Definition of Probability** is given as follows:

If a random experiment can produce $n$ mutually exclusive and equally likely outcomes and if $m$ of these outcomes are considered favourable to the occurrence of a certain event $A$, then the probability

e event $A$, denoted by $P(A)$, is defined as the ratio $\frac{m}{n}$. Symbolically, we write

$$P(A) = \frac{m}{n} = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

This definition was formulated by the French mathematician P.S. Laplace (1749–1827) and can be conveniently used in experiments where the total number of possible outcomes and the number of comes favourable to an event can be determined.

The classical definition has the following shortcomings:

i) This definition is said to involve circular reasoning as the term *equally likely* really means *equally probable*. Thus probability is defined by introducing concepts that presume a prior knowledge of the meaning of probability.

ii) This definition is not applicable when the assumption of equally likely does not hold.

iii) This definition becomes vague when the number of possible outcomes may be infinite.

**The Relative Frequency or *A Posteriori* Definition of Probability.** If a random experiment is ed a large number of times, say $n$, under identical conditions and if an event $A$ is observed to occur es, then the probability of the event $A$ is defined as the limit of the relative frequency $\frac{m}{n}$ as $n$ tends inity. Symbolically, we write

$$P(A) = \lim_{n\to\infty} \frac{m}{n}.$$

This definition assumes that as $n$ increases indefinitely, the ratio $\frac{m}{n}$ tends to become stable at the cal value $P(A)$. To investigate this kind of the long-run stability, several coin-tossing experiments been performed. The results of a few well known experiments are shown below:

| Experimenter | Number of times coin tossed ($n$) | Number of heads obtained ($m$) | Ratio $\frac{m}{n}$ |
|---|---|---|---|
| Buffon | 4,040 | 2,048 | 0.5069 |
| K. Pearson | 12,000 | 6,019 | 0.5016 |
| K. Pearson | 24,000 | 12,012 | 0.5005 |

is quite obvious that the value of the ratio $\frac{m}{n}$ fluctuates about the number 0.5 and becomes to 0.5 as the number of throws increases. This sort of a long-run frequency property provides a basis of the theory of probability.

is definition is also called the *statistical* or *empirical* definition of probability as it is based on onal data. It is more useful for practical problems.

This definition too has certain limitations as the conditions under which an experim performed, may change from trial to trial and it is not possible, in practice, to repeat the experim infinite number of times and hence the ratio $\frac{m}{n}$ may not be unique.

(c) **The Axiomatic Definition of Probability.** This definition, introduced in 1933 by the R mathematician Andrei N. Kolmogorov (1903–1987), is based on a set of axioms, where an axiom statement that is assumed to be true. Let $S$ be a sample space with the sample points $E_1, E_2, \ldots, E_n$. To each sample point, we assign a real number, denoted by the symbol $P(E_i)$, and called the probab $E_i$, that must satisfy the following basic axioms:

**Axiom** (i). For any event $E_i$, $0 \le P(E_i) \le 1$.

**Axiom** (ii). $P(S) = 1$ for the sure event $S$.

**Axiom** (iii). If $A$ and $B$ are mutually exclusive events (subsets), then $P(A \cup B) = P(A) + P(B)$.

It is to be emphasized that the axiomatic theory of probability assumes that *some* prob defined as a non-negative real number is to be attached to each sample point $E_i$ such that the sum such numbers must equal one. The assignment of probabilities may be based on past evidence or on other underlying conditions.

**Probability of an event.** If an event $A$ is defined in a sample space $S$, then its probability equal to the sum of the probabilities of all sample points that are included in $A$, *i.e.* $P(A) = \sum P(E_i)$. all the $n$ possible outcomes of a random experiment are equally likely to occur, then each out assigned the same probability $\frac{1}{n}$, *e.g.* in throwing a fair die once, $P(1) = \frac{1}{6}, P(2) = \frac{1}{6}, \ldots, P(6) = \frac{1}{6}$

It follows from axiom (iii) that for any event $A$ containing $m$ equally likely outcomes (sample po have

$$P(A) = \frac{m}{n} = \frac{\text{Number of sample points in } A}{\text{Number of sample points in } S} = \frac{n(A)}{n(S)}.$$

It is interesting to note that in the classical definition of probability,

i) $P(A) = \frac{m}{n} \ge 0$;

ii) $P(S) = \frac{n}{n} = 1$;

iii) If $A$ and $B$ are mutually exclusive events, and $m_1$ and $m_2$ outcomes are favourable to $A$ an then

$$P(A \cup B) = \frac{m_1 + m_2}{n} = P(A) + P(B).$$

Thus the classical definition of probability also satisfies Kolmogorov's axioms. It is reason conclude that the probability is always a number between zero and one (inclusive).

**6.4.1 Subjective or Personalistic Probability.** As its name suggests, the *subjective* or personalistic probability is a measure of the strength of a person's belief regarding the occurrence of an event $A$. Probability in this sense is purely subjective and is based on whatever evidence is available to the individual. This definition being flexible, may be applied to those real-world situations where neither the equally likely nor a relative frequency approach is possible. The subjective probability has a disadvantage that two or more persons faced with the same evidence may arrive at different probabilities.

**Example 6.4** If a card is drawn from an ordinary deck of 52 playing cards, find the probability that (i) the card is a red card, (ii) the card is a diamond, (iii) the card is a 10.

The total number of possible outcomes is 52, and we assume that all possible outcomes are equally likely.

(i) Let $A$ represent the event that the card drawn is a red card. Then the number of outcomes favourable to the event $A$ is 26 since there are 26 red cards.

Hence $P(A) = \frac{m}{n} = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$

$$= \frac{26}{52} = \frac{1}{2}$$

(ii) Let $B$ denote the event that the card drawn is a diamond. Then the number of outcomes favourable to the event $B$ is 13 since there are 13 diamonds.

Hence $P(B) = \frac{13}{52} = \frac{1}{4}$.

(iii) Let $C$ denote the event that the card drawn is a 10. Then the number of outcomes favourable to $C$ is 4 as there are four 10's.

Thus $P(C) = \frac{4}{52} = \frac{1}{13}$.



**Example 6.5** A fair coin is tossed three times. What is the probability that *at least* one head appears?

The sample space for this experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

whence $n(S) = 8$.

As the coin is fair, we assume that each of these outcomes is equally likely to appear. Therefore we assign equal probability of $\frac{1}{8}$ to each outcome.

Let $A$ denote the event that *at least* one head appears. Then

$$A = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$$

Therefore $n(A) = 7$.

Hence $P(A) = \frac{n(A)}{n(S)} = \frac{7}{8}$.

**Example 6.6** If two fair dice are thrown, what is the probability of getting (i) a double six? sum of 8 or more dots?

The sample space S is represented by the following 36 outcomes:

$$
\begin{aligned}
S = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\\
&(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)\\
&(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\\
&(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\\
&(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\\
&(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}
\end{aligned}
$$

As the dice are fair, therefore each of these 36 outcomes is equally likely and a probability of is attached with each outcome.

i) Let $A$ represent the event that a double six occurs.

Then $A = \{(6, 6)\}$ and thus $P(A) = \dfrac{1}{36}$.

ii) Let $B$ denote the event that a sum of 8 or more dots occurs.

Then $B = \{(6, 2), (5, 3), (4, 4), (3, 5), (2, 6), (6, 3), (5, 4), (4, 5), (3, 6), (6, 4), (5, 5), (4, 6), (6, 5), (5, 6), (6, 6)\}$,
*i.e.* $n(B) = 15$.

Hence $P(B) = \dfrac{15}{36} = \dfrac{5}{12}$.



**Example 6.7** Six white balls and four black balls, which are indistinguishable apart from are placed in a bag. If six balls are taken from the bag, find the probability of their being three three black.

The total number of possible equally likely outcomes in $S$ is

$$\binom{10}{6} = \frac{10!}{6!(10-6)!} = 210$$

Let $A$ represent the event that three white and three black balls are taken. Then the num outcomes that correspond to the event $A$ is $\binom{6}{3} \times \binom{4}{3} = 80$.

Therefore $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{80}{210} = \dfrac{8}{21}$.

**Example 6.8** An employer wishes to hire three people from a group of 15 applicants, 8 me women, all of whom are equally qualified to fill the position. If he selects the three at random, wh probability that (i) all three will be men, (ii) *at least* one will be a woman? (P.U., M.A. Econ

The sample space $S$ contains $\binom{15}{3} = 455$ sample points, the number of ways in which 3 people can be selected from 15 applicants.

i) Let $A$ represent the event that the three selected will be men. Then $A$ contains $\binom{8}{3} = 56$ sample points, the number of ways in which 3 men can be selected from 8 men.

Therefore $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{56}{455} = \dfrac{8}{65}$

ii) *At least* one woman means one, two or three women. Let $B$ denote the event that *at least* one woman is selected.

Then $n(B) = \binom{7}{1}\binom{8}{2} + \binom{7}{2}\binom{8}{1} + \binom{7}{3}$

$= 196 + 168 + 35 = 399$ sample points.

Hence $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{399}{455} = \dfrac{57}{65} = 0.877.$

**Example 6.9** Four items are taken at random from a box of 12 items and inspected. The box is rejected if more than 1 item is found to be faulty. If there are 3 faulty items in the box, find the probability that the box is accepted.

The sample space $S$ contains $\binom{12}{4} = 495$ sample points.

The box contains 3 faulty and 9 good items. The box is accepted if there is (i) no faulty item, or (ii) one faulty item in the sample of 4 items selected.

Let $A$ denote the event the number of faulty items chosen is 0 or 1. Then

$$n(A) = \binom{3}{0}\binom{9}{4} + \binom{3}{1}\binom{9}{3}$$

$= 126 + 252 = 378$ sample points.

$$P(A) = \frac{378}{495} = 0.76$$

Hence the probability that the box is accepted is 0.76.

## LAWS OF PROBABILITY

Following are some of the basic rules for the calculations of probability. These rules have important applications.

**Theorem 6.1** If $\phi$ is the impossible event, then $P(\phi) = 0$.

**Proof.** The sure event $S$ and the impossible event $\phi$ are mutually exclusive and their union is

$$S \cup \phi = S$$

Then $\quad P(S) = P(S \cup \phi) = P(S) + P(\phi)$

Subtracting $P(S)$ from both sides, we get

$$P(\phi) = 0$$

Thus the probability of the impossible event is zero. It is to be kept in mind that the conve this rule is not generally true. Moreover, a theorem is a statement derived either from axio previously proved theorems.

**Theorem 6.2 Law of Complementation.** If $\overline{A}$ is the complement of an event $A$ relative sample space $S$, then

$$P(\overline{A}) = 1 - P(A).$$

**Proof.** Since the event $A$ and $\overline{A}$ are mutually exclusive and collectively exhaustive, together make up the entire sample space $S$, therefore, we have

$$A \cup \overline{A} = S$$

Thus $\quad P(A \cup \overline{A}) = P(S)$

or $\quad P(A) + P(\overline{A}) = 1 \qquad [\because P(S) = 1 \text{ by axiom (ii)}]$

or $\quad P(\overline{A}) = 1 - P(A).$



Hence the probability of the complement of an event is equal to one minus the probabili event. Complementary probabilities are very useful when the question asks for the probability of " *one*".

**Example 6.10** A coin is tossed 4 times in succession. What is the probability that *at least* occurs?

The sample space $S$ for this experiment consists of $2^4 = 16$ sample points, as each toss can 2 outcomes, and we assume that each outcome is equally likely.

Let $A$ represent the event that *at least* one head occurs. Then $A$ consists of many sample p the other hand, $\overline{A}$ is the event that *no* head occurs and $\overline{A}$ has the single sample point $\{TTTT\}$. T $P(\overline{A}) = \frac{1}{16}$.

Hence by the law of complementation, we have

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{16} = \frac{15}{16}.$$

**Example 6.11** A coin is biased so that the probability that it falls showing tails is $\frac{3}{4}$.

(a) Find the probability of obtaining *at* least one head when the coin is tossed five times.

(b) How many times must the coin be tossed so that the probability of obtaining *at least* one head is greater than 0.98?

Here $P$ (a head appears) $= \dfrac{1}{4}$ and

$P$ (no head or tail appears) $= \dfrac{3}{4}$.

(a) Let A be the event that *at least* one head is obtained when the coin is tossed 5 times, and $\overline{A}$ is the event that no head is obtained. Then by the law of complementation, we have

$$P(A) = 1 - P(\overline{A})$$
$$= 1 - \left(\frac{3}{4}\right)^5 = 1 - 0.237 = 0.763$$

(b) Let the coin be tossed $n$ times to obtain the probability of *at least* one head greater than 0.98.

Then $1 - \left(\dfrac{3}{4}\right)^n \geq 0.98$ i.e. $\left(\dfrac{3}{4}\right)^n \leq 0.02$

Taking logs, we have

$$n \log\left(\frac{3}{4}\right) \leq \log 0.02$$

Dividing both sides by $\log\left(\dfrac{3}{4}\right)$ and reversing the inequality sign as $\log\left(\dfrac{3}{4}\right)$ is negative, we have

$$n \geq \frac{\log 0.2}{\log(3/4)} \geq \frac{\bar{2}.3010}{\bar{1}.8751} \geq \frac{-1.6990}{-0.1249} \geq 13.6$$

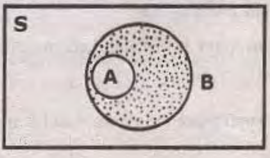
$n = 14$.

the coin should be tossed 14 times so that the probability of obtaining *at least* one head is greater than 0.98.

**Theorem 6.3 Probability of Subevent.** If $A$ and $B$ are two events such that $A \subset B$, then $P(A) \leq P(B)$.

**Proof.** For $A \subset B$, the event $B$ may be written as the union of two mutually exclusive events $B \cap A$ and $B \cap \overline{A}$,

$B = (B \cap A) \cup (B \cap \overline{A})$



B∩Ā is shaded

$B \cap A = A$ so $B = A \cup (B \cap \overline{A})$

$P(B) = P(A) + P(B \cap \overline{A})$

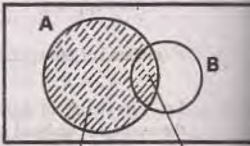
$P(B \cap \overline{A}) \geq 0$.

$P(A) \leq P(B)$.

It is to be noted that an event such as $A \cap B$ is called a *joint event* and probability associated with a joint event is called a *joint probability*.

**Theorem 6.4** If $A$ and $B$ are any two events defined in a sample space $S$, then

$$P(A \cap \overline{B}) = P(A) - P(A \cap B).$$



A is shaded

**Proof.** The events $A \cap \overline{B}$ and $A \cap B$ are mutually exclusive and their union is $A$ (see the Venn diagram). That is

$$A = (A \cap \overline{B}) \cup (A \cap B).$$

$$\therefore \quad P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

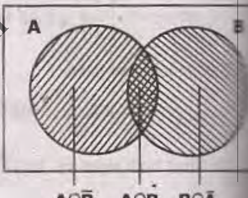
Hence $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

**Theorem 6.5 Addition Law** If $A$ and $B$ are any two events defined in a sample space $S$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Proof.** The event $A \cup B$ may be written as the union of two mutually exclusive events $A$ and $B \cap \overline{A}$ (see the diagram). That is

$$A \cup B = A \cup (B \cap \overline{A})$$



A∪B is shaded

Then $P(A \cup B) = P(A) + P(B \cap \overline{A})$

Again the event $B$ may also be decomposed into two mutually exclusive events as

$$B = (A \cap B) \cup (\overline{A} \cap B)$$

Thus $\quad P(B) = P(A \cap B) + P(\overline{A} \cap B)$

Subtracting this result from the former, we get

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

Hence $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B).$

This law, often called the *General Rule of Addition* for probabilities may be stated as below

"If two events $A$ and $B$ are *not* mutually exclusive, then the probability that *at least* one of them occurs, is given by the sum of the separate probabilities of events $A$ and $B$ minus the probability of the joint event $A \cap B$."

**Corollary 1.** If $A$ and $B$ are mutually exclusive events, then

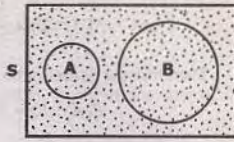
$$P(A \cup B) = P(A) + P(B)$$

**Proof:** Since the events $A$ and $B$ are mutually exclusive, therefore

$$A \cap B = \phi \text{ and } P(A \cap B) = P(\phi) = 0$$

Hence $\quad P(A \cup B) = P(A) + P(B)$, which is just a restatement of axiom (iii).

Alternatively. Let $n$ be the total number of sample points, and let $A$ and $B$ be two events; A consisting of $m_1$ sample points and $B$ of $m_2$ sample points. The occurrence of $A \cup B$ is an event which consists of all the sample points belonging to either $A$ or $B$.




Since $A$ and $B$ are mutually exclusive events, therefore they have no sample points in common. Then obviously, the number of points contained in $A \cup B$ is $m_1 + m_2$.

$$\text{Hence } P(A \cup B) = \frac{\text{number of sample points in } A \cup B}{\text{number of sample points in } S}$$

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

Corollary 2. If $A_1, A_2, ..., A_k$ are $k$ mutually exclusive events, then the probability that one of them occurs, is the sum of the probabilities of the separate events, *i.e.*

$$P(A_1 \cup A_2 \cup ... \cup A_k) = P(A_1) + P(A_2) + ... + P(A_k).$$

It is to be noted that if $A_1, A_2, ..., A_k$ are mutually exclusive and collectively exhaustive, then $P(A_1) + P(A_2) + ... + P(A_k) = 1$.

Corollary 3. If $A$ and $B$ are any two events, then

$$P(A \cup B) \le P(A) + P(B).$$

Proof. The addition law for any two events $A$ and $B$ may be written as

$$P(A \cup B) + P(A \cap B) = P(A) + P(B)$$

Therefore $\quad P(A \cup B) \le P(A) + P(B)$

In general, for any $k$ events $A_1, A_2, ..., A_k$, the relation

$$P(A_1 \cup A_2 \cup ... \cup A_k) \le P(A_1) + P(A_2) + ... + P(A_k).$$

This result is known as *Boole's inequality*.

**Example 6.12** If one card is selected at random from a deck of 52 playing cards, what is the probability that the card is a club or a face card or both?

Let $A$ represent the event that the card selected is a club, $B$, the event that the card selected is a face card and $A \cap B$, the event that the card selected is both a club and a face card. Then we need $P(A \cup B)$.

Now $\quad P(A) = \frac{13}{52}$, as there are 13 clubs,

$\quad P(B) = \frac{12}{52}$, as there are 12 face cards,

and $\quad P(A \cap B) = \dfrac{3}{52}$, since 3 of clubs are also face cards.

Therefore the desired probability is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}.$$

**Example 6.13** An integer is chosen at random from the first 200 positive integers. What is the probability that the integer chosen is divisible by 6 or by 8?

The sample space for this experiment is

$$S = \{1, 2, 3, \ldots, 199, 200\}, \text{ and therefore } n(S) = 200$$

Let A represent the event that the integer chosen is divisible by 6, B, the event that the integer chosen is divisible by 8, and $A \cap B$, the event that the integer chosen is divisible by both 6 and 8, i.e. 24.

Then we need $P(A \cup B)$.

Now $\quad n(A) = \left[\dfrac{200}{6}\right] = 33, \; n(B) = \left[\dfrac{200}{8}\right] = 25$, and

$n(A \cap B) = \left[\dfrac{200}{24}\right] = 8$, where [x] stands for the highest integer in $x$.

Hence $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{50}{200} = \frac{1}{4}$$



**Example 6.14** A pair of dice are thrown. Find the probability of getting a total of either 5 or 11.

The sample space has 36 outcomes when two dice are thrown. (see example 6.6 on page 188)

Let $A$ be the event that a total of 5 occurs and $B$ be the event that a total of 11 occurs. Then the two events are

A= {(1, 4), (2, 3), (3, 2), (4, 1)}, and

B = {(5, 6), (6, 5)}.

The events $A$ and $B$ are mutually exclusive as a total of 5 and 11 cannot both occur together. Therefore

$$P(A \cup B) = P(A) + P(B) = \frac{4}{36} + \frac{2}{36} = \frac{1}{6}.$$

**Example 6.15** Three horses $A$, $B$ and $C$ are in a race; $A$ is twice as likely to win as $B$ and $B$ is twice as likely to win as $C$. What is the probability that $A$ or $B$ wins?

Let $\quad P(C) = p$ as the events are not equally likely.

Then $P(B) = 2p$ as $B$ is twice as likely to win as $C$.

Similarly $P(A) = 2P(B) = 4p$.

Since $A$, $B$ and $C$ are mutually exclusive and collectively exhaustive, therefore the sum of their probabilities must be equal to 1. Thus,

$$p + 2p + 4p = 1 \quad \text{or} \quad p = \frac{1}{7}$$

$$\therefore \quad P(A) = \frac{4}{7},\ P(B) = \frac{2}{7} \quad \text{and} \quad P(C) = \frac{1}{7}$$

Hence $P(A \cup B) = P(A) + P(B) = \frac{4}{7} + \frac{2}{7} = \frac{6}{7}$.

**Theorem 6.6** If $A$, $B$ and $C$ are any three events in a sample space $S$, then the probability of *at least one* of them occurring is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

**Proof:** Let us write $A \cup B \cup C = A \cup (B \cup C)$

$$= A \cup D, \text{ where } D = B \cup C.$$

Then $P(A \cup B \cup C) = P(A \cup D)$

$$= P(A) + P(D) - P(A \cap D)$$

$$= P(A) + P(B \cup C) - P[A \cap (B \cup C)]$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P[A \cap (B \cup C)]$$

The distributive law of sets gives

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Since the two sets are not disjoint, their intersection is given by $A \cap B \cap C$.

Therefore $\quad P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

Hence substitution of this result gives

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

The result may be written as

$$P(A_1 \cup A_2 \cup A_3) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3).$$

In general, the formula for k event is

$$P(A_1 \cup A_2 \cup \ldots A_k) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<l} P(A_i \cap A_j \cap A_l) - \ldots + (-1)^{k+1} P(A_1 \cap A_2 \cap \ldots \cap A_k).$$

**Example 6.16** A card is drawn at random from a deck of ordinary playing cards. What is the probability that it is a diamond, a face card or a king?

Let $A$ represent the event that the card drawn is a diamond, $B$, the event that the card drawn is a face card, $C$, the event that the card drawn is a king, $A \cap B$, the event that the card drawn is both diamond and face card, and so on. Then we need

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

Now $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{13}{52}$, (there are 13 diamonds)

$P(B) = \dfrac{n(B)}{n(S)} = \dfrac{12}{52}$, (there are 12 face cards)

$P(C) = \dfrac{n(C)}{n(S)} = \dfrac{4}{52}$, (there are 4 kings)

$P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{3}{52}$, (diamonds and face card)

$P(B \cap C) = \dfrac{n(B \cap C)}{n(S)} = \dfrac{4}{52}$, (face cards and kings)

$P(A \cap C) = \dfrac{n(A \cap C)}{n(S)} = \dfrac{1}{52}$, (diamond and king)

$P(A \cap B \cap C) = \dfrac{n(A \cap B \cap C)}{n(S)} = \dfrac{1}{52}$, (diamond and face card and king)

Hence, we get

$$P(A \cup B \cup C) = \frac{13}{52} + \frac{12}{52} + \frac{4}{52} - \frac{3}{52} - \frac{4}{52} - \frac{1}{52} + \frac{1}{52} = \frac{22}{52} = 0.423$$



## 6.6 CONDITIONAL PROBABILITY

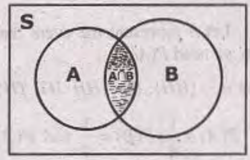
The sample space for an experiment must often be changed when some additional information pertaining to the outcome of the experiment is received. The effect of such information is to *reduce* the sample space by excluding some outcomes as being impossible which before receiving the information were believed possible. The probabilities associated with such a *reduced* sample space are called *conditional probabilities*. The following example illustrates the concept of conditional probability.

Let us consider the die-throwing experiment with sample space $S = \{1, 2, 3, 4, 5, 6\}$. Suppose we wish to know the probability of the outcome that the *die shows 6*, say event $A$. If before seeing the outcome, we are told that the die shows an even number of dots, say event $B$, then the information that the *die shows an even number* excludes the outcomes 1, 3 and 5, and thereby *reduces* the original sample space to a sample space that consists of 3 outcomes 2, 4 and 6, *i.e.* the *reduced* sample space is $B = \{2, 4, 6\}$. Then the desired probability in the reduced sample space $B$ is $\dfrac{1}{3}$, since each outcome in the reduced

sample space is equally likely. We call the $\frac{1}{3}$ as the *conditional probability* of the event $A$ because it is computed under the condition that the die has shown even number of dots. In other words,

$$P(\text{die shows 6/die shows even numbers}) = \frac{1}{3},$$

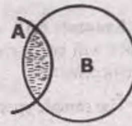
where the vertical line is read as *given that* and the information following the vertical line describes the conditioning event. This is in fact the probability of getting a 6 in the *reduced sample space B*, and is symbolized as $P(A/B)$. Thus we have the following;



$$P(A/B) = \frac{\text{number of sample points in } A \cap B}{\text{number of sample points in } B}$$

$$= \frac{n(A \cap B)}{n(B)}$$

Dividing the numerator and the denominator by the number of sample points in original sample space i.e. $n(S)$, we get



Reduced Sample Space B

$$P(A/B) = \frac{n(A \cap B)}{n(S)} \cdot \frac{n(S)}{n(B)} = \frac{P(A \cap B)}{P(B)}$$

This illustration leads us to make the following definition of conditional probability.

If $A$ and $B$ are two events in a sample space $S$ and if $P(B)$ is not equal to zero, then the *conditional probability* of the event $A$ *given that* event $B$ has occurred, written as $P(A/B)$, is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, the conditional probability $P(A/B)$ remains undefined.

Similarly, $P(B/A) = \frac{P(A \cap B)}{P(A)}$, where $P(A) > 0$.

It should be noted that $P(A/B)$ satisfies all the basic axioms of probability, namely

(i) $0 \le P(A/B) \le 1$.

(ii) $P(S/B) = 1$ $\quad [\because S \cap B = B, \therefore P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1]$

(iii) $P(A_1 \cup A_2 / B) = P(A_1/B) + P(A_2/B)$, provided the events $A_1$ and $A_2$ are mutually exclusive

Thus to determine the *conditional* probability $P(A/B)$, either we directly calculate the probability of $A$ relative to the *reduced* sample space $B$ or we use $P(A \cap B)$ and $P(B)$, the probabilities of the events in the original sample space $S$.

**Example 6.17** Two coins are tossed. What is the conditional probability that two heads result given that there is at least one head?

The sample space $S$ for this experiment is

$$S = \{HH, HT, TH, TT\}.$$

Let $A$ represent the event that two heads appear, and $B$, the event that there is at least one head. Then we need $P(A/B)$.

Since $A = \{HH\}$, $B = \{HH, HT, TH\}$ and $A \cap B = \{HH\}$.

$\therefore \quad P(A) = \frac{1}{4}, P(B) = \frac{3}{4}$ and $P(A \cap B) = \frac{1}{4}$

Hence $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$.

**Example 6.18** A man tosses two fair dice. What is the conditional probability that the sum of the two dice will be 7, given that (i) the sum is odd, (ii) the sum is greater than 6, (ii) the two dice had the same outcome? (P.U., B.A./B.Sc.



The sample space $S$ for this experiment consists of the following 36 equally likely outcomes

$$\begin{aligned} S = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ &(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\ &(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\ &(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\ &(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), \\ &(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}. \end{aligned}$$

Let $A = \{\text{the sum is 7}\}$, $B = \{\text{the sum is odd}\}$,

$C = \{\text{the sum is greater than 6}\}$, and

$D = \{\text{the two dice had the same outcomes}\}$. Then

$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$,

$B = \{(1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 4), \ldots, (6, 5)\}$,

$C = \{(1, 6), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 3), (4, 4), \ldots, (6, 6)\}$,

$D = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$,

$A \cap B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$,

$A \cap C = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$,

$A \cap D = \phi$

$$\therefore \quad P(A) = \frac{6}{36},\ P(B) = \frac{18}{36},\ P(C) = \frac{21}{36},\ P(D) = \frac{6}{36}$$

$$P(A \cap B) = \frac{6}{36},\ P(A \cap C) = \frac{6}{36} \text{ and } P(A \cap D) = 0.$$

using the definition of conditional probability, we get

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{6}{36} \times \frac{36}{18} = \frac{1}{3}$$

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{6}{36} \times \frac{36}{21} = \frac{2}{7}$$

$$P(A/D) = \frac{P(A \cap D)}{P(D)} = 0 \times \frac{36}{6} = 0.$$

**Example 6.19** What is the probability that a randomly selected poker hand, contains exactly 3 given that it contains *at least* 2 aces?

Let $A$ represent the event that exactly 3 aces are selected and $B$, the event that *at least* 2 aces are . Then we need $P(A/B)$.

Since a poker hand consists of 5 cards, therefore the sample space $S$ contains $\binom{52}{5} = 2{,}598{,}960$

$$n(A) = \binom{4}{3}\binom{48}{2} \text{ outcomes;}$$

$$n(B) = \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}$$

*at least* 2 aces means 2 or 3 or 4 aces; and

$$n(A \cap B) = \binom{4}{3}\binom{48}{2} \text{ as } A \subset B.$$

$$P(A \cap B) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}, \text{ and}$$

$$P(B) = \frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}},$$

Hence $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$= \frac{\binom{4}{3}\binom{48}{2}}{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}$$

$$= \frac{4{,}512}{108{,}336} = 0.0416.$$

**Theorem 6.7 Multiplication Law.** If $A$ and $B$ are any two events defined in a sample sp then

$$P(A \cap B) = P(A)\, P(B/A), \text{ provided } P(A) \neq 0,$$

$$= P(B)\, P(A/B), \text{ provided } P(B) \neq 0,$$

The conditional probability of $B$ *given* that $A$ has occurred is

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ where } P(A) \neq 0$$

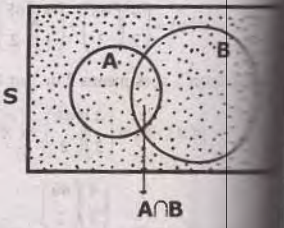
Multiplying both sides by $P(A)$, we get

$$P(A \cap B) = P(A).\, P(B/A).$$

The second form is easily obtained by interchanging $A$ and $B$.

This is called the *general rule of multiplication* for probabilities.

**Alternative Proof.** Let $S$ represent a sample space of an experiment having $n$ equally likely outcomes (sample points). Let $m_1$ be the number of sample points contained in $A$ (including those common to $B$); $m_2$ be the number of sample points in $B$ and $m_3$ be the number of sample points belonging both to $A$ and $B$. Then (assuming $m_1 > 0, m_2 > 0$),




$$P(A \cap B) = \frac{m_3}{n}$$

The fraction $\dfrac{m_3}{n}$ may be written as $\dfrac{m_3}{n} = \dfrac{m_3}{m_1}.\dfrac{m_1}{n}$

But $\dfrac{m_1}{n} = P(A)$ and $\dfrac{m_3}{m_1} =$ conditional probability of $B$, given that $A$ has occurred, *i.e.* $P$

Hence $P(A \cap B) = P(A)\, P(B/A)$.

Since the joint event $A \cap B$ involves $A$ and $B$ symmetrically, therefore interchanging $A$ and $B$, we get

$$P(A \cap B) = P(B)\, P(A/B).$$

This rule may be stated as below:

"The probability that two events A and $B$ will *both* occur is equal to the probability that one of the events will occur multiplied by the conditional probability that the other event will occur *given* that the first event has occurred."

**Corollary.** This rule may be extended to several events. In case of three events $A$, $B$ and $C$, we have

$$\begin{aligned} P(A \cap B \cap C) &= P(D \cap C), \text{ were } D = A \cap B \\ &= P(D)\, P(C/D) \\ &= P(A \cap B)\, P(C/A \cap D) \\ &= P(A)\, P(B/A)\, P(C/A \cap B); \end{aligned}$$

Similarly, for more than three events, the formula may be proved by mathematical induction.

**Example 6.20** A box contains 15 items, 4 of which are defective and 11 are good. Two items are selected. What is the probability that the *first* is good and the *second* defective.

Let $A$ represent the event that the first item selected is good and $B$, the event that the second item is defective.

Then we need to calculate the probability of the joint event $A \cap B$ by the rule $P(A \cap B) = P(A).\ P(B/A)$.

Now $P(A) = \dfrac{11}{15}$

Given the event $A$ has occurred, there remains 14 items of which 4 are defective. Therefore the probability of selecting a defective after a good has been selected, *i.e.* $P(B/A) = \dfrac{4}{14}$.

Hence $P(A \cap B) = P(A).\ P(B/A) = \dfrac{11}{15} \times \dfrac{4}{14} = \dfrac{44}{210} = 0.16.$

**Example 6.21** Two cards are dealt from a pack of ordinary playing cards. Find the probability that the second card dealt is a *heart*.

Let $H_1$ represent the event that the first card dealt is a *heart*, and $H_2$, the event that the second card dealt is a *heart*. Then

$P$(second card is a heart) = $P$(first card is a *heart* and second card is a *heart*) + $P$(first card is *not a heart* and second card is a *heart*).

$$\begin{aligned} P(H_2) &= P(H_1 \cap H_2) + P(\overline{H}_1 \cap H_2) \\ &= P(H_1)P(H_2/H_1) + P(\overline{H}_1)P(H_2/\overline{H}_1) \end{aligned}$$

$$=\left(\frac{13}{52}\times\frac{12}{51}\right)+\left(\frac{39}{52}\times\frac{13}{51}\right)$$

$$=\frac{1}{17}+\frac{13}{68}=\frac{17}{68}=\frac{1}{4}$$

**Example 6.22** Box $A$ contains 5 green and 7 red balls. Box $B$ contains 3 green, 3 red and 6 yel balls. A box is selected at random and a ball is drawn at random from it. What is the probability th ball drawn is green?

Let $E$ represent the event that the green ball is drawn. Then $E$ can occur in one of the followi mutually exclusive ways:

i) Box $A$ is selected and a green ball is drawn, *i.e.* $A\cap E$, or

ii) Box $B$ is selected and a green ball is drawn, *i.e.* $B\cap E$.

Therefore $P$(green ball) $= P$(box $A$ and green ball) + $P$(box $B$ and green ball)

$= P$(box $A$) $P$(green ball/box $A$) + $P$(box $B$) x $P$(green ball/box $B$)

In symbols, $P(E)=P(A\cap E)+P(B\cap E)$

$$=P(A)\,P(E/A)+P(B)\,P(E/B)$$

$$=\frac{1}{2}\cdot\frac{5}{12}+\frac{1}{2}\cdot\frac{3}{12}=\frac{1}{3}$$

**Example 6.23** An urn contains 10 white and 3 black balls. Another urn contains 3 whit black balls. Two balls are transferred from first urn and placed in the second and then one ball i from the latter. What is the probability that it is a white ball? (P.U., B.A./B.Sc.

Let $A$ represent the event that 2 balls are drawn from the first urn and transferred to the seco Then $A$ can occur in the following three mutually exclusive ways:

$A_1$ = 2 white balls,
$A_2$ = 1 white ball and 1 black ball,
$A_3$ = 2 black balls

Thus $P(A_1)=\binom{10}{2}\div\binom{13}{2}=\frac{45}{78}$,

$P(A_2)=\binom{10}{1}\binom{3}{1}\div\binom{13}{2}=\frac{30}{78}$, and

$P(A_3)=\binom{3}{2}\div\binom{13}{2}=\frac{3}{78}$.

The second urn after having transferred 2 balls from the first urn, contains

| | | |
|---|---|---|
| i) | 5 white and 5 black balls | (2 white balls transferred) |
| ii) | 4 white and 6 black balls | (1 white and 1 black ball transferred) |
| iii) | 3 white and 7 black balls | (2 black balls transferred) |

Let $W$ represent the event that a white ball is drawn from the second urn after having transferred 2 balls from the first urn. Then

$$P(W) = P(W \cap A_1) + P(W \cap A_2) + P(W \cap A_3)$$

Now $P(W \cap A_1) = \frac{5}{10} \times \frac{45}{78} = \frac{15}{52}$.

$P(W \cap A_2) = \frac{4}{10} \times \frac{30}{78} = \frac{2}{13}$, and

$P(W \cap A_3) = \frac{3}{10} \times \frac{3}{78} = \frac{3}{260}$.

Hence the required probability is

$$P(W) = \frac{15}{52} + \frac{2}{13} + \frac{3}{260} = \frac{59}{130} = 0.4538$$

**Example 6.24** A card is drawn at random from a deck of ordinary playing cards. What is the probability that it is a diamond, a face card or a king? (P.U., B.A./B.Sc. 1992)

There are two ways of solving this problem. One is given in Example 6.16. A second approach is given below:

Let $A$ = the card drawn is a diamond,

$B$ = the card drawn is a face card, and

$C$ = the card drawn is a king.

Then we need

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

Now $P(A) = \frac{13}{52}$, $P(B) = \frac{12}{52}$, $P(C) = \frac{4}{52}$

$P(A \cap B) = P(A)P(B/A) = \frac{13}{52} \times \frac{3}{13} = \frac{3}{52}$,

$P(B \cap C) = P(B)P(C/B) = \frac{12}{52} \times \frac{4}{12} = \frac{4}{52}$,

$P(A \cap C) = P(A)P(C/A) = \frac{13}{52} \times \frac{1}{13} = \frac{1}{52}$, $\left(\text{or } P(C \cap A) = \frac{4}{52} \times \frac{1}{4}\right)$

$P(A \cap B \cap C) = P(A)\ P(B/A)\ P(C/A \cap B)$

$$= \frac{13}{52} \times \frac{3}{13} \times \frac{1}{3} = \frac{1}{52}$$

Thus, we get

$$P(A \cup B \cup C) = \frac{13}{52} + \frac{12}{52} + \frac{4}{52} - \frac{3}{52} - \frac{4}{52} - \frac{1}{52} + \frac{1}{52}$$

$$= \frac{22}{52} = 0.423$$

**Example 6.25** Three urns of the same appearance are given as follows:

Urn $A$ contains 5 red and 7 white balls.

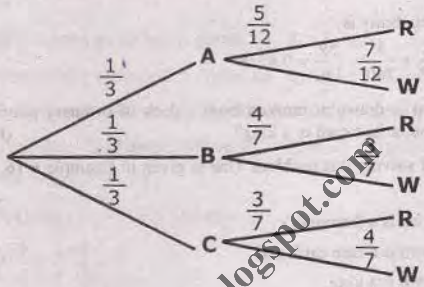
Urn $B$ contains 4 red and 3 white balls.

Urn $C$ contains 3 red and 4 white balls.

An urn is selected at random and a ball is drawn from the urn.

i) What is the probability that the ball drawn is red?

ii) If the ball drawn is red, what is probability that it came from urn $A$?






Here we first select one of the three urns and then we draw a ball which is either red ($R$) or (W). In other words, we perform a sequence of two experiments. This process is described probability *tree diagram*, (see page 204) in which each branch of the tree gives the respective pr

Now the probability of selecting urn $A$, for instance, and then a red ball ($R$) is $\frac{1}{3}.\frac{5}{12}=\frac{5}{36}$.

the probability that any particular path of the tree occurs is, by the multiplication law, the produ probability of each branch of the path.

i) Now the probability of drawing a red ball is given by the relation

$$P(R)=P(A)P(R/A)+P(B)\ P(R/B)+P(C)\ P(R/C)$$

as there are three mutually exclusive paths leading to the drawing of a red ball.

$$\text{Hence } P(R)=\frac{1}{3}.\frac{5}{12}+\frac{1}{3}.\frac{4}{7}+\frac{1}{3}.\frac{3}{7}$$

$$=\frac{119}{252}=0.4722$$

ii) Here we need the probability that urn $A$ is selected, given that the ball drawn is re $P(A/R)$.

By definition, $P(A/R)=\dfrac{P(A\cap R)}{P(R)}$

But $P(A \cap R)$ = Probability that urn $A$ is selected and a red ball is drawn

$$= \frac{1}{3} \times \frac{5}{12} = \frac{5}{36}$$

Hence $P(A/R) = \dfrac{5/36}{119/252}$

$$= \frac{35}{119} = 0.294$$

## 6.7 INDEPENDENT AND DEPENDENT EVENTS

Two events $A$ and $B$ in the same sample space $S$, are defined to be *independent* (*or statistically independent*) if the probability that one event occurs, is not affected by whether the other event has or has not occurred, that is

$$P(A/B) = P(A) \quad \text{and} \quad P(B/A) = P(B).$$

It then follows that two events $A$ and $B$ are independent *if and only if*

$$P(A \cap B) = P(A)\ P(B)$$

The events $A$ and $B$ are defined to be *dependent if* $P(A \cap B) \neq P(A) \times P(B)$. This means that the occurrence of one of the events in some way affects the probability of the occurrence of the other event.

It is to be emphasized that two mutually exclusive events $A$ and $B$ are independent if and only if $P(A)\ P(B) = 0$, which is true when either $P(A) = 0$ or $P(B) = 0$. If both events $A$ and $B$ have non-zero probabilities, they must have a sample point in common. Thus two events that are independent, can never be mutually exclusive. Moreover, two events that are mutually exclusive, are also dependent events, and two events that are non-mutually exclusive, may either be independent or dependent events.

Three events $A$, $B$ and $C$, all defined on the same sample space, are said to be *mutually independent* if they satisfy the following conditions:

i) They are pairwise independent, *i.e.* $P(A \cap B) = P(A)\ P(B)$;

$P(A \cap C) = P(A).\ P(C)$; $P(B \cap C) = P(B).\ P(C)$.

ii) They are mutually independent, *i.e.*

$P(A \cap B \cap C) = P(A).\ P(B).\ P(C)$.

In general, the $k$ events $A_1, A_2, \ldots, A_k$ are defined to be *mutually independent* if and only if the probability of the intersection of any 2, 3, …, or $k$ of them equals the product of their respective probabilities.

**Example 6.26** Two events $A$ and $B$ are such that $P(A) = \frac{1}{4}, P(A/B) = \frac{1}{2}$, and $P(B/A) = \frac{2}{3}$.

i) Are $A$ and $B$ independent events?

ii) Are $A$ and $B$ mutually exclusive events?

iii) Find $P(A \cap B)$ and $P(B)$.

i) If $A$ and $B$ are independent events, then $P(A/B) = P(A)$

Now $P(A) = \frac{1}{4}$ and $P(A/B) = \frac{1}{2}$ i.e. $P(A/B) \neq P(A)$

Hence $A$ and $B$ are not independent events.

ii) If $A$ and $B$ are mutually exclusive events, then $P(A/B) = 0$.

But it is given that $P(A/B) = \frac{1}{2}$

Hence $A$ and $B$ are not mutually exclusive events.

iii) Now $P(A \cap B) = P(A)\ P(B/A)$

$$= \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$

By definition, we have

$$P(B)\ P(A/B) = P(A)\ P(B/A)$$

or $P(B)\left(\frac{1}{2}\right) = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)$ so that $P(B) = \frac{1}{4} \times \frac{2}{3} \times \frac{2}{1} = \frac{1}{3}$



**Example 6.27** Two fair dice, one red and one green, are thrown. Let $A$ denote the event that red die shows an even number and $B$, the event that the green die shows a 5 or a 6. Show that the events $A$ and $B$ are independent.

The sample space $S$ contains 36 equally likely outcomes when two dice are thrown.

Given $A$ = event that red die shows an even number.

$B$ = event that green die shows a 5 or a 6, and therefore

$A \cap B$ = event that red die shows an even number and green die shows a 5 or a 6.

Then $A$ contains 18 outcomes, $B$ contains 12 and $A \cap B$ contains only 6 outcomes.

Associating with each outcome a probability of $\frac{1}{36}$, we get

$$P(A) = \frac{18}{36} = \frac{1}{2},\ P(B) = \frac{12}{36} = \frac{1}{3} \text{ and } P(A \cap B) = \frac{6}{36} = \frac{1}{6}$$

Since $P(A \cap B) = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = P(A)\ P(B)$, therefore the events $A$ and $B$ are independent.

**Alternatively.**

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/3} = \frac{1}{2} = P(A), \text{ and}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3} = P(B).$$

Hence the events $A$ and $B$ are independent.

**Example 6.28** Let $A$ be the event that a family has children of both sexes and $B$ be the event that a family has at most one boy. If a family is known to have (i) three children, then show that $A$ and $B$ are independent events, (ii) four children, then show that $A$ and $B$ are dependent events.

(P.U., B.A. (Hons.) Part-I, 1970)

Let $b$ denote a boy and $g$ a girl. Then

i) the equiprobable sample space $S$ would be

$$S = \{bbb, bbg, bgb, gbb, bgg, gbg, ggb, ggg\}$$

The two events are

$A$ = {children of both sexes},

= {bbg, bgb, gbb, bgg, gbg, ggb}, and

$B$ = {at most one body},

= {bgg, gbg, bgg, ggb, ggg}

The event $A \cap B$ is

$A \cap B$ = {bgg, gbg, ggb}

Thus their respective probabilities are

$$P(A) = \frac{6}{8} = \frac{3}{4},\ P(B) = \frac{4}{8} = \frac{1}{2},\ \text{and}\ P(A \cap B) = \frac{3}{8}$$

Now $P(A)\ P(B) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} = P(A \cap B)$.

Hence $A$ and $B$ are independent.

ii) the sample space $S$ may be represented by the following 16 equally likely outcomes:

$S$ = {bbbb, bbbg, bbgb, bgbb, gbbb, bbgg, bgbg, gbbg, gbgb, bggb, ggbb, bggg, gbgg, ggbg, gggb, gggg}



The events are:

$A$ = {bbbg, bbgb, bgbb, gbbb, bbgg, bgbg, gbbg, gbgb, bggb, ggbb, bggg, gbgg, ggbg, gggb}

$B$ = {bggg, gbgg, ggbg, gggb, gggg}, and

$A \cap B$ = {bggg, gbgg, ggbg, gggb}

Their probabilities are

$$P(A) = \frac{14}{16} = \frac{7}{8},\ P(B) = \frac{5}{16}\ \text{and}\ P(A \cap B) = \frac{4}{16} = \frac{1}{4}.$$

Now $P(A)\ P(B) = \frac{7}{8} \cdot \frac{5}{16} \neq P(A \cap B)$.

Hence $A$ and $B$ are dependent events.

**Theorem 6.8** If $A$ and $B$ are two independent events, then

$$P(A \cap B) = P(A).\ P(B).$$

**Proof:** Since $A$ and $B$ are independent events, therefore

$$P(A) = P(A/B) \quad \text{and} \quad P(B) = P(B/A).$$

Substituting these results in the general rule of multiplication, we obtain

$$P(A \cap B) = P(A)\ P(B).$$

**Theorem 6.9** If $A$ and $B$ are two independent events in a sample space $S$, then (i) $A$ and $\overline{B}$ are independent, (ii) $\overline{A}$ and $B$ are independent, and (iii) $\overline{A}$ and $\overline{B}$ are independent.

**Proof.** (i) The events $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive and their union is $A$, i.e. $A = (A \cap B) \cup (A \cap \overline{B})$.

Therefore $P(A) = P(A \cap B) + P(A \cap \overline{B})$

or $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

$= P(A) - P(A)P(B)$ [$\because$ $A$ and $B$ are independent ]

$= P(A)\,[1 - P(B)] = P(A)\ P(\overline{B})$.

Hence $A$ are $\overline{B}$ are independent.

(ii) Similarly,

$$P(B) = P(B \cap A) + P(B \cap \overline{A})$$

or $P(B \cap \overline{A}) = P(B) - P(B \cap A)$

$= P(B) - P(B)\,P(A)$ [$\because$ $A$ and $B$ are independent ]

$= P(B)\,[1 - P(A)] = P(B)\ P(\overline{A})$

Therefore $\overline{A}$ and $B$ are independent.

(iii) Using De Morgan's law, $\overline{A} \cap \overline{B} = \overline{A \cup B}$, we have

$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= P(\overline{A \cup B}) \\ &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)\,P(B) \\ &= [1 - P(A)][1 - P(B)] = P(\overline{A})\,P(\overline{B}) \end{aligned}$$

which shows that $\overline{A}$ and $\overline{B}$ are independent.

**Example 6.29** Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is (i) replaced, (ii) not replaced.

(P.U., B.A./B.Sc., 1967; M.A. Econ. 1969)

Let $A$ denote the event *ace* on first draw and $B$ denote the event *ace* on the second draw.

(i) In case of replacement, event $A$ and $B$ are independent.

Thus $P(\text{both cards are aces}) = P(A \cap B) = P(A)\,P(B)$

$$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}.$$

(ii) If the first card is not replaced, then $A$ and $B$ are dependent events and therefore

$P(\text{both cards are aces}) = P(\text{first card is an ace}) \times P(\text{second card is an ace given that the first card is an ace})$

*i.e.* $P(A \cap B) = P(A)\,P(B/A) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.

**Example 6.30** A pair of fair dice is thrown twice. What is the probability of getting totals of 5 and

(P.U., B.A./B.Sc. 1978)

Let $A$ represent the event of getting a total of 5 and $B$, the event of getting a total of 11, when two dice are thrown.

Then $A$ can occur in the following two ways:

$A_1$ = {a total of 5 occurs on the first throw},

$A_2$ = {a total of 5 occurs on the second throw},

and $B$ can occur in the following two ways:

$B_1$ = {a total of 11 occurs on the first throw},

$B_2$ = {a total of 11 occurs on the second throw}.

The joint event $A \cap B$ can occur in two mutually exclusive ways $A_1 \cap B_2$ or $B_1 \cap A_2$, *i.e.* $A \cap B = (A_1 \cap B_2) \cup (B_1 \cap A_2)$.

Therefore $P(A \cap B) = P(A_1 \cap B_2) + P(B_1 \cap A_2)$

$$= P(A_1)\,P(B_2) + P(B_1)\,P(A_2)$$

($\because$ events are independent)

$$= \frac{4}{36} \cdot \frac{2}{36} + \frac{2}{36} \cdot \frac{4}{36} = \frac{1}{162} + \frac{1}{162} = \frac{1}{81}.$$

**Example 6.31** The probability that a man will be alive in 25 years is 3/5, and the probability t his wife will be alive in 25 years is 2/3. Find the probability that (i) both will be alive, (ii) only the m will be alive, (iii) only the wife will be alive, (iv) at least one will be alive and (v) neither will be alive 25 years. (P.U., B.A./B.Sc. 197

Let $A$ be the event that the man will be alive and $B$ be the event that his wife will be alive in years. Then

$$P(A)=\frac{3}{5}, \text{ and } P(B)=\frac{2}{3}.$$

i) We need the probability that both will be alive, *i.e.* $P(A\cap B)$.

Since $A$ and $B$ are independent, therefore

$$P(A\cap B)=P(A).\ P(B)=\frac{3}{5}\times\frac{2}{3}=\frac{2}{5}.$$

ii) We need the probability that only the man will be alive, *i.e.* $P(A\cap\overline{B})$.

Since $A$ and $\overline{B}$ are independent and $P(\overline{B})=1-P(B)$, therefore

$$P(A\cap\overline{B})=P(A)\ P(\overline{B})=\frac{3}{5}\times\left(1-\frac{2}{3}\right)=\frac{1}{5}.$$

iii) We require the probability that only the wife will be alive, *i.e.* $P(\overline{A}\cap B)$. Thus

$P(\overline{A}\cap B)=P(\overline{A})\ P(B)=\frac{2}{5}\times\frac{2}{3}=\frac{4}{15}$, as the event $\overline{A}$ and $B$ are independent $P(\overline{A})=1-P(A)$.

iv) We require the probability that *at least* one will be alive, *i.e.* $P(A\cup B)$.

Since the events $A$ and $B$ are independent and not mutually exclusive, therefore

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{3}{5}+\frac{2}{3}-\frac{2}{5}=\frac{13}{15}.$$

v) We need the probability that neither will be alive, *i.e.* $P(\overline{A}\cap\overline{B})$.

Since $\overline{A}$ and $\overline{B}$ are independent, therefore

$$P(\overline{A}\cap\overline{B})=P(\overline{A})P(\overline{B})$$

$$=[1-P(A)]\,[1-P(B)]$$

$$=\frac{2}{5}\times\frac{1}{3}=\frac{2}{15}.$$

**Theorem 6.10. Independent Repeated Trials with two Outcomes.** If the probability of an event $A$ occurring in a ingle trial is $p$, then the probability of its occurring $k$ times in $n$ independent trials is given by $P(A_k) = \binom{n}{k} p^k q^{n-k}$, where $q = 1 - p$.

**Proof:** If the event $A$ occurs $k$ times in $n$ independent trials, then the event $\overline{A}$ (not-$A$) will occur in the remaining ($n$-$k$) trials, that is

$$\underbrace{AAA...A}_{k \text{ times}}\ \underbrace{\overline{A}\overline{A}\overline{A}...\overline{A}}_{n-k \text{ times}}$$

The probability of this sequence would be

$$\underbrace{ppp...p}_{k \text{ times}}\ \underbrace{qqq...q}_{n-k \text{ times}}$$

$$= p^k q^{n-k}.$$

But this is one sequence in which the event $A$ occurs $k$ times and fails to occur (n – k) times. The number of possible sequences is $\binom{n}{k}$.

Hence the required probability is

$$P(A_k) = \binom{n}{k} p^k q^{n-k}.$$

It is interesting to note that this is a special case of a very general result, called the *binomial law*, which is discussed in a bit detail in chapter 8.



**Example 6.32** Five coins are tossed once (or one coin is tossed five times). What is the probability of getting precisely 3 heads?

We know that the probability of getting a head is $\frac{1}{2}$,

$$\therefore \quad P(3 \text{ heads on a single toss of 5 coins}) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3}$$

$$= 10 \times \left(\frac{1}{2}\right)^5 = \frac{5}{16}.$$

**Example 6.33** If 60 percent of the voters in the City of Lahore prefer candidate $X$, what is the probability that in a sample of 12 voters exactly 7 will prefer $X$?

Here $n = 12,\ k = 7,\ p = 0.60$ and $q = 0.40$

Therefore $P(7 \text{ out of 12 prefer } X) = \binom{12}{7}(0.60)^7(0.40)^{12-7}$

$$= (792)\,(0.02799)\,(0.01024) = 0.227$$

**Theorem 6.11 Bayes' theorem.** If the events $A_1, A_2, ..., A_k$ form a partition of sample space S, that is, the events $A_i$ are mutually exclusive and their union is $S$, and if $B$ is any other event of $S$ such that it can occur only if one of the $A_i$ occurs, then for any $i$,

$$\boxed{P(A_i / B) = \frac{P(A_i)P(B / A_i)}{\sum_{i=1}^{k} P(A_i)P(B / A_i)}, \text{ for } i = 1, 2, ..., k.}$$

**Proof:** By the multiplicative law of probabilities, we have

$$P(B \cap A_i) = P(B)P(A_i / B)$$
$$= P(A_i)P(B / A_i).$$

Equating the equivalent relations of $P(B \cap A_i)$ and solving for $P(A_i / B)$, we get

$$P(A_i / B) = \frac{P(A_i)P(B / A_i)}{P(B)}$$


**B is shaded**

We may write the event $B$ as $B = S \cap B$ (see the Venn diagram)

$$= (A_1 \cup A_2 \cup ... \cup A_k) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup ... \cup (A_k \cap B),$$

where the $A_i \cap B$ are also mutually exclusive.

Therefore $P(B) = P(A_1 \cap B) + P(A_2 \cap B) + ... + P(A_k \cap B)$

Using the multiplicative law of probabilities, we may express each term $P(A_i \cap B)$ as $P(A_i)P(B / A_i)$. Then

$$P(B) = P(A_1)P(B / A_1) + P(A_2)P(B / A_2) + ... + P(A_k)P(B / A_k)$$
$$= \sum_{i=1}^{k} P(A_i)\, P(B / A_i)$$

This result is generally known as the *theorem on total probability*. Replacing $P(B)$ by the total probability formula for the event $B$, we obtain *Bayes' formula* as

$$P(A_i / B) = \frac{P(A_i)P(B / A_i)}{\sum_{i=1}^{k} P(A_i)P(B / A_i)}$$

This result is known as Bayes' theorem after an English clergyman Thomas Bayes (1702–1761) who derived it and first used in a paper that was published posthumously in 1763. It should be noted that the original probabilities $P(A_i)$ are known as the *a priori* probabilities and the conditional proba

$B$) are called the *a posteriori* or *inverse* probabilities because probabilities are revised after some [illegible]al information has been obtained. Bayes' formula is also called the *formula for probabilities of* [illegible]*eses* on account of the reason that the events $A_1, A_2, A_k$ may be thought of as hypotheses to [illegible] for occurrence of the event $B$.

**Example 6.34** In a bolt factory, machines $A$, $B$ and $C$ manufacture 25, 35 and 40 percent of the [illegible] output, respectively. Of their outputs, 5, 4, 2 percent, respectively, are defective bolts. $A$ bolt is [illegible] at random and found to be defective. What is the probability that the bolt came from machine $A$?

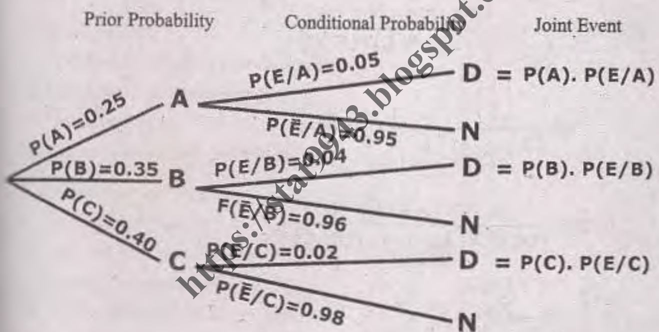
The *a priori* probabilities (before the information that the bolt is defective) are [illegible] $= 0.25$, $P(B) = 0.35$, and $P(C) = 0.40$.

Let $E$ represent the event that a bolt is defective ($D$).

Then the conditional probabilities are

$$P(E/A) = 0.05,\ P(E/B) = 0.04,\ \text{and}\ P(E/C) = 0.02.$$

The outcomes with their respective probabilities may be shown by a tree diagram as below:




[illegible] $E$) is the *a posteriori* probability that the selected defective bolt came from machine $A$. [illegible] by Bayes' theorem, we get

$$P(A/E) = \frac{P(A).P(E/A)}{P(A).P(E/A) + P(B).P(E/B) + P(C).P(E/C)}$$

$$= \frac{(0.25)(0.05)}{(0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02)}$$

$$= \frac{0.0125}{0.0345} = 0.362$$

Similarly, the posterior probabilities of machines $B$ and $C$ are

$$P(B/E) = 0.406, \quad \text{and} \quad P(C/E) = 0.232$$

**Example 6.35** An urn contains four balls which are known to be either (i) all white or (ii) white and two black. A ball is drawn at random and is found to be white. What is the probability the balls are white? (P.U., B.A./B.Sc. (Hons.) Part-III, 196

Let $A_1$ be the hypothesis that all the balls are white and $A_2$ be the hypothesis that two are and two black. Then the *a priori* probabilities must be

$$P(A_1) = P(A_2) = \frac{1}{2}, \text{ as the selection of a hypothesis is random.}$$

Again let $B$ the event that the ball drawn is white. Then the conditional probabilities are

$$P(B/A_1) = \frac{^4C_1}{^4C_1} = 1 \quad \text{and} \quad P(B/A_2) = \frac{^2C_1}{^4C_1} = \frac{1}{2}.$$

Therefore by Bayes' theorem, we get the *a posteriori* probabilities

$$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{P(A_1)P(B/A_1) + P(A_2)P(B/A_2)}$$

$$= \frac{\left(\frac{1}{2}\right)(1)}{\left(\frac{1}{2}\right)(1) + \frac{1}{2}\left(\frac{1}{2}\right)} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3};$$

$$P(A_2/B) = \frac{P(A_2)P(B/A_2)}{P(A_1)P(B/A_1) + P(A_2)P(B/A_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}.$$

Hence the first hypothesis, *i.e.* all the balls are white, is preferred as it has large probability.



## EXERCISES

### OBJECTIVE

Answer 'True' or 'False'. If the statement is not true then replace the underlined words wit make the statement true:

i) The probability of an event is a <u>whole number</u>.

If two events are mutually exclusive, they are also independent.

If A and B are mutually exclusive events, the sum of their probabilities equal to one.

The sample points of a sample space are equally likely events.

If the sets of sample points belonging to two different events do not intersect, the events are independent.

Probability of head on tossing a coin is ½.

If P(A/B) = P(A) and P(B/A) = P(B) then the two events A and B are independent.

It is always true that $P(A) = 1 - P(\overline{A})$.

The probabilities of complementary events always are equal.

If events A and B are statistically independent then $P(A \cap B) = P(A) + P(B)$.

## TIPLE CHOICE QUESTIONS

A simple event is
- a) a collection of exactly two outcomes.
- b) does not include any outcome.
- c) includes one and only one outcome.
- d) includes more than one events.

A compound event includes
- a) at least four outcomes
- b) one and only one outcome
- c) at least two outcomes
- d) all the outcomes of an experiment

The probability of an event is always
- a) greater than zero
- b) less than 1
- c) in the range zero to 1
- d) greater than 1

The classical probability method is applied to an experiment that
- a) cannot be repeated.
- b) has equally likely outcomes.
- c) has all independent outcomes.
- d) does not have more than two outcomes.

v) The relative frequency method is applied to an experiment that

a) does not have equally likely outcomes but can be repeated.

b) does not have equally likely outcomes and cannot be repeated.

c) has equally likely outcomes and cannot be repeated.

d) has all independent outcomes.

vi) Which of the following values cannot be the probability of an event?

a) .82

b) 0

c) 1.75

d) 0.36

vii) In a group of 400 families, 300 own houses. If one family is randomly selected from group the probability that this family owns a house is:

a) .75

b) .25

c) .80

d) .40

viii) Two mutually exclusive events

a) always occur together.

b) cannot occur together.

c) can sometimes occur together.

d) can never occur together.

ix) The two events A and B are mutually exclusive. Which one of the following statements be true?

a) $P(A \cap B) = 0$.

b) $P(A \cap B) = 1$.

c) $P(A \cup B) = 0$.

d) $P(A \cup B) = 1$.

x) $P(A) = 0.6$ and $P(B) = 0.5$. Which of the following statements is true?

a) A and B are mutually exclusive.

b) A and B are not mutually exclusive.

c) A and B are independent.

d) A and B are dependent.

The conditional probability of event A given that the event B has already occurred is written as:

a) $P(A \cup B)$.

b) $P(B/A)$

c) $P(A \cap B)$

d) $P(A/B)$

Two complementary events

a) have no common outcomes

b) have common outcomes

c) contain the same outcomes

d) can have common outcomes.

The union of two events A and B is written as:

a) (A or B)

b) (A and B)

c) (B/A)

d) (A/B)

The intersection of two events A and B is written as:

a) (A or B)

b) (A and B)

c) (AB)

d) (A/B)



The joint probability of two independent events A and B is:

a) $P(A) + P(B)$

b) $P(A) + P(B) - P(A \cap B)$

c) $P(A)P(B)$

d) $P(A)P(A/B)$

## SUBJECTIVE

List all the proper subsets of the universal set
S = {chair, student, pen}.

Construct a Venn diagram to illustrate the following subsets of
S = {ball, pen, table, coin, die, card, book},
A = {ball, pen, book}, B = {pen, table, coin}, C = {card}.

6.3 a) Let A = {1,2}, B = {2,3} and C = {3} be subsets of the universal set S = {1,2
Determine the elements of the following sets:

i) A x A, ii) A x B, iii) B x A,
iv) $(A \times B) \cup (B \times C)$ v) $(A \times B) \cap (B \times C)$

b) Let A = {2,3}, B = {1,3,5} and C = {4,6}. Construct the "tree diagram" of A x B x C

6.4 a) Explain what is meant by a Random Experiment, a Sample Space and an Event.

b) Let A, B and C be events (subsets) in a sample space S defined by

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{2, 3, 4\},\ B = \{3, 4, 5\},\ C = \{5, 6, 7\}$$

List the members of the following events:

i) $\overline{A} \cap B$, ii) $\overline{A} \cup B$, iii) $\overline{\overline{A} \cap \overline{B}}$, iv) $\overline{A \cap (B \cup C)}$.

6.5 If $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{0, 2, 4, 6, 8\}$,

$B = \{1, 3, 5, 7, 9\}$, $C = \{2, 3, 4, 5)$ and $D = \{1, 6, 7)$, list the elements in the following

i) $A \cup C$, ii) $A \cap B$, iii) $\overline{C}$, iv) $(\overline{C} \cap D) \cup B$,

v) $(\overline{S \cap C})$, vi) $A \cap C \cap \overline{D}$

6.6 a) A pair of dice is rolled. List the elements of the sample space S. Let A denote the
"the sum is less than 5" and B the event "a 6 occurs on either die". List the elem
corresponding to event A and to event B.

b) Two dice are rolled. Let A be the event that the sum of dots on the faces show
and B the event that there is at least one 3 shown. Describe $A \cup B$; $A \cap B$; A –
$(A \cap \overline{B}) \cup \overline{A}$.

6.7 a) Enumerate all the possible (i) combinations and (ii) permutation of 3 letters cho
the four letters A, B, C and D.

b) A club consists of 15 members. In how many ways can;

i) the three officers; president, vice-president, and secretary-treasurer, be chose
ii) a committee of three members be selected?

6.8 How many different bridge hands can be selected from an ordinary deck of 52 play

6.9 We have n = 10 persons and we wish to divide them at random into 3 groups consis
and 2 persons respectively. In how many ways is this possible?

6.10 a) Define the terms: Experiment, Outcome, Event, Sample Space, Simple and C
Events, Mutually Exclusive Events.

b) Show that a sample space consisting of 4 elements has $2^4$ different events.

c) Prove that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$.

using the fact that a sample space with n elements has $2^n$ subsets

11 Distinguish among classical or *a priori* probability, relative frequency or a posteriori probability, axiomatic probability and subjective or personalistic probability. What is the disadvantage of each? Why do we study probability theory?

12 Explain what is wrong with each of the following statements;

a) An investment counselor claims that the probability that a stock's price will go up is 0.60, remain unchanged is 0.38, or go down is 0.25.

b) If two coins are tossed, there are 3 possible outcomes: 2 heads, one head and one tail, and 2 tails. Hence probability of each of these outcomes is $\frac{1}{3}$.

c) The probabilities that a certain truck driver would have no, one and two or more accidents during the year are 0.90, 0.02, and 0.09.

d) $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{4}$, $P(C) = \frac{1}{6}$ for the probabilities of three mutually exclusive events A, B and C.

Find the probability for each of the following events:

i) An odd number appears in a single toss of a fair die.
ii) The sum 8 appears in a single toss of a pair of fair dice.
iii) At least one head appears in three tosses of a fair coin.
iv) A king, ace, jack of clubs or queen of diamonds appears in drawing a single card from a well-shuffled ordinary deck of 52 cards. (P.U., B.A./B.Sc. 1970)

a) Describe the classical, relative frequency, and subjective concepts of probability.

b) A marble is drawn at random from a box containing 10 red, 30 white, 20 blue and 15 orange marbles. Find the probability that it is (i) orange or red, (ii) not-'red or blue', (iii) not blue, (iv) white, (v) red, white or blue. (P.U., B.A./B.Sc. 1970)

a) If two dice are thrown, what are the various total number of dots that may turn up? What are the probabilities of each of them? What is the probability that the number of dots will total at least four?

(b) Show that in a single throw of two dice, the probability of throwing more than 7 is equal to that of throwing less than 7, and hence find the probability of throwing exactly 7. State clearly what assumptions you are making. (P.U., B.A./B.Sc. 1981)

a) Two dice are thrown. Let A be the event that the sum of the upper face numbers is odd, and B the event of at least one ace. Assuming a sample space of 36 points, list the sample points which belong to the events $A \cap B$, $A \cup B$ and $A \cap \overline{B}$. Find the probabilities of these events, assuming equally likely events.

b) Two good dice are rolled simultaneously. Let A denote the event "the sum shown is 8" and B the event "the two show the same number." Find $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A \cup B)$.

A box contains six discs numbered 1 to 6. Find for each integer k from 3 to 11, the probability that the numbers on two discs drawn without replacement have a sum equal to k. (P.U., B.A./B.Sc. 1977)

6.18 In a single throw of two fair dice, find the probability that the product of the numbers on dice is (i) between 8 and 16 (both inclusive), (ii) divisible by 4. (P.U., B.A./B.Sc. 1

6.19 a) Elaborate the statement that "two mutually exclusive events need not be equally li by giving suitable examples.

b) Compare the probabilities of at least one 6 in 4 tosses of a fair die with the probab at least one double 6 in 24 tosses of two fair dice.

c) Compare the probability of a total of 9 with that of a total of 10 when three fair d tossed once.

*Hint*: To find the sample points in different events, combine the faces of the third di relevant sums of the first two dice.

6.20 a) From a pack of 52 cards, two are drawn at random. Find the probability that one is and the other a queen. (P.U., B.A./B.Sc

b) A set of eight cards contains one joker. *A* and *B* are two players and A chooses 5 ca random, B taking the remaining 3 cards. What is the probability that A has the jok

6.21 a) Of 12 eggs in a refrigerator, 2 are bad. From these 4 eggs are chosen at random to cake. What are the probabilities that (i) exactly one is bad? (ii) at least one is bad?

b) A certain carton of eggs has 3 bad eggs and 9 good eggs. An omelette is made of randomly chosen from the carton. What is the probability that there are (i) no ba (ii) at least 1 bad egg, (iii) exactly 2 bad eggs in the omelette?

Imp 6.22 a) An integer between 3 and 12 inclusive is chosen at random. What is the probabil is an even number? That it is an even and divisible by 3?

b) Three distinct integers are chosen at random from the first 20 positive integers. C the probability that (i) their sum is even, (ii) their product is even.

c) A box contains 4 red, 4 white and 5 green balls. Three balls are drawn from together. Find the probability that they may be (i) all of different colours, (ii) a same colours.

Imp 6.23 a) Find the probability of obtaining *at least* one 6 when (i) 5 dice are thrown, (ii) thrown.

b) How many dice must be thrown so that the probability of obtaining *at least* least 0.99?

c) A missile is fired at a target and the probability that the target is hit is 0.7. Find missiles should be fired so that the probability that the target is hit *at least* once than 0.995.

6.24 A bag contains 14 identical balls, 4 of which are red, 5 black and 5 white. Six balls from the bag. Find the probability that (i) 3 are red, (ii) *at least* two are white. (P.U., B.A. Hons. in E

6.25 a) Three applicants are to be selected at random out of 4 boys and 6 girls. probability of selecting (i) all girls, (ii) all boys, (iii) *at least* one boy?

b) From a group of 6 men and 8 women, 5 people are chosen at random. Find the probability that there are more men chosen than women.

6.26 A firm buys 3 shipments of parts each month. The purchasing agent selects at random from among four in-state suppliers and six out-of-state suppliers. What is the probability that the orders are placed with

i) the in-state suppliers only?

ii) the out-of-state suppliers only?

iii) *at least* one in-state supplier? (P.U., M.A. Econ. 1979)

6.27 a) Using a sample space or otherwise, show that for any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

b) A class contains 10 men and 20 women of which half the men and half the women have brown eyes. Find the probability that a person chosen at random is a man or has brown eyes. (P.U., B.A./B.Sc. 1974)

c) In a group of 20 adults, 4 out of the 7 women and 2 out of the 13 men wear glasses. What is the probability that a person chosen at random from the group is a woman or someone who wear glasses?

6.28 a) The events $E_1$ and $E_2$ are neither independent nor mutually exclusive. Denote by $p_{12}$ the probability that $E_1$ and $E_2$ both happen. Prove that the probability that at least one of $E_1$ and $E_2$ happens, is $p_1 + p_2 - p_{12}$. (P.U., B.A./B.Sc., 1975)

b) One integer is chosen at random from the numbers 1, 2, 3, .... , 50. What is the probability that the chosen number is divisible by 6 or by 8?

a) Define the probability of an event.

b) State and prove the addition law of probability for any two events A and B.

c) A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a bolt?

a) *Define Mutually Exclusive Events*. State and prove the theorem of addition of probabilities concerning mutually exclusive events.

b) What is the probability of throwing either 7 or 11 with two dice?

c) If A and B are mutually exclusive events and $P(A) = 0.4$ and $P(B) = 0.5$, find (i) $P(A \cup B)$, (ii) $P(\overline{A})$.

a) If A and B are any two events defined on a sample space S, show that $P[(A \cap \overline{B}) \cup (B \cap \overline{A})] = P(A) + P(B) - 2P(A \cap B)$. (P.U., B.A./B.Sc. 1989)

b) Let A and B be events with $P(A \cup B) = \frac{3}{4}$, $P(\overline{A}) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{4}$, find (i) $P(A)$, (ii) $P(B)$, (iii) $P(A \cap \overline{B})$.

c) If $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4}$ and $P(\overline{B}) = \frac{5}{8}$, then find (i) $P(A \cap B)$, (ii) $P(\overline{A} \cap \overline{B})$, (iii) $P(\overline{A} \cup \overline{B})$ and (iv) $P(B \cap \overline{A})$.

6.32 Using the Venn diagram, show that

i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

(P.U., B.A./B.Sc., 1976, 7

6.33 a) Explain what is meant by conditional probability.

b) A pair of fair dice is thrown. If the two numbers appearing are different, find the probability that (i) the sum is 6, (ii) the sum is four or less. (P.U., M.A. Econ. 197

Imp 6.34 a) A box contains 4 bad and 6 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

b) If two balanced dice are rolled, find the conditional probability that sum of dots will be 7 given that it is odd.

c) In a firm, 20 percent of the employees have accounting background, while 5 percent of the employees are executives and have accounting background. If an employee has an accounting background, what is the probability that the employee is an executive?

6.35 A box contains 10 red and 12 white rose flowers. Flowers are picked up at random one by one without replacement. What is the probability that;

i) the first 3 flowers are red?

ii) there are 2 red and 2 white flowers in the first four picked up?

iii) the third one is red given that the first 2 are white? (P.U., B.A./B.Sc., 198

6.36 a) State and prove the multiplication law of probabilities for any two events A and B.

b) Define Independent Events and Dependent Events. Give examples.

c) Given $P(A) = 0.60$, $P(B) = 0.40$, $P(A \cap B) = 0.24$, find $P(A/B)$, $P(A \cup B)$, $P(\overline{A}$ ... $P(B/A)$, $P(\overline{B})$. What is the relation between $A$ and $B$?

6.37 a) Differentiate between *independent* and *mutually exclusive* events. Are independent events mutually exclusive?

b) Let $A$ and $B$ be two events associated with an experiment. Suppose that $P(A) = 0$... $P(A \cup B) = 0.7$. Let $P(B) = p$.

i) For what choice of $p$ are $A$ and $B$ mutually exclusive?

ii) For what choice of $p$ are $A$ and $B$ independent? (P.U., M.A. Stat.

c) Given $P(A) = 0.5$ and $P(A \cup B) = 0.6$, find $P(B)$ if

i) A and B are mutually exclusive. ii) A and B are independent.

iii) $P(A/B) = 0.4$

6.38 Indicate whether each of the following statements is true or false. If false, indicate why.

a) If $P(A/B) = 0$, then $A$ and $B$ are mutually exclusive.

b) If $P(A/B) = 0$, then $A$ and $B$ are independent.

c) If $P(A/B) = P(B/A)$, then $P(A) = P(B)$.

d) If $A$ and $B$ are independent, then $P(A) = P(B)$.

6.39 a) State and prove the multiplication law of probabilities for independent and dependent events.

b) Let $A$ and $B$ be two independent events such that the probability is $\frac{1}{8}$ that they will occur simultaneously and $\frac{3}{8}$ that neither of them will occur. Find $P(A)$ and $P(B)$.

(B.Z.U. & P.U., B.A/B.Sc. 1976)

6.40 Two dice are cast: $E_1$ is the event that a6 appears on at least one die, $E_2$ is the event that a 5 appears on exactly one die and $E_3$ is the event that same number appears on both dice.

i) Are $E_1$ and $E_2$ independent?

ii) Are $E_2$ and $E_3$ independent?

iii) Are $E_3$ and $E_1$ independent? (P.U., B.A/B.Sc. 1980)

a) A can solve 75% of the problems in this book and B can solve 70%. What is the probability that either A or B can solve a problem chosen at random?

b) Three cards are drawn in succession without replacement from an ordinary deck of playing cards. Find the probability that the first card is a red ace, the second card is a ten or jack, and the third card is greater the 3 but less than 7.

The probability that A will be alive after 10 years to come is 5/7 and for B it is 7/9. Find out the probability that (i) both of them will die, (ii) A will be alive and B dead, (iii) B will be alive and A dead, (iv) both of them will be alive, in 10 years to come.

a) A bag contains 3 red and 5 black balls and another 4 red and 7 black balls. A ball is drawn from a bag selected at random. Find the probability that it is red.

b) One urn contains 3 white and 2 black balls, another contains 5 white and 3 black balls. If an urn is chosen at random and a ball is taken from it, what is the probability that it is white?

Two drawings each of three balls are made from a bag containing 5 white and 8 black balls; the balls are not being replaced before the next trial. What are the probabilities that the first drawing will give 3 white balls and the second 3 black balls?

a) Show that the multiplication law $P(A \cap B) = P(A/B)\,P(B)$, established for two events, may be generalized to three events as follows;

$$P(A \cap B \cap C) = P(A/B \cap C)\ P(B/C)\,P(C)$$

b) A farmer has a box containing 30 eggs, 5 of which have blood spots. He checks eggs by taking them at random one after another from the box. What is the probability that the first two eggs have spots and the third will be clear?

6.46 Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child is selected at random from each group. Show that the probability that the three selected consists of 1 girl and 2 boys is 13/32.

6.47 There are three families, each having four children; 2 boys and 2 girls; 3 boys and 1 girl; and 1 boy and 3 girls. A child from each family is invited to a party. Find the probability (i) that only girls turn up for the party, (ii) that two girls and one boy turn up for the party.

6.48 The odds that a book will be favourably reviewed by three independent critics are 3 to 2, 4 to 3 and 2 to 3 respectively. What is the probability that of three reviews a majority will be favourable?

*Hint*: If we are given the odds that an event $A$ will occur, as $a$ to $b$, then $p = \frac{a}{a+b}$ and $q = \frac{b}{a+b}$.

6.49 a) A can hit a target four times in 5 shots; B three times in 4 shots; C twice in 3 shots. They fire a volley. What is the probability that two shots at least hit?

b) A committee of three –A, B, and C, – is to make a decision on the basis of a majority vote. What is the probability of a wrong decision by the committee if the probabilities of a wrong decision by each member are $P(A) = 0.05$, $P(B) = 0.05$, and $P(C) = 0.10$?

(P.U., B.A/B.Sc. ...)

6.50 The probability that three men hit a target are respectively $\frac{1}{6}, \frac{1}{4}$ and $\frac{1}{3}$. Each shoots once at the target.

i) Find the probability that exactly one of them hits the target.

ii) If only one hit the target, what is the probability that it was the first man?

(P.U., B.A/B.Sc. ...)

6.51 Three missiles are fired at a target. If the probabilities of hitting the target are 0.4, 0.5 and ... respectively, and if the missiles are fired independently, what is the probability?

i) that all the missiles hit the target?

ii) that at leas one of the three hits the target?

iii) that exactly one hits the target?

iv) that exactly 2 hit the target?

6.52 $A$ and $B$ play 12 games of chess, of which 6 are won by $A$, 4 are won by $B$, and 2 end in a tie. They agree to play a tournament consisting of 3 games. Find the probability that (a) A wins all three games, (b) two games end in a tie, (c) $A$ and $B$ win alternately, (d) $B$ wins at least one game. (P.U., B.A/B.Sc. ...)

6.53 The contents of two urns are as follows:

Urn A contains 3 red and 2 white balls.
Urn B contains 2 red and 5 white balls.

An urn is selected at random; a ball is drawn and put into the other urn; then a ball is drawn from the second urn. Find the probability that both balls drawn are of the same colour.
*Hint*: Construct the tree diagram.

6.54 A man invited 5 friends. He was born in April as also all the invited friends. What is the probability that none of the friends was born on the same day of the month as the host. (C.S.S., 1962)

6.55 a) What are respective chances of winning of A and B who toss a coin alternately on the understanding that the first to obtain heads, wins the toss?

b) Three men toss in succession for a prize to be given to the one who first obtains heads. Show that their respective chances of winning are 4/7, 2/7 and 1/7.

6.56 a) Find the probability of getting exactly 4 heads when 6 coins are tossed. (B.Z.U., B.A./B.Sc. 1976)

b) Sixteen coins are tossed once. What is the probability of obtaining (i) exactly 8 heads, (ii) exactly 11 heads?

The national pass rate for an examination is 40%. A school enters 6 candidates. Calculate the probability that (i) 2 candidates will pass, and (ii) 5 candidates will pass. Explain why the probability of all passing is not equal to the probability of all failing.

a) State and prove Bayes' theorem.

b) Three urns of the same appearance have the following proportions of white and black balls.

Urn $A$: 1 white, 2 black balls.
Urn $B$: 2 white, 1 black ball.
Urn $C$: 2 white, 2 black balls.

One of the urns is selected and a ball is drawn from it. It turns out to be white. What is the probability that urn $C$ was chosen? (P.U., B.A., (Hons.) – Part III, 1965, B.A/B.Sc. 2007)

There are three coins, identical in appearance, one of which is ideal and the other two biased with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively for a head. One coin is taken at random and tossed twice. If a head appears both the times, what is the probability that the ideal coin was chosen?

In a certain college, 4% of the men and 1% of the women are taller than 6 feet. Further more, 60% of the students are women. Now if a student is selected at random and is taller than 6 feet, what is the probability that the student is a woman? (P.U., B.A. Hons. 1974)

Three cooks $A$, $B$ and $C$, bake a special kind of cake, and with respective probabilities 0.02, 0.03 and 0.05 it fails to rise. In the restaurant where they work, $A$ bakes 50 percent of these cakes, $B$ 30 percent, and $C$ 20 percent. What proportions of "failures" is caused by $A$?

6.62 The stock of a warehouse consists of boxes of high, medium and low quality lightbulbs in respective proportions 1:2:2. The probabilities of bulbs of three types being unsatisfactory are 0.0, 0.1 and 0.2 respectively. If a box is chosen at random and two bulbs in it are tested and found to be satisfactory, what is the probability that it contains bulbs (i) of high quality, (ii) of medium quality, (iii) of low quality?

6.63 A patient is thought to have one of three diseases $A_1, A_2$ and $A_3$ whose probabilities under the given conditions are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$ respectively. A test is carried out to help the diagnosis and it yields a positive result with a probability of 0.1 for disease $A_1$, a probability of 0.2 for disease $A_2$ and a probability of 0.9 for disease $A_3$. The test is conducted 5 times and the results are positive 4 times and negative once. What is the probability of each disease after testing?

(P.U., B.A. (Hons.) – Part-III, 1967)

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# CHAPTER 7

# RANDOM VARIABLES

# RANDOM VARIABLES

## INTRODUCTION

Usually, we are not interested in a particular outcome of a random experiment but our interest ... to some *numerical* description of the outcome. For example, when two coins are tossed, we may ...ested only in the *number* of heads which appear and not in the actual sequence of heads and tails ... is not a numerical quantity. To express the outcomes in numbers, we assign to each non-numerical ...e of the sample space S = {*HH, HT, TH, TT*}, one of the *numbers* $i$ ($i = 0, 1, 2$) corresponding to ...ber of heads appearing. That is, we express the outcomes in terms of numerical values as

| | |
|---|---|
| Domain or Sample Space ($E_i$): | ($HH$),($HT$),($TH$),($TT$) |
| Range or Corresponding Value $X = f(E)$: | 2, 1, 1, 0 |

Again in the experiment of throwing a pair of dice, if we are interested only in the sum of the dots ... upper faces of the two dice and not in the particular dots, we assign to each possible outcome of ...eriment, one of *numbers* $i$ ($i = 2, 3, 4, \ldots, 12$) corresponding to the *sum* of dots appearing on their ...

It is clear that the *numbers* 0, 1, 2 and 2, 3, 4, …, 12 in the above cited examples, are random ...s determined by the outcomes of the random experiments. Such a numerical quantity whose ... is determined by the outcome of a random experiment, is called a *random variable*. ...ically, we assign a single real number to each outcome of the sample space, and hence we state ...ndom variable is a real-valued function defined on a sample space*. Thus the number of heads ... in tossing of two coins and the *sum* of the dots obtained with a pair of dice in the above ...s are the values of random variables.

It should be noted that in the above definition, a *function* has been named a *random variable*. This ...ogy, though inappropriate and somewhat unfortunate, is universally accepted and used.

A random variable is also called a *chance variable*, a *stochastic variable* or simply a *variate* and is ...ed as *r.v.* The random variables are usually denoted by capital Latin letters such as $X$, $Y$, $Z$; ... values taken by them are represented by the corresponding small letters such as $x$, $y$, $z$. It is to ...sized that more than one *r.v.* can be defined on the same sample space. There are two types of ... *discrete* and the *continuous*.

## ...STRIBUTION FUNCTION

The *distribution function* of a random variable $X$, denoted by $F(x)$, is defined by $F(x) = P(X \leq x)$. ...tion $F(x)$ gives the probability of the event that $X$ takes a value *less than* or *equal to* a specified ... The distribution function is abbreviated to d.f. and is also called the *cumulative distribution* (*cdf*) ... cumulative probability function of the $X$ from the smallest upto specific values of $x$.

...ce $F(x)$ is a probability, it is quite obvious that

$$F(-\infty) = P(\phi) = 0 \quad \text{and} \quad F(+\infty) = P(S) = 1.$$

Let $a$ and $b$ be two real numbers such that $a < b$. Then the probability of the interval $(a, b]$ is

$$F(b) - F(a) = P(X \leq b) - P(X \leq a)$$
$$= P(a < X \leq b),$$

which is non-negative and hence $F(x)$ is a non-decreasing function of $x$.

Again $\lim_{h \to 0} F(x+h) = F(x)$, *i.e.* the function $F(x)$ is continuous on the right at each value of

A *d.f.* $F(x)$ thus has the following properties:

i) $F(-\infty) = 0, F(+\infty) = 1$.

ii) $F(x)$ is a non-decreasing function of $x$, *i.e.* $F(x_1) \leq F(x_2)$ if $x_1 \leq x_2$.

iii) $F(x)$ is continuous at least on the right of each $x$.

All random variables have distribution functions. Distribution functions for differen distinguished by using the notation $F_x$, $F_y$, etc.

## 7.3 DISCRETE RANDOM VARIABLE AND ITS PROBABILITY DISTRIBUTI

A random variable $X$ is defined to be *discrete* if it can assume values which are finite or infinite. When $X$ takes on a finite number of values, they may be listed as $x_1, x_2, \ldots, x_n$. countably infinite case, the values may be listed as $x_1, x_2, x_3, \ldots, x_n, \ldots$ The number of heads coin tossing experiments, the number of defective items observed in a consignment, the number accidents, the number of bacteria in 1cc of water, etc. are the examples of discrete *r.v.*

Lat $X$ be a discrete *r.v.* taking on distinct values $x_1, x_2, \ldots, x_n, \ldots$ Then the function, deno or $f(x)$, and defined by

$$f(x_i) = P(X = x_i) \quad \text{for } i = 1, 2, \ldots, n, \ldots$$
$$= 0, \quad \text{for } x \neq x_i$$

is called the *probability function* (*pf*) of the r.v. $X$, and the values $x_1, x_2, \ldots, x_n, \ldots$ among which probability 1 is distributed, are called the *probability points* or *jump points*, $P(X = x_i)$ probability that the discrete r.v. $X$ takes the value $x_i$.

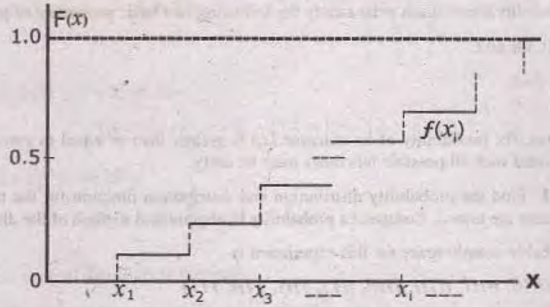
The distribution function for a discrete r.v. is

$$F(x) = \sum_i f(x_i),$$

where sum is taken over all $x_i$ that are *less than* or *equal to* x.

Letting $x$ tend to $+\infty$, we have $F(+\infty) = \sum_i f(x_i) = 1$.

$F(x)$ in case of a discrete r.v. is a *step function*. That is, its graph consists of hor segments between any two successive values and has a *step* or *jump* of height $f(x_i)$, at each val figure on page 229). It should be noted that $F(x)$ is continuous but between jumps, it is constan




A *discrete* r.v. may also be defined as a r.v. whose d.f. *jumps* at the possible values of $X$ and remains constant between adjacent jump points. The height of a jump at each point $x$ is the probability of the event $X = x$.

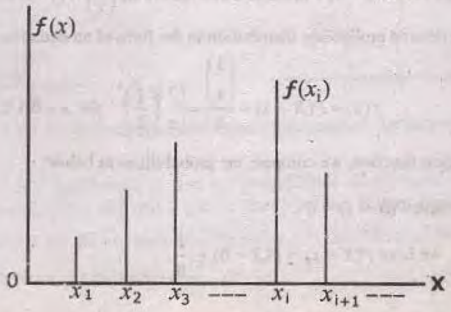
Thus $$f(x_i) = [\text{jump at } x_i] = P(X = x_i).$$

The set whose elements are the ordered pairs $[x_i, f(x_i)]$, $i = 1, 2, \ldots$ defines the *probability function*. Some writers do not make any distinction between the terms *probability function* and *probability distribution* but they use them interchangeably. The probability distribution of a r.v. may conveniently be expressed *either* in a tabular form by showing all the possible values of $X$ and the associated probabilities $f(x_i) = P(X = x_i)$ as

| Values $(x_i)$ | $x_1$ | $x_2$ | ... | $x_n$ | ... |
|---|---|---|---|---|---|
| Probability $f(x_i)$ | $f(x_1)$ | $f(x_2)$ | ... | $f(x_n)$ | ... |

or the form of an equation for $f(x)$ with a list of the possible values of $X$.

The graph of a probability distribution is obtained by locating the values $x_1, x_2, \ldots, x_i, \ldots$ along the horizontal axis and drawing vertical lines of heights equal to $f(x_1), f(x_2), \ldots, f(x_i), \ldots$ above them. A probability distribution can also be graphically displayed by a probability histogram.



A probability distribution must satisfy the following two basic properties of probability:

i) $f(x_i) \geq 0$, for all $i$.

ii) $\sum_i f(x_i) = 1$.

In other words, the probability of an outcome $(x_i)$ is greater than or equal to zero and the sum probabilities associated with all possible outcomes must be unity.

**Example 7.1** Find the probability distribution and distribution function for the number of when 3 balanced coins are tossed. Construct a probability histogram and a graph of the distribution.

The equiprobable sample space for this experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Let $X$ be the r.v. that denotes the number of heads. Then the values of $x$ are 0, 1, 2 and 3, and probabilities are:

$$f(0) = P(X = 0) = P[\{TTT\}] = \frac{1}{8},$$

$$f(1) = P(X = 1) = P[\{HTT, THT, TTH\}] = \frac{3}{8}$$

$$f(2) = P(X = 2) = P[\{HHT, HTH, THH\}] = \frac{3}{8}$$

$$f(3) = P(X = 3) = P[\{HHH\}] = \frac{1}{8}$$

Putting this information in the tabular form, we obtain the desired probability distribution of

| Number of heads $(x_i)$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability $f(x_i)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

To obtain a formula, we need expressions for $x$ heads to be selected out of 3 heads and denominators for all values. Now, $x$ heads can be selected in $\binom{3}{x}$ ways and the total number of is 8. Therefore the desired probability distribution in the form of an equation is

$$f(x) = P(X = x) = \frac{\binom{3}{x}}{8} = \binom{3}{x}\left(\frac{1}{2}\right)^3 \quad for\ x = 0, 1, 2, 3.$$

For distribution function, we compute the probabilities as below:

If $x < 0$, we have $P(X < x) = 0$

If $0 \leq x < 1$, we have $P(X < x) = P(X = 0) = \frac{1}{8}$.

For $1 \leq x < 2$, we have

$$P(X < x) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$$

Similarly, for $2 \leq x < 3$, we have

$$P(X < x) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{4}{8} + \frac{3}{8} = \frac{7}{8}$$

Finally for $x \geq 3$, we have

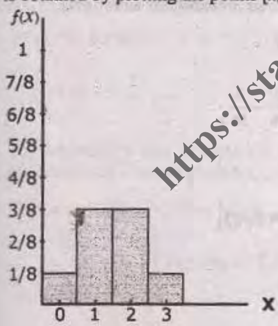
$$P(X < x) = \sum_{i=0}^{3} P(X = i) = 1.$$

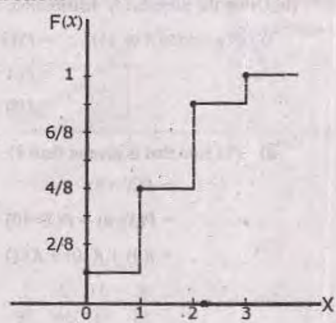
Hence the desired distribution function is

$$F(x) = \begin{cases} 0 \;, & \text{for } x < 0 \\ \frac{1}{8} \;, & \text{for } 0 \leq x < 1 \\ \frac{4}{8} \;, & \text{for } 1 \leq x < 2 \\ \frac{7}{8} \;, & 2 \leq x < 3 \\ 1 \;, & \text{for } x \geq 3 \end{cases}$$



The probability histogram is obtained by plotting the points $[x, f(x)]$, while the graph of distribution tion is obtained by plotting the points $[x, F(x)]$ as shown below:




**Examples 7.2 (a)** Find the probability distribution of the sum of the dots when two fair dice are

(b) Use the probability distribution to find the probabilities of obtaining (i) a sum of 8 or 11, sum that is greater than 8, (iii) a sum that is greater than 5 but less than or equal to 10.

(a) The sample space $S$ for the experiment of throwing two dice contains 36 sample points, which ly likely, i.e. each point has probability $\frac{1}{36}$.

Let $X$ be the random variable representing the sum of dots which appear on the dice. Then values of the r.v. are 2, 3, 4, ..., 12. The probabilities of these values are computed as below:

$f(2) = P(X = 2) = P[\{1,1\}] = \frac{1}{36}$, as there is only one point resulting in a sum of 2,

$$f(3) = P(X = 3) = P[\{(1,2),(2,1)\}] = \frac{2}{36},$$

$$f(4) = P(X = 4) = P[\{(1,3),(2,2),(3,1)\}] = \frac{3}{36},$$

Similarly, $f(5) = \frac{4}{36}$, $f(6) = \frac{5}{36}$, $f(7) = \frac{6}{36}$, $f(8) = \frac{5}{36}$, $f(9) = \frac{4}{36}$,

$f(10) = \frac{3}{36}$, $f(11) = \frac{2}{36}$ *and* $f(12) = \frac{1}{36}$.

Therefore the desired probability distribution of the r.v. $X$ is

| $x_i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x_i)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

It is interesting to note that this result may also be expressed by the equation as

$$f(x) = \frac{6-|7-x|}{36}, \quad \text{for } x = 2, 3, 4, \ldots, 12.$$



(b) Using the probability distribution, we get the required probabilities as follows:

i) $P$(a sum of 8 or 11) $= P[(X=8) \text{ or } (X=11)]$

$= P(X=8) + P(X=11)$

$= f(8) + f(11) = \frac{5}{36} + \frac{2}{36} = \frac{7}{36}$

ii) $P$(a sum that is greater than 8)

$= P(X > 8)$

$= P(X=9) + P(X=10) + P(X=11) + P(X=12)$

$= f(9) + f(10) + f(11) + f(12)$

$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}$

iii) $P$(a sum that is greater than 5 but less than or equal to 10)

$= P(5 < X \leq 10)$

$= P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$

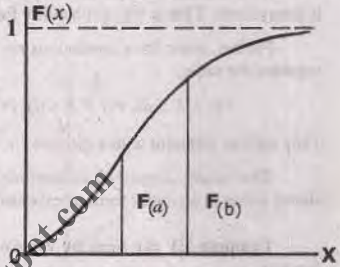
$= f(6) + f(7) + f(8) + f(9) + f(10)$

$= \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} = \frac{23}{36}$

## CONTINUOUS RANDOM VARIABLE AND ITS PROBABILITY DENSITY FUNCTION

A random variable $X$ is defined to be *continuous* if it can assume every possible value in an interval (a, b), a < b, where $a$ and $b$ may be $-\infty$ and $+\infty$ respectively. The height of a person, the temperature at ace, the amount of rainfall, time to failure for an electronic system, etc. are examples of continuous m variable.

A r.v. $X$ may also be defined as *continuous* if its d.f. $F(x)$ is continuous and is differentiable where except at isolated points in the given range. The graph of $F(x)$ has no jumps or steps but is a ous function for all $x$.




Let the derivative of $F(x)$ be denoted by $f(x)$, i.e.

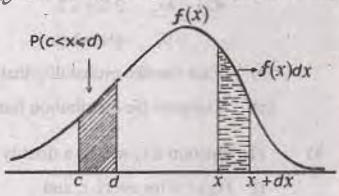
$$\frac{dF(x)}{dx} = f(x)$$

Since $F(x)$ is a non-decreasing function of $x$, we have

(i) $f(x) \geq 0$,

(ii) $F(x) = \int_{-\infty}^{x} f(x)\,dx$, for all $x$.

The function $f(x)$ is called the *probability density function*, abbreviated to p.d.f., or simply *density* n of the r.v. $X$.




A p.d.f. has the following properties:

(i) $f(x) \geq 0$, for all $x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) The probability that $X$ takes on a value in the interval $[c, d]$, $c < d$ is given by

$$P(c < x \leq d) = F(d) - F(c)$$

$$= \int_{-\infty}^{d} f(x)dx - \int_{-\infty}^{c} f(x)dx$$

$$= \int_{c}^{d} f(x)dx$$

is the *area* under the curve $y = f(x)$ between $X = c$ and $X = d$. e figure above).

In other words, $f(x)$ is a non-negative function, the integration takes place over all possible values r.v. $X$ between the specified limits and the probabilities are given by appropriate areas under the

Also $\quad P(x < X \le x + dx) = F(x + dx) - F(x)$

$$\cong f(x)dx$$

The quantity $f(x)dx$ is called the *probability differential* or *probability element* (p.e.) of $X$.

Since $\quad P(X = k) = \int_k^k f(x)\,dx = 0,$

it should therefore the noted that the probability of a continuous r.v. $X$ taking any particular [value] is always zero. That is why probability for a continuous r.v. is measurable only over a given interv[al].

Further, since for a continuous r.v. $X$, $P(X=x) = 0$ for every $x$, the following four probab[ilities are] regarded the same:

$$P(c \le X \le d),\ P(c < X \le d),\ P(c \le X < d) \text{ and } P(c < X < d).$$

They may be different with a discrete r.v.

The values (expressed as intervals) of a continuous r.v. and their associated probabiliti[es are] shown either in a tabular form or expressed by means of a formula.

**Example 7.3** (a) Find the value of $k$ so that the function $f(x)$ defined as follows, may be a [density] function

$$f(x) = kx, \quad 0 \le x \le 2$$
$$= 0, \quad \text{elsewhere}$$

(b) Find also the probability that both of two sample values will exceed 1.

(c) Compute the distribution function $F(x)$.

a) The function f(x) will be a density function, if

i) $f(x) \ge 0$ for every $x$, and

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

The first condition is satisfied when $k \ge 0$. The second condition will be satisfied, if $\int_{-\infty}^{\infty} f$

i.e. if $\quad 1 = \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx$

i.e. if $\quad 1 = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} kx\,dx + \int_{2}^{\infty} 0\,dx$

i.e. if $\quad 1=0+\left[k\dfrac{x^2}{2}\right]_0^2+0=2k$

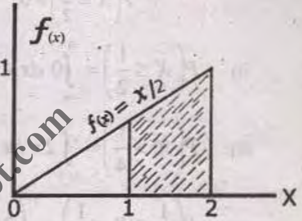
This gives $k=\dfrac{1}{2}$

Hence $f(x)=\begin{cases}\dfrac{x}{2}, & \text{for } 0\le x\le 2\\ 0, & \text{elsewhere}\end{cases}$

$P(X>1)$ = areas of shaded region

$$=\int_1^2 f(x)\,dx$$

$$=\int_1^2 \frac{x}{2}\,dx=\left[\frac{x^2}{4}\right]_1^2=\frac{3}{4}$$




$\therefore\quad P(\text{two sample values exceeding one})=\dfrac{3}{4}\times\dfrac{3}{4}=\dfrac{9}{16}$

To compute the distribution function, we find

$$F(x)=P(X<x)=\int_{-\infty}^{x} f(x)\,dx$$

... $x$ such that $-\infty<x\le 0, F(x)=\displaystyle\int_{-\infty}^{x} 0\,dx=0,$

... $x\le 2$, we have $F(x)=\displaystyle\int_{-\infty}^{0} 0\,dx+\int_0^x\left(\frac{x}{2}\right)dx=\left[\frac{x^2}{4}\right]_0^x=\frac{x^2}{4},$

...lly for $x>2$, we have $F(x)=\displaystyle\int_{-\infty}^{0} 0\,dx+\int_0^2\frac{x}{2}\,dx+\int_2^{\infty}0\,dx=1$

... $F(x)=0\;,\qquad \textit{for } x<0$

$\qquad=\dfrac{x^2}{4},\qquad \textit{for } 0\le x\le 2$

$\qquad=1\;,\qquad \textit{for } x>2$

**...ample 7.4** A r.v. $X$ is of continuous type with p.d.f.

$f(x)=2x,\quad 0<x<1,$

$\quad=0,\quad$ elsewhere.

Find (i) $P\left(X=\frac{1}{2}\right)$, (ii) $P\left(X\le\frac{1}{2}\right)$, (iii) $P\left(X>\frac{1}{4}\right)$, iv) $P\left(\frac{1}{4}\le X<\frac{1}{2}\right)$

(v) $P\left(X\le\frac{1}{2}\mid\frac{1}{3}\le X\le\frac{2}{3}\right)$.

Clearly $f(x)\ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx=\int_0^1 2x\,dx=1.$

i) Since $f(x)$ is a continuous probability function, therefore

$$P\left(X=\frac{1}{2}\right)=0.$$

ii) $P\left(X\le\frac{1}{2}\right)=\int_{-\infty}^{0} 0\,dx+\int_0^{1/2} 2x\,dx=0+[x^2]_0^{1/2}=\frac{1}{4}$

iii) $P\left(X>\frac{1}{4}\right)=\int_{1/4}^{1} 2x\,dx+\int_1^{\infty} 0\,dx=[x^2]_{1/4}^{1}+0=\frac{15}{16}$

iv) $P\left(\frac{1}{4}\le X<\frac{1}{2}\right)=\int_{1/4}^{1/2} 2x\,dx=[x^2]_{1/4}^{1/2}=\frac{3}{16}$

v) Applying the definition of conditional probability, we get

$$P(X\le\frac{1}{2}\mid\frac{1}{3}\le X\le\frac{2}{3})=\frac{P\left(\frac{1}{3}\le X\le\frac{1}{2}\right)}{P\left(\frac{1}{3}\le X\le\frac{2}{3}\right)}=\frac{\int_{1/3}^{1/2}2x\,dx}{\int_{1/3}^{2/3}2x\,dx}$$

$$=[x^2]_{1/3}^{1/2}\div[x^2]_{1/3}^{2/3}$$

$$=\frac{5}{36}\times\frac{9}{3}=\frac{5}{12}.$$



**Example 7.5** A continuous r.v. $X$ has the *d.f.* $F(x)$ as follows:

$$
\begin{aligned}
F(X) &= 0, && \text{for } x<0,\\
&=\frac{2x^2}{5}, && \text{for } 0<x\le 1,\\
&=-\frac{3}{5}+\frac{2}{5}\left(3x-\frac{x^2}{2}\right), && \text{for } 1<x\le 2,\\
&=1 && \text{for } x>2.
\end{aligned}
$$

Find the p.d. and $P(|X|<1.5)$.

By definition, we have $f(x) = \dfrac{d}{dx}F(x)$.

Therefore
$$f(x) = \frac{4x}{5} \quad \text{for } 0 < x \le 1$$
$$= \frac{2}{5}(3-x) \quad \text{for } 1 < x \le 2$$
$$= 0 \quad \text{elsewhere.}$$

$P(|X| < 1.5) = P(-1.5 < X < 1.5)$

$$= \int_{-\infty}^{-1.5} 0\, dx + \int_{-1.5}^{0} 0\, dx + \int_{0}^{1} \frac{4x}{5} dx + \int_{1}^{1.5} \frac{2(3-x)}{5} dx$$

$$= 0 + 0 + \left[\frac{2x^2}{5}\right]_0^1 + \left[\frac{2}{5}\left(3x - \frac{x^2}{2}\right)\right]_1^{1.5}$$

$$= \frac{2}{5} + \frac{2}{5}\left[\left(4.5 - \frac{2.25}{2}\right) - \left(3 - \frac{1}{2}\right)\right]$$

$$= 0.40 + 0.35 = 0.75.$$



## OINT DISTRIBUTIONS

he distribution of two or more random variables which are observed simultaneously when an ment is performed, is called their *joint distribution*. It is customary to call the distribution of a r.v. as *univariate*. Likewise, a distribution involving two, three or many r.v.'s simultaneously is d to as *bivariate, trivariate* or *multivariate*.

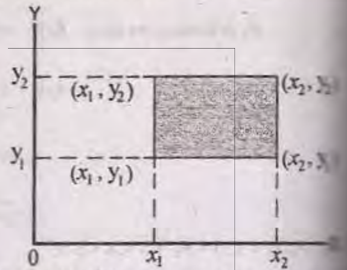
**5.1 Bivariate Distribution Function.** Let $X$ and $Y$ be two r.v.'s defined on the same sample Then the function $F(x, y)$ defined by $F(x, y) = P(X \le x$ and $Y \le y)$, where $F(x, y)$ gives the ity that $X$ will take on a value *less than* or *equal to* $x$ and, at the same time, $Y$ will take on a value or *equal to* $y$, is called a *bivariate* or *joint distribution function* of $X$ and $Y$.

bivaraite d.f. $F(x, y)$ possesses the following properties:

- $F(x, -\infty) = F(-\infty, y) = 0$, $F(+\infty, +\infty) = 1$
- $F(x, y)$ is a non-decreasing function of $x$ and $y$, and is continuous on the right.
- If $x_1 < x_2$ and $y_1 < y_2$, then

$$P(x_1 \le X < x_2;\ y_1 \le Y < y_2) = P(X<x_2, Y<y_2) - P(X<x_2, Y<y_1) - P(X<x_1, Y<y_2) + P(X<x_1, Y<y_1)$$
$$= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) > 0$$

The probability that a random point $(X, Y)$ falls in the interval $(x_1 \le X < x_2; y_1 \le Y < y_2)$ is shown graphically.




A bivariate distribution may be *discrete* when the possible values of $(X, Y)$ are finite or countably infinite. It is *continuous* if $(X, Y)$ can assume all values in some *non-countable set of the plane*. A bivariate distribution is said *mixed* when one r.v. is discrete and the other is continues.

**7.5.2 Bivariate Probability Function.** Let $X$ and $Y$ be two *discrete* r.v.'s defined on the sa sample space $S$, $X$ taking the values $x_1, x_2, \ldots, x_m$ and $Y$ taking the values $y_1, y_2, \ldots, y_n$. The probability that $X$ takes on the values $x_i$ and, at the same time, $Y$ takes on the value $y_j$, denoted by or $p_{ij}$, is defined to be the *joint probability function* or simply the *joint distribution* of $X$ and $Y$. Th *joint probability function*, also called the *bivariate probability function* $f(x, y)$ is a function whose the point $(x_i, y_j)$ is given by

$$f(x_i, y_j) = P(X = x_i \text{ and } Y = y_j), \qquad \begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n. \end{array}$$



The joint or bivariate probability distribution consisting of all pairs of values $(x_i, y_j)$ and their as probabilities $f(x_i, y_j)$ *i.e.* the set of triples $[x_i, y_j, f(x_i, y_j)]$ can either be shown in a two-way table as

*Joint Probability Distribution of X and Y*

| X \ Y | $y_1$ | $y_2$ | ... | $y_j$ | ... | $y_n$ | $P(X=x_i)$ |
|---|---|---|---|---|---|---|---|
| $x_1$ | $f(x_1, y_1)$ | $f(x_1, y_2)$ | .... | $f(x_1, y_j)$ | .... | $F(x_1, y_n)$ | $g(x_1)$ |
| $x_2$ | $f(x_2, y_1)$ | $f(x_2, y_2)$ | .... | $f(x_2, y_j)$ | .... | $f(x_2, y_n)$ | $g(x_2)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ |
| $x_i$ | $f(x_i, y_1)$ | $f(x_i, y_2)$ | ... | $f(x_i, y_j)$ | .... | $f(x_i, y_n)$ | $g(x_i)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ |
| $x_m$ | $f(x_m, y_1)$ | $f(x_m, y_2)$ | ... | $f(x_m, y_j)$ | ... | $f(x_m, y_n)$ | $g(x_m)$ |
| $P(Y=y_j)$ | $h(y_1)$ | $h(y_2)$ | | $h(y_j)$ | | $h(y_n)$ | 1 |

or be expressed by means of a formula for $f(x, y)$. The probabilities $f(x, y)$ can be obtained by s appropriate values of $x$ and $y$ in the table or formula.

A joint probability function has the following properties;

i) $f(x_i, y_j) > 0$, for all $(x_i, y_j)$, i.e. for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

ii) $\sum_i \sum_j f(x_i, y_j) = 1$

**7.5.3 Marginal Probability Functions.** From the joint probability function for $(X,$ obtain the individual probability function of $X$ and $Y$. Such individual probability functions *marginal probability function*.

Let $f(x, y)$ be the joint probability function of two discrete r.v.'s $X$ and $Y$. Then the marginal ...ability function of $X$ is defined as

$$g(x_i) = \sum_{j=1}^{n} f(x_i, y_j)$$

$= f(x_i, y_1) + f(x_i, y_2) + \ldots + f(x_i, y_n)$ as $x_i$ must occur either with $y_1$ or $y_2$ or ... or $y_n$.

$= P(X = x_i)$;

...s, the individual probability function of $X$ is found by adding over the rows of the two-way table.

Similarly, the marginal probability function for $Y$ is obtained by adding over the column as

$$h(y_j) = \sum_{i=1}^{m} f(x_i, y_j) = P(Y = y_j)$$

The values of *marginal probabilities* are often written in the margins of the joint table as they are ...w and column totals in the table. The probabilities in each marginal probability function add to 1.

**7.5.4 Conditional Probability Functions.** Let $X$ and $Y$ be two discrete r.v.'s with joint probability ...ction $f(x, y)$. Then the conditional probability function for $X$ given $Y=y$, denoted as $f(x|y)$, is defined by

$$f(x_i|y_j) = P(X=x_i \mid Y=y_j)$$

$$= \frac{P(X = x_i \text{ and } Y = y_j)}{P(Y = y_j)}$$

$$= \frac{f(x_i, y_j)}{h(y_i)}, \qquad \text{for } i = 1, 2, \ldots, j = 1, 2, \ldots$$

...re $h(y)$ is the marginal probability and $h(y) > 0$. It gives the probability that $X$ takes on the values $x_i$ ...en that $Y$ has taken on the values $y_j$. The conditional probability $f(x_i \mid y_j)$ is non-negative and (for a ...en fixed $y_j$) adds to 1 on $i$ and hence is a probability function.

Similarly, the *conditional probability* function for $Y$ *given* $X = x$ is

$$f(y_j \mid x_i) = P(Y=y_j \mid X=x_i)$$

$$= \frac{P(Y = y_j \text{ and } X = x_i)}{P(X = x_i)}$$

$$= \frac{f(x_i, y_j)}{g(x_i)}, \qquad \text{where } g(x) > 0.$$

**7.5.5 Independence**. Two discrete r.v.'s $X$ and $Y$ are said to be *statistically independent*, if and ...ly if for all possible pairs of values $(x_i, y_j)$ the joint probability function $f(x, y)$ can be expressed as the ...duct of the two marginal probability functions. That is, $X$ and $Y$ are independent, if

$$f(x, y) = P(X=x_i \text{ and } Y=y_j)$$

$$= P(X=x_i).\ P(Y=y_j) \qquad \text{for all } i \text{ and } j.$$

$$= g(x)\ h(y).$$

It should be noted that the joint p.f. of $X$ and $Y$ when they are independent, can be obtained by multiplying together their marginal probability functions.

**Example 7.6** An urn contains 3 black, 2 red and 3 green balls and 2 balls are selected at random from it. If $X$ is the number of black balls and $Y$ is the number of red balls selected, then find

i) the joint probability function $f(x, y)$;
ii) $P(X + Y \le 1)$;
iii) the marginal p.d. $g(x)$ and $h(y)$;
iv) the conditional p.d. $f(x \mid 1)$,
v) $P(X=0 \mid Y = 1)$, and
vi) Are $X$ and $Y$ independent?

i) The sample space $S$ for this experiment contains $\binom{8}{2} = 28$ sample points. The possible values of $X$ are 0, 1 and 2, and those for $Y$ are 0, 1 and 2. The values that $(X, Y)$ can take on are (0, 0), (0, 1), (1, 0), (1, 1), (0, 2) and (2, 0). We desire to find $f(x, y)$ for each value $(x, y)$.

Now $f(0, 0) = P(X=0 \text{ and } Y=0)$, where the event $(X=0 \text{ and } Y=0)$ represents that neither black nor red ball is selected, implying that the 2 selected are green balls. This event therefore contains $\binom{3}{0}\binom{2}{0}\binom{3}{2} = 3$ sample points, and

$$f(0, 0) = P(X=0 \text{ and } Y=0) = \frac{3}{28}.$$

Again
$$\begin{aligned} f(0, 1) &= P(X=0 \text{ and } Y=1) \\ &= P(\text{none is black, 1 is red and 1 is green}) \\ &= \frac{\binom{3}{0}\binom{2}{1}\binom{3}{1}}{28} = \frac{6}{28} \end{aligned}$$

Similarly,
$$\begin{aligned} f(1, 1) &= P(X = 1 \text{ and } Y = 1) \\ &= P(\text{1 is black, 1 is red and none is green}) \\ &= \frac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{28} = \frac{6}{28} \end{aligned}$$

Similar calculations give the probabilities of other values and the joint p.f. of $X$ and $Y$ is given as:

| $(x, y)$ | (0, 0) | (0, 1) | (1, 0) | (1, 1) | (0, 2) | (2, 0) |
|---|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{3}{28}$ | $\frac{6}{28}$ | $\frac{9}{28}$ | $\frac{6}{28}$ | $\frac{1}{28}$ | $\frac{3}{28}$ |

These probabilities can also be represented in another tabular form as follows:

**Joint Probability Distribution**

| X \ Y | 0 | 1 | 2 | $P(X=x_i)$ $g(x)$ |
|---|---|---|---|---|
| 0 | $\frac{3}{28}$ | $\frac{6}{28}$ | $\frac{1}{28}$ | 10/28 |
| 1 | $\frac{9}{28}$ | $\frac{6}{28}$ | 0 | 15/28 |
| 2 | $\frac{3}{28}$ | 0 | 0 | 3/28 |
| $P(Y=y_j)$ $h(y)$ | 15/28 | 12/28 | 1/28 | 1 |

Clearly, this joint p.d. of the two r.v.'s $(X, Y)$ can be represented by the formula.

$$f(x,y)=\frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{28}, \qquad \begin{array}{l} x=0,1,2 \\ y=0,1,2 \\ 0\le x+y\le 2 \end{array}$$

ii) To compute $P(X + Y \leq 1)$, we see that $x + y \leq 1$ for the cells (0, 0), (0, 1) and (1, 0). Therefore

$$P(X+Y\le 1) = f(0,0)+f(0,1)+f(1,0)$$
$$=\frac{3}{28}+\frac{6}{28}+\frac{9}{28}=\frac{18}{28}=\frac{9}{14}$$

iii) The marginal p.d.'s are

| x | 0 | 1 | 2 |
|---|---|---|---|
| g(x) | 10/28 | 15/28 | 3/28 |

| y | 0 | 1 | 2 |
|---|---|---|---|
| h(y) | 15/28 | 12/28 | 1/28 |

iv) By definition the conditional p.d. $f(x \mid 1)$ is

$$f(x\mid 1) = P(X=x \mid Y=1)$$
$$=\frac{P(X=x \text{ and } Y=1)}{P(Y=1)}=\frac{f(x,1)}{h(1)}$$

Now $$h(1)=\sum_{x=0}^{2} f(x,1)=\frac{6}{28}+\frac{6}{28}+0=\frac{12}{28}=\frac{3}{7}$$

Therefore $$f(x|1)=\frac{f(x,1)}{h(1)}=\frac{7}{3}f(x,1),\ x=0,1,2$$

Thus $$f(0\mid 1)=\frac{7}{3}f(0,1)=\left(\frac{7}{3}\right)\left(\frac{6}{28}\right)=\frac{1}{2}$$

$$f(1 \mid 1) = \frac{7}{3} f(1,1) = \left(\frac{7}{3}\right)\left(\frac{6}{28}\right) = \frac{1}{2}$$

$$f(2 \mid 1) = \frac{7}{3} f(2,1) = \left(\frac{7}{3}\right)(0) = 0$$

Hence the conditional p.d. of $X$ given the $Y = 1$, is

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(x \mid 1)$ | 1/2 | 1/2 | 0 |

v) Finally, $P(X=0 \mid Y=1) = f(0 \mid 1) = \frac{1}{2}$

vi) We find that $f(0, 1) = \frac{6}{28}$,

$$g(0) = \sum_{y=0}^{2} f(0, y) = \frac{3}{28} + \frac{6}{28} + \frac{1}{28} = \frac{10}{28}$$

$$h(1) = \sum_{x=0}^{2} f(x,1) = \frac{6}{28} + \frac{6}{28} + 0 = \frac{12}{28}$$

Now $\quad \frac{6}{28} \neq \frac{10}{28} \times \frac{12}{28}$,

i.e. $\quad f(0, 1) \neq g(0)\, h(1)$,

and therefore $X$ and $Y$ are not statistically independent.



**Example 7.7** The joint p.d. of two discrete r.v.'s $X$ and $Y$ is given by

$$f(x, y) = \frac{xy^2}{30} \qquad \text{for } x = 1, 2, 3 \text{ and } y = 1, 2.$$

Are $X$ and $Y$ independent?

The *marginal p.d.* for $X$ is

$$g(x) = \sum_{y} f(x, y)$$

$$= \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x(1)^2}{30} + \frac{x(2)^2}{30} = \frac{x}{6}, \text{ for } x = 1, 2, 3;$$

and the *marginal p.d.* for $Y$ is

$$h(y) = \sum_{x} f(x, y)$$

$$= \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{1y^2}{30} + \frac{2y^2}{30} + \frac{3y^2}{30} = \frac{y^2}{5}, \text{ for } y = 1, 2$$

Clearly, $\dfrac{xy^2}{30} = \dfrac{x}{6} \times \dfrac{y^2}{5}$, for $x = 1, 2, 3$ and $y = 1, 2$,

i.e. $f(x, y) = g(x) \,.\, h(y)$

Hence $X$ and $Y$ are independent.

**7.5.6 Continuous Bivariate Distributions.** The *bivariate probability density function* of continuous r.v.'s $X$ and $Y$ is an integrable function $f(x, y)$ satisfying the following properties:

(i) $f(x, y) \geq 0$ for all $(x, y)$.

(ii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dx\, dy = 1$, and

(iii) $P(a \leq X \leq b, c \leq Y \leq d) = \displaystyle\int_a^b\int_c^d f(x, y)dy\, dx$.

The distribution function (d.f) of the bivariate r.v. $(X, Y)$ is defined by

$$F(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u, v)dv\, du.$$

It should be noted that analogous to the relationship $\dfrac{d}{dx}F(x) = f(x)$, we have $\dfrac{\partial^2 F(x,y)}{\partial x\, \partial y} = f(x, y)$, wherever $F$ is differentiable.



The *marginal* p.d.f. of the continuous r.v. $X$ is

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

and of the r.v. $Y$ is

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

That is the marginal p.d.f. of any of the variables is obtained by integrating out the other variable from the joint p.d.f. between the limits $-\infty$ and $+\infty$.

The *conditional* p.d.f. of the continuous r.v. $X$ given that $Y$ takes the value y, is defined to be

$$f(x|y) = \frac{f(x, y)}{h(y)},$$

where $f(x, y)$ and $h(y)$ are respectively the joint p.d.f. of $X$ and $Y$, and the marginal p.d.f. of $Y$ and $h(y) > 0$.

Similarly, the *conditional* p.d.f. of the continuous r.v. $Y$ given that $X = x$, is

$$f(y|x) = \frac{f(x, y)}{g(x)}, \text{ provided that } g(x) > 0.$$

It is worth noting that the conditional p.d.f's satisfy all the requirements for a univariate function.

Finally, two continuous r.v.'s $X$ and $Y$ are said to be statistically independent, if and only joint density $f(x, y)$ can be factored in the form $f(x, y) = g(x). h(y)$ for all possible values of $X$ and

**Example 7.8** Given the following joint p.d.f.

$$f(x,y) = \frac{1}{8}(6 - x - y),\ 0 \le x \le 2;\ 2 \le y \le 4,$$

$$= 0, \text{ elsewhere.}$$

a) Verify that $f(x, y)$ is a joint density function.

b) Calculate (i) $P\left(X \le \frac{3}{2}, Y \le \frac{5}{2}\right)$, (ii) $P(X + Y < 3)$.

c) Find the marginal p.d.f. $g(x)$ and $h(y)$.

d) Find the conditional p.d.f. $f(x \mid y)$ and $f(y \mid x)$. (P.U., B.A. H

a) The joint density $f(x, y)$ will be a p.d.f. if

i) $f(x, y) \ge 0$ and

ii) $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y)dx\, dy = 1.$

Now $f(x, y)$ is clearly $\ge 0$ for all x and y in the given region, and



$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y)dx\,dy = \frac{1}{8}\int\limits_{0}^{2}\int\limits_{2}^{4}(6 - x - y)dy\, dx$$

$$= \frac{1}{8}\int\limits_{0}^{2}\left[6y - xy - \frac{y^2}{2}\right]_2^4 dx$$

$$= \frac{1}{8}\int\limits_{0}^{2}(6 - 2x)dx = \frac{1}{8}[6x - x^2]_0^2$$

$$= \frac{1}{8}[12 - 4] = 1.$$

Thus $f(x, y)$ has the properties of a joint p.d.f.

b) (i) To determine the probability of a value of the r.v. $(X, Y)$ falling in the region

we find

$$P(X \le \frac{3}{2}, Y \le \frac{5}{2}) = \int_{x=0}^{\frac{3}{2}} \int_{y=2}^{\frac{5}{2}} \frac{1}{8}(6 - x - y)dy\, dx$$

$$= \frac{1}{8}\int_0^{\frac{3}{2}} \left[6y - xy - \frac{y^2}{2}\right]_2^{\frac{5}{2}} dx$$

$$= \frac{1}{8}\int_0^{\frac{3}{2}} \left(\frac{15}{8} - \frac{x}{2}\right)dx = \frac{1}{64}\left[15x - 2x^2\right]_0^{\frac{3}{2}} = \frac{9}{32}$$

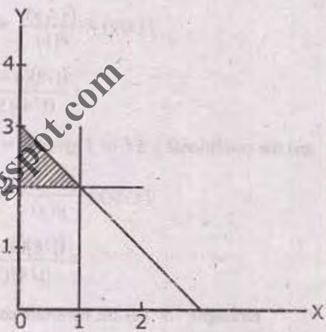
(ii) $P(X + Y < 3) = \frac{1}{8}\int_0^1 \int_2^{3-x} (6 - x - y)dy\, dx (\because x + y \le 3, \therefore y = 3 - x)$

$$= \frac{1}{8}\int_0^1 \left[6x - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1 \left(\frac{x^2}{2} - 4x + \frac{7}{2}\right)dx$$

$$= \frac{1}{8}\left[\frac{x^3}{6} - 2x^2 - \frac{7x}{2}\right]_0^1$$

$$= \frac{1}{8} \times \frac{10}{6} = \frac{5}{24}$$




Event "$X + Y \le 3$" is shaded



The marginal p.d.f. of $X$ is

$g(x) \quad = \int_{-\infty}^{\infty} f(x,y)dy, \qquad -\infty < x < \infty$

$= \frac{1}{8}\int_2^4 (6 - x - y)dy, \qquad 0 \le x < 2$

$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4 \qquad 0 \le x \le 2.$

$= \frac{1}{4}(3 - x) \qquad 0 \le x \le 2.$

$= 0 \qquad x < 0 \; or \; x \ge 2.$

Similarly, the marginal p.d.f. of $Y$ is

$$h(y)=\frac{1}{8}\int_0^2(6-x-y)\,dx, \qquad 2\le y\le 4$$

$$=\frac{1}{4}(5-y) \qquad 2\le y\le 4$$

$$=0, \text{ elsewhere.}$$

d) The conditional p.d.f. of X given Y = $y$, is

$$f(x|y)=\frac{f(x,y)}{h(y)}, \text{ where } h(y)>0.$$

$$=\frac{(1/8)(6-x-y)}{(1/4)(5-y)}=\frac{6-x-y}{2(5-y)}$$

and the conditional p.d.f. of $Y$ given X = x, is

$$f(y|x)=\frac{f(x,y)}{g(x)}, \text{ where } g(x)>0.$$

$$=\frac{(1/8)(6-x-y)}{(1/4)(3-x)}=\frac{6-x-y}{2(3-x)}.$$



**Example 7.9** Let the bivariate continuous random variable $(X, Y)$ have the joint probability function given by

$$f(x,y)=x^2+\frac{xy}{3},\ 0\le x\le 1,\ 0\le y\le 2,$$

$$=0, \text{ elsewhere.}$$

a) Check that $f(x, y)$ is a p.d.f.

b) Find the marginal p.d.f.'s.

c) Find the conditional p.d.f.'s and verify that $f(x|\, y)$ is a p.d.f.

a) The function $f(x, y)$ will be a p.d.f. if

i. $f(x, y) \geq 0$ and

ii. $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dx\,dy=1.$

Now $f(x, y)$ is clearly $\geq 0$ for all $x$ and y in the given interval and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dx\ dy = \int_0^2\int_0^1\left(x^2+\frac{xy}{3}\right)dx\ dy$$

$$= \int_0^2\left[\frac{x^3}{3}+\frac{x^2y}{6}\right]_0^1 dy = \int_0^2\left(\frac{1}{3}+\frac{y}{6}\right)dy$$

$$= \left[\frac{y}{3}+\frac{y^2}{12}\right]_0^2 = \frac{2}{3}+\frac{4}{12} = 1.$$

The marginal p.d.f.'s are

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^2\left(x^2+\frac{xy}{3}\right)dy$$

$$= \left[x^2y+\frac{xy^2}{6}\right]_0^2 = 2x^2+\frac{2}{3}x = \frac{2}{3}x(1+3x);\ 0 \leq x \leq 1$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1\left(x^2+\frac{xy}{3}\right)dx$$

$$= \left[\frac{x^3}{3}+\frac{x^2y}{6}\right]_0^1 = \frac{1}{3}+\frac{y}{6} = \frac{1}{6}(2+y) \qquad 0 \leq y \leq 2$$

The conditional p.d.f. of X for given $Y = y$ is

$$f(x/y) = \frac{f(x,y)}{h(y)} = \frac{x^2+\frac{xy}{3}}{\frac{1}{6}(2+y)} = \frac{6x^2+2xy}{2+y},\ 0 \leq x \leq 1,\ 0 \leq y \leq 2.$$

The conditional p.d.f. of Y for given $X = x$ is

$$f(y/x) = \frac{f(x,y)}{g(x)} = \frac{x^2+\frac{xy}{3}}{\frac{2}{3}x(1+3x)} = \frac{3x^2+xy}{6x^2+2x} = \frac{3x+y}{6x+2},\ 0 \leq y \leq 2,\ 0 \leq x \leq 1.$$

To verify that the conditional p.d.f. $f(x/y)$ is a p.d.f., we have

$$\int_0^1 \frac{6x^2+2xy}{2+y}dx = \left[\frac{1}{2+y}(2x^3+x^2y)\right]_0^1$$

$$=\frac{1}{2+y}(2+y)=1, \text{ for all } y.$$

Hence the conditional p.d.f. $f(x \mid y)$ satisfies the requirements for a univariate density function.

## 7.6 MATHEMATICAL EXPECTATION OF A RANDOM VARIABLE

Let a discrete r.v. X have possible values $x_1, x_2, \ldots, x_n, \ldots$ with corresponding probabilities $f(x_1)$, $f(x_2), \ldots, f(x_n), \ldots$ such that $\sum f(x)=1$. Then the mathematical expectation or the expectation or the *expected value* of X, denoted by $E(X)$, is defined as

$$E(X) = x_1 f(x_1) + x_2 f(x_2) + \ldots + x_n f(x_n) + \ldots$$

$$= \sum_{i=1}^{\infty} x_i f(x_i), \text{ provided it converges absolutely}$$

The sum converges absolutely if and only if $\sum |x| f(x)$ is finite.

When $X$ takes on only a finite number of values, we have $E(X)=\sum_{i=1}^{n} x_i f(x_i)$ which may be regarded as a *weighted mean* of the variable's possible values $x_1, x_2, \ldots, x_n$, each being weighted by its respective probability. In case the values are equally likely, $E(X)=\frac{1}{n}\sum x_i$, which represents the ordinary arithmetic mean of the $n$ possible values. $E(X)$ is also called the *mean* of $X$ and is usually denoted by the letter $\mu$. It should be noted that $E(X)$ is the average value of the r.v. $X$ over a very large number of trials.

If the r.v. $X$ is continuous with p.d.f. $f(x)$, then

$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx$, provided the integral converges absolutely, i.e. $\int_{-\infty}^{\infty} |x| f(x)\,dx$ is finite.

Clearly the definition of mathematical expectation in the case of a continuous r.v. is essentially the same with summation being replaced by integral.

**Example 7.10** **(a)** What is the mathematical expectation of the number of heads when 3 coins are tossed?

**(b)** What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability of success? (P.U., B.Sc.)

a) Let the r.v. $X$ represent the number of heads when three fair coins are tossed. Then X has the following p.d.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

Hence the mathematical expectation of X is

$$E(X)=0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=\frac{12}{8}=1.5$$

It should be noted that $E(X)$ is 1.5, which is not an integer and not any value $X$ could actually have. ...ly means that, if the experiment of tossing 3 coins is repeated a very large number of times, we ... expect *on the average* to get 1.5 heads.

(b) Let the r.v. $X$ denote the number of failures preceding the first success. Then as $X$ takes the value 0, 1, 2, 3, …, the respective probabilities are $p, qp, q^2p, q^3p, \ldots$ where $q = 1-p$.

$$\begin{aligned}\text{Hence } E(X) &= x_1f(x_1) + x_2f(x_2) + x_3f(x_3) + \ldots\\ &= 0.p + 1.qp + 2.q^2p + 3.q^3p + \ldots\\ &= qp\,(1 + 2q + 3q^2 + \ldots)\\ &= qp(1-q)^{-2} = qp(p)^{-2} = \frac{q}{p}\end{aligned}$$

**Example 7.11** (a) If it rains, an umbrella salesman can earn \$30 per day. If it is fair, he can lose \$6 ... What is his expectation if the probability of rain is 0.3? (P.U., B.A./B.Sc. 1982)

(b) A man draws 2 balls from a bag containing 3 white and 5 black balls. If he receives Rs. 70 for ... ball he draws and Rs. 7 for every black ball, find his expectation. (P.U., B.A./B.Sc. 1987)

(a) Let $X$ represent the number of dollars the salesman earns. Then X is a r.v. with possible values 30 and −6, where −6 corresponds to the fact that salesman loses, and the corresponding probabilities are 0.3 and 0.7 respectively.

$$\begin{aligned}\text{Hence } E(X) &= 30 \times 0.3 + (-6) \times 0.7\\ &= \$\,(9.00 - 4.20) = \$4.80 \text{ per day.}\end{aligned}$$

(b) Two balls from a bag containing 3 white and 5 black balls can be drawn in the following three mutually exclusive ways:

i) 2 white balls.

ii) 1 white and 1 black ball,

iii) 2 black balls.

Let p denote the probability of drawing 2 balls.

$$\text{Then}\quad p_1=\binom{3}{2}\div\binom{8}{2}=\frac{3}{28}$$

$$p_2=\binom{3}{1}\binom{5}{1}\div\binom{8}{2}=\frac{15}{28},\text{ and}$$

$$p_3=\binom{5}{2}\div\binom{8}{2}=\frac{10}{28}.$$

Let X denote the amount to be received. Then

$$x_1 = 2 \times \text{Rs. } 70 + 0 \times \text{Rs. } 7 = \text{Rs. } 140,$$

$$x_2 = 1 \times \text{Rs. } 70 + 1 \times \text{Rs. } 7 = \text{Rs. } 77,$$

$$x_3 = 0 \times \text{Rs. } 70 + 2 \times \text{Rs. } 7 = \text{Rs. } 14,$$

Hence the required expectation $= x_1p_1 + x_2p_2 + x_3p_3$

$$= \frac{3}{28} \times 140 + \frac{15}{28} \times 77 + \frac{10}{28} \times 14$$

$$= 15 + 41.25 + 5 = \text{Rs. } 61.25.$$

**Example 7.12** Find the expected value of the r.v. X having the p.d.f.

$$f(x) = 2(1 - x), \quad 0 < x < 1$$
$$= 0, \quad \text{elsewhere,}$$

Now $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= 2\int_0^1 x(1-x)\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}$$



**7.6.1 Expectation of a Function of a Random variable.** Let $H(X)$ be a function of the r.v. Then $H(X)$ is also a r.v. and also has an expected value, as any function of a r.v. is also a r.v. If X is a discrete r.v. with p.d. $f(x)$ then since $H(X)$ takes the value $H(x_i)$ when $X = x_i$, the expected value of the function $H(X)$ is

$$E[H(X)] = H(x_1)\, f(x_1) + H(x_2)\, f(x_2) + \ldots + H(x_n)\, f(x_n)$$

$$= \sum_i H(x_i) f(x_i), \text{ provided the series converges absolutely.}$$

Similarly, if $X$ is a continuous r.v. with p.d.f. $f(x)$, then

$$E[H(X)] = \int_{-\infty}^{\infty} H(x_i)\, f(x_i)\, dx, \text{ provide the integral exists.}$$

In particular, if $H(X) = X^2$, then $E(X^2) = \sum x_i^2 f(x_i)$

It is relevant to note that $E(X^2)$ is not the same as $[E(X)]^2$.

Again if $H(X) = (X - \mu)^2$, where $\mu$ is the population mean, then

$$E(X - \mu)^2 = \Sigma(x_i - \mu)^2 f(x)$$

We call this expected value the variance and denote it by Var($X$) or $\sigma^2$. That is $\sigma^2 = E(X-\mu)^2 = E(X^2) - [E(X)]^2$. The positive square root of the variance, as before, is called the *standard deviation*.

It is useful to note the following important results about variance.

i) Var($X$) cannot be negative.

ii) Var($a$) = 0, where $a$ is a constant.

iii) Var($aX$) = $a^2$ Var($X$), where $a$ is a constant.

iv) Var($aX + b$) = $a^2$ Var($X$), where $a$ and $b$ are constants.

More generally, if $H(X) = X^k$, $k = 1, 2, 3, \ldots$, then

$$E(X^k) = \sum x_i^k f(x_i)$$

which we call the *kth moment about the origin* of the r.v. $X$ and we denote it by $\mu'_k$ .

Similarly, if $H(X) = (X - \mu)^k$, $k = 1, 2, 3, \ldots$, then we get an expected value, called the *kth moment about* the mean of the r.v. X, which we denote by $\mu_k$. That is

$$\mu_k = E(X-\mu)^k = \sum (x_i - \mu)^k \; f(x_i)$$

In case of a continuous r.v., the summations are replaced by integrals.

The skewness is often measured by

$\gamma_1 = \dfrac{E(X-\mu)^3}{\sigma^3}$ and Kurtosis by $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$ as discussed earlier.



**Example 7.13** Let X have the following probability distribution:

| $x_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x_i)$ | 0.2 | 0.3 | 0.2 | 0.2 | 0.1 |

Find the probability functions of $3X - 1$, $X^2$ and $X^2 + 2$; and find $E(3X - 1)$, $E(X^2)$ and $E(X^2 + 2)$.

The probability distribution of the r.v. $H(X) = 3X - 1$, is

| Values of $X$, $x_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Probabilities, $f(x_i)$ | 0.2 | 0.3 | 0.2 | 0.2 | 0.1 |
| Values of $3X-1$, $(3x_i-1)$ | 2 | 5 | 8 | 11 | 14 |

and

$$E(3X-1) = \sum H(x_i) f(x_i) = \sum (3x_i - 1) f(x_i)$$

$$= 2 \times 0.2 + 5 \times 0.3 + 8 \times 0.2 + 11 \times 0.2 + 14 \times 0.1$$

$$= 0.4 + 1.5 + 1.6 + 2.2 + 1.4 = 7.1$$

The p.d. of $H(X) = X^2$ is

| $x_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x_i)$ | 0.2 | 0.3 | 0.2 | 0.2 | 0.1 |
| $x_i^2$ | 1 | 4 | 9 | 16 | 25 |

and $\quad E(X^2) = \sum x_i^2 f(x_i)$

$$= 1 \times 0.2 + 4 \times 0.3 + 9 \times 0.2 + 16 \times 0.2 + 25 \times 0.1$$

$$= 0.2 + 1.2 + 1.8 + 3.2 + 2.5 = 8.9$$

Similarly, $\quad E(X^2 + 2) = \sum (x_i^2 + 2) f(x_i)$

$$= 3 \times 0.2 + 6 \times 0.3 + 11 \times 0.2 + 18 \times 0.2 + 27 \times 0.1$$

$$= 0.6 + 1.8 + 2.2 + 3.6 + 2.7 = 10.9$$

**Example 7.14** Let $X$ be a r.v. with p.d.f.

$$f(x) = 2(x-1), \quad 1 < x < 2$$
$$= 0, \quad \text{elsewhere.}$$

Find the expected values of $H(X) = 2X - 1$ and $H(X) = X^2$.

Now $\quad E(2X-1) = \int_{-\infty}^{\infty} (2x-1) f(x)\,dx = 2\int_1^2 (2x-1)(x-1)\,dx$

$$= 2\int_1^2 (2x^2 - 3x + 1)\,dx = 2\left[\frac{2x^3}{3} - \frac{3x^2}{2} + x\right]_1^2$$

$$= 2\left[\left(\frac{16}{3} - 6 + 2\right) - \left(\frac{2}{3} - \frac{3}{2} + 1\right)\right]$$

$$= 2\left[\frac{4}{3} - \frac{1}{6}\right] = \frac{7}{3}$$

and $\quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= 2\int_1^2 x^2(x-1)\,dx = 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= 2\left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= 2\left[\frac{4}{3} + \frac{1}{12}\right] = \frac{17}{6}$$

**Example 7.15** If the continuous r.v. $X$ has p.d.f.

$$f(x) = \frac{3}{4}(3-x)(x-5), \quad 3 \le x \le 5$$

$$= 0, \qquad \text{elsewhere.}$$

...ate the arithmetic mean, variance and standard deviation of $X$.

$$\text{Now} \quad E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{3}{4}\int_3^5 x.(3-x)(x-5)\, dx$$

$$= \frac{3}{4}\int_3^5 (-x^3 + 8x^2 - 15x)\, dx = \frac{3}{4}\left[-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{15x^2}{2}\right]_3^5$$

$$= \frac{3}{4}\left[\left(-\frac{625}{4} + \frac{1000}{3} - \frac{375}{2}\right) - \left(-\frac{81}{4} + \frac{216}{3} - \frac{135}{2}\right)\right]$$

$$= \frac{3}{4}\left[\left(-\frac{125}{12} + \frac{63}{4}\right)\right] = \frac{3}{4}\left(\frac{64}{12}\right) = 4$$

$$\text{Again} \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \frac{3}{4}\int_3^5 x^2(3-x)(x-5)\, dx$$

$$= \frac{3}{4}\int_3^5 (-x^4 + 8x^3 - 15x^2)\, dx = \frac{3}{4}\left[-\frac{x^5}{5} + \frac{8x^4}{4} - \frac{15x^3}{3}\right]_3^5$$

$$= \frac{3}{4}\left[\frac{-1}{5}(3125) + 2(625) - 5(125) - \left\{\frac{-1}{5}(243) + 2(81) - 5(27)\right\}\right]$$

$$= \frac{3}{4}\left[0 + \frac{243}{5} - 162 + 135\right] = \frac{3}{4}\left[\frac{108}{5}\right] = \frac{81}{5}$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{81}{5} - (4)^2 = \frac{1}{5} = 0.2, \text{ and}$$

$$S.D.(X) = \sqrt{0.2} = 0.447$$

...the mean = 4, variance = 0.2 an standard deviation = 0.447.

**Example 7.16** The continuous *r.v.* $X$ has p.d.f. $f(x)$, where $f(x) = \frac{3}{4}(1+x^2)$ for $0 \le x \le 1$. If $E(X) = \mu$ and $Var(X) = \sigma^2$, find $P(|X-\mu| < \sigma)$.

Now $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \frac{3}{4}\int_0^1 x(1+x^2)\,dx = \frac{3}{4}\int_0^1 (x+x^3)\,dx$$

$$= \frac{3}{4}\left[\frac{x^2}{2}+\frac{x^4}{4}\right]_0^1 = \frac{3}{4}\left[\frac{1}{2}+\frac{1}{4}\right] = \frac{9}{16} = 0.5625$$

And $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \frac{3}{4}\int_0^1 x^2(1+x^2)\,dx = \frac{3}{4}\int_0^1 (x^2+x^4)\,dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1 = \frac{3}{4}\left[\frac{1}{3}+\frac{1}{5}\right]$$

$$= \frac{3}{4}\left(\frac{8}{15}\right) = \frac{2}{5} = 0.4$$

$\therefore\ Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{5} - \left(\frac{9}{16}\right)^2$

$$= \frac{107}{1280} = 0.0836, \text{ so that}$$

S.D. (X) or $\sigma = \sqrt{0.0836} = 0.289$

Now $P(|X-\mu| < \sigma) = P(-\sigma < X-\mu < \sigma)$

$$= P(\mu-\sigma < X < \mu+\sigma)$$

$$= P(0.5625 - 0.289 < X < 0.5625 + 0.289)$$

$$= P(0.2735 < X < 0.8515), \text{ and}$$

$$P(0.2735 < X < 0.8515) = \frac{3}{4}\int_{0.2735}^{0.8515}(1+x^2)\,dx$$

$$= \frac{3}{4}\left[x + \frac{x^3}{3}\right]_{0.2735}^{0.8515}$$

$$= \frac{3}{4}\left[0.8515 + \frac{(0.8515)^3}{3} - (0.2735 + \frac{(0.2735)^3}{3})\right]$$

$$= \frac{3}{4}[1.05729 - 0.28032] = 0.8527$$

Hence $P(|X - \mu| < \sigma) = 0.8527$.

**Example 7.17** A continuous *r.v.* $X$ has the *p.d.f.*

$$f(x) = \frac{3}{4}x(2-x), \quad 0 \le x \le 2.$$

$$= 0, \qquad \text{elsewhere}$$

Find the first four moments about the mean and the coefficient of skewness.

We first calculate the moments about origin as:

$$\mu_1' = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x(2x - x^2)\,dx = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4}\left[\frac{16}{12}\right] = 1;$$

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^2(2x - x^2)\,dx = \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right] = \frac{3}{4}\left[\frac{8}{5}\right] = \frac{6}{5};$$

$$\mu_3' = E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^3(2x - x^2)\,dx = \frac{3}{4}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{64}{5} - \frac{64}{6}\right] = \frac{3}{4}\left[\frac{64}{30}\right] = \frac{8}{5};$$

$$\mu_4' = E(X^4) = \int_{-\infty}^{\infty} x^4 f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^4(2x - x^2)\,dx = \frac{3}{4}\left[\frac{2x^6}{6} - \frac{x^7}{7}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{64}{3} - \frac{128}{7}\right] = \frac{3}{4}\left[\frac{64}{21}\right] = \frac{16}{7}.$$

Then we find the moments about the mean as;

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{6}{5} - (1)^2 = \frac{1}{5}$$

$$\mu_3 = \mu_3' - 3\mu_1'\,\mu_2' + 2(\mu_1')^3$$

$$= \frac{8}{5} - 3(1)\left(\frac{6}{5}\right) + 2(1)^3 = \frac{8}{5} - \frac{18}{5} + 2 = 0;$$

$$\mu_4 = \mu_4' - 4\mu_1'\,\mu_3' + 6(\mu_1')^2\,\mu_2' - 3(\mu_1')^4$$

$$= \frac{16}{7} - 4(1)\left(\frac{8}{5}\right) + 6(1)^2\left(\frac{6}{5}\right) - 3(1)^4$$

$$= \frac{16}{7} - \frac{32}{5} + \frac{36}{5} - 3 = \frac{3}{35}.$$



The coefficient of skewness is

$$\gamma_1 = \frac{E(X-\mu)^3}{\sigma^3} = 0 \qquad [\because E(X-\mu)^3 = \mu_3]$$

**7.6.2 Properties of Expected Values.** The important properties of the expected values of a given below:

1. If $a$ is a constant, then $E(a) = a$.

**Proof:** Let $X$ be a discrete r.v. with p.d. $[x_i, f(x_i)]$, $i = 1, 2, \ldots, n$.

Then $E(X)$ for $X = a$ is given by

$$E(a) = \sum_{i=1}^{n} a\, f(x_i)$$

$$= a f(x_1) + a f(x_2) + \dots + a f(x_n)$$

$$= a\,[f(x_1) + f(x_2) + \dots + f(x_n)]$$

$$= a \sum_i f(x_i) = a \qquad (\because \Sigma f(x_i) = 1)$$

Thus the expected value of a constant is constant itself.

2. If $X$ is a discrete r.v. and if $a$ and $b$ are constants, then

$$E(aX + b) = a\,E(X) + b.$$

**Proof.** Let the p.d. of the r.v. $X$ be $[x_i, f(x_i)]$, $i = 1, 2, \dots, n$.

Then by definition of expected value, we have

$$E(aX + b) = \sum_{i=1}^{n} (ax_i + b)\, f(x_i)$$

$$= (ax_1 + b) f(x_1) + (ax_2 + b) f(x_2) + \dots + (ax_n + b) f(x_n)$$

$$= a[x_1 f(x_1) + x_2 f(x_2) + \dots + x_n f(x_n)] + b[f(x_1) + f(x_2) + \dots + f(x_n)]$$

$$= a \sum_i x_i\, f(x_i) + b \sum_i f(x_i) = a\,E(X) + b$$

This result is often called the expected value of a linear transformation of the r.v. $X$.

If $b = 0$, then $E(aX) = a\,E(X)$.

If $a = 1$, $b = -\mu$, then $E(X - \mu) = E(X) - \mu = \mu - \mu = 0$.

That is, the expected value of the deviation of any r.v. from its mean is always equal to zero.

3. The expected value of the sum of any two random variables is equal to the sum of their expected values, i.e.

$$E(X + Y) = E(X) + E(Y)$$

**Proof.** Let $X$ and $Y$ be two discrete r.v.'s defined on the same sample space $S$. Let $X$ assume m values $x_1$, $x_2$, ..., $x_m$ with probabilities $g(x_1)$, $g(x_2)$, ... $g(x_m)$ and $Y$ assume n values $y_1, y_2, \dots, y_n$ with probabilities $h(y_1)$, $h(y_2)$, ..., $h(y_n)$. The sum $X + Y$ is a r.v. taking the values $x_i + y_j$ with probabilities $f(x_i, y_j)$ for all combinations of values of $i$ and $j$. Hence by definition, we have

$$E(X+Y) = \sum_i \sum_j (x_i + y_j)\, f(x_i, y_j)$$

$$= \sum_i \sum_j x_i f(x_i, y_j) + \sum_i \sum_j y_j f(x_i, y_j)$$

But $\sum_i \sum_j x_i f(x_i, y_j) = \sum_i x_i \sum_j f(x_i, y_j)$

$$= \sum_i x_i[f(x_i, y_1) + f(x_i, y_2) + ... + f(x_i, y_n)]$$

$$= \sum_i x_i g(x_i) \qquad (\because \text{ all possible values of y are included in } \sum_j)$$

$$= E(X)$$

Similarly, $\sum_i \sum_j y_j f(x_i, y_j) = \sum_j y_j \sum_i f(x_i, y_j)$

$$= \sum_j y_j[f(x_1, y_j) + f(x_2, y_j) + ... + f(x_m, y_j)]$$

$$= \sum_j y_j\, h(y_j) = E(Y)$$

Hence $E(X + Y) = E(X) + E(Y)$

It is interesting to note that this result holds in general, that is, the expected value of a number of r.v.'s is the sum of their expected values. In other words,

$$E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$$

or $\quad E(\sum X_i) = \sum_i [E(X_i)]$

The result also holds for the difference of r.v.'s, i.e.

$$E(X - Y) = E(X) - E(Y).$$

4. The expected value of the product of two independent r.v.'s is equal to the product of expected values, i.e.

$$E(XY) = E(X)\, E(Y).$$

**Proof.** Let the r.v. $X$ assume m values $x_1, x_2, \dots, x_m$ and the r.v. $Y$, the n values $y_1, y_2, \dots, y_n$. Let the probability of $X$ assuming the value $x_i$ be $g(x_i)$ and the probability of $Y$ assuming the values $y_j$ be The product $XY$ is a r.v. taking the value $x_i y_j$ with probabilities $f(x_i, y_j)$. As $X$ and $Y$ are independent, joint p.d. $f(x_i, y_j)$ can be factored as $f(x_i, y_j) = g(x_i)\, h(y_j)$.

Hence $\qquad E(XY) = \sum_i \sum_j x_i y_j f(x_i, y_j)$

$$= \sum_i \sum_j x_i y_j g(x_i) h(y_j)$$

$$= \sum_i x_i g(x_i) \sum_j y_j h(y_j)$$

$$= E(X)\, E(Y).$$

This result can easily be extended to several independent r.v.'s.

It should be noted that these properties are valid for continuous r.v.'s in which case the su are replaced by integrals.

**Example 7.18** Let $X$ and $Y$ be two discrete r.v.'s with the following joint p.d.

| $y$ \ $x$ | 2 | 4 |
|---|---|---|
| 1 | 0.10 | 0.15 |
| 3 | 0.20 | 0.30 |
| 5 | 0.10 | 0.15 |

Find $E(X)$, $E(Y)$, $E(X + Y)$, $E(2X - 3Y)$ and $E(XY)$.

To determine the expected values of $X$ and $Y$, we first find the marginal p.d. $g(x)$ and $h(y)$ by ...ing over the columns and rows of the two-way table as below:

| $y$ \ $x$ | 2 | 4 | $h(y)$ |
|---|---|---|---|
| 1 | 0.10 | 0.15 | 0.25 |
| 3 | 0.20 | 0.30 | 0.50 |
| 5 | 0.10 | 0.15 | 0.25 |
| $g(x)$ | 0.40 | 0.60 | 1.00 |

$$E(X) = \sum x_i g(x_i) = 2 \times 0.40 + 4 \times 0.60 = 0.80 + 2.40 = 3.2$$

$$E(Y) = \sum y_j h(y_j) = 1 \times 0.25 + 3 \times 0.50 + 5 \times 0.25$$

$$= 0.25 + 1.50 + 1.25 = 3.0$$

$$E(X + Y) = \sum_i \sum_j (x_i + y_j) f(x_i, y_j)$$

$$= (2 + 1)(0.10) + (2 + 3)(0.20) + (2 + 5)(0.10) + (4 + 1)(0.15) + (4 + 3)(0.30) + (4 + 5)(0.15)$$

$$= 0.30 + 1.00 + 0.70 + 0.75 + 2.10 + 1.35 = 6.20$$

$$= E(X) + E(Y)$$

$$E(2X-3Y) = 2\,E(X) - 3\,E(Y)$$

$$= 2\,(3.2) - 3(3.0) = -2.6$$

...$X$ and $Y$ are independent, therefore

$$E(XY) = E\,(X)\,E(Y) = (3.2)\,(3.0) = 9.6$$

**Example 7.19** $X$ and $Y$ are two independent r.v.'s such that

$$g(x) = \frac{1}{3} \text{ for } x = 1, 2, 3 \text{ and } h(y) = \frac{1}{2} \text{ for } y = 0.1.$$

...$Z = 2X - Y$, then verify that $E(Z) = 2E(X) - E(Y)$.

The joint distribution of the two independent r.v.'s $X$ and $Y$ is

| $Y$ | $X$ | | | $h(y)$ |
|---|---|---|---|---|
| | 1 | 2 | 3 | |
| 0 | 1/6 | 1/6 | 1/6 | 1/2 |
| 1 | 1/6 | 1/6 | 1/6 | 1/2 |
| $g(x)$ | 1/3 | 1/3 | 1/3 | 1 |

Now $$E(X) = \sum xg(x) = \left(1\times\frac{1}{3}\right) + \left(2\times\frac{1}{3}\right) + \left(3\times\frac{1}{3}\right) = 2,$$

$$E(Y) = \sum y\,h(y) = \left(0\times\frac{1}{2}\right) + \left(1\times\frac{1}{2}\right) = \frac{1}{2}, \text{ and}$$

$$E(Z) = E[2X - Y] = 2E(X) - E(Y) = 2\times 2 - \frac{1}{2} = 3\frac{1}{2}$$

For verification, we find the value of $E(2X-Y)$ directly as below:

$$\begin{aligned} E[2X - Y] &= \sum_i\sum_j (2x_i - y_j) f(x_i, y_j) \\ &= (2\times1-0)\frac{1}{6} + (2\times1-1)\frac{1}{6} + (2\times2-0)\frac{1}{6} \\ &\quad + (2\times2-1)\frac{1}{6} + (2\times3-0)\frac{1}{6} + (2\times3-1)\frac{1}{6} \\ &= \frac{2}{6} + \frac{1}{6} + \frac{4}{6} + \frac{3}{6} + \frac{6}{6} + \frac{5}{6} \\ &= \frac{21}{6} = 3\frac{1}{2} \end{aligned}$$

Hence the result.



**Example 7.20** Three dice, each numbered in the usual way from one to six, are colour[ed white,] red and blue respectively. After casting them, a boy 'scores' in the following way. To the white n[umber] he adds twice the red number and then subtracts the blue number. Thus a white three, a red f[our and a] blue two would score $3 + 8 - 2 = 9$. Assume that the boy casts the dice a large number of times. [Find] the mean and the variance of the scores.

Let $X_w$, $X_r$ and $X_b$ denote the numbers when the white, red and blue dice are cast respect[ively and] let $Y$ represent the score a boy gets by casting the dice. Then his score according to the conditi[on is] $Y = X_w + 2X_r - X_b$, and we need $E(Y)$ and $\text{Var}(Y)$.

Now $$\begin{aligned} E(Y) &= E[X_w + 2X_r - X_b] \\ &= E(X_w) + 2\,E(X_r) - E(X_b), \text{ where} \end{aligned}$$

$$E[X_i, i = w,r,b] = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6} = \frac{21}{6}$$

Therefore $$E(Y) = \frac{21}{6} + 2\left(\frac{21}{6}\right) - \frac{21}{6} = \frac{21}{3} = 7$$

... by definition, $Var(Y) = E(Y^2) - [E(Y)]^2$.

$$E(Y^2) = E[X_w + 2X_r - X_b]^2$$
$$= E(X_w^2) + 4\,E(X_r^2) + E(X_b^2) + 4\,E(X_wX_r) - 2\,E(X_wX_b) - 4\,E(X_rX_b)$$
$$= E(X_w^2) + 4\,E(X_r^2) + E(X_b^2) + 4\,E(X_w)\,E(X_r) - 2\,E(X_w)\,E(X_b) - 4\,E(X_r)\,E(X_b)$$

$$E[X_i^2, \text{i=w, r, b}] = 1\times\frac{1}{6}+4\times\frac{1}{6}+9\times\frac{1}{6}+16\times\frac{1}{6}+25\times\frac{1}{6}+36\times\frac{1}{6}=\frac{91}{6}$$

$$E(Y^2)=\frac{91}{6}+4\left(\frac{91}{6}\right)+\frac{91}{6}+4\left(\frac{21}{6}\right)\left(\frac{21}{6}\right)-2\left(\frac{21}{6}\right)\left(\frac{21}{6}\right)-4\left(\frac{21}{6}\right)\left(\frac{21}{6}\right)$$
$$=\frac{91}{6}+\frac{364}{6}+\frac{91}{6}+49-\frac{49}{2}-49=\frac{133}{2}$$

$$Var(Y) = \frac{133}{2}-(7)^2 = 17.5.$$

**Example 7.21** Let $X$ and $Y$ be independent r.v's with joint p.d.f.

$$f(x,y) = \frac{x(1+3y^2)}{4}, \qquad 0 < x < 2, 0 < y < 1$$
$$= 0, \qquad \text{elsewhere}$$

Find $E(X)$, $E(Y)$, $E(X + Y)$ and $E(XY)$.

To determine $E(X)$ and $E(Y)$, we first find the marginal p.d.f. $g(x)$ and $h(y)$ as below:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{x(1+3y^2)}{4}\,dy$$
$$= \frac{1}{4}[xy + xy^3]_0^1 = \frac{x}{2}, \qquad \text{for } 0 < x < 2.$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_0^2 \frac{x(1+3y^2)}{4}\,dx = \frac{1}{4}\left[\frac{x^2}{2}+3xy^2\right]_0^2$$
$$= \frac{1}{2}(1+3y^2), \qquad \text{for } 0 < y < 1.$$

$$E(X) = \int_{-\infty}^{\infty} x\,g(x)\,dx$$
$$= \int_0^2 x\left(\frac{x}{2}\right)dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}, \text{ and}$$

$$E(Y) = \int_{-\infty}^{\infty} y\,h(y)\,dy = \frac{1}{2}\int_0^1 y(1+3y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} + \frac{3y^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2} + \frac{3}{4}\right] = \frac{5}{8}.$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)\, f(x,y)\, dx\, dy$$

$$= \int_0^2\int_0^1 (x+y)\frac{x(1+3y^2)}{4}\, dy\, dx$$

$$= \int_0^2\int_0^1 \frac{x^2+3x^2y^2}{4}\, dy\, dx + \int_0^2\int_0^1 \frac{xy+3xy^3}{4}\, dy\, dx$$

$$= \int_0^2 \frac{1}{4}\left[x^2y + x^2y^3\right]_0^1 dx + \int_0^2 \frac{1}{4}\left[\frac{xy^2}{2} + \frac{3xy^4}{4}\right]_0^1 dx$$

$$= \int_0^2 \frac{1}{4}\left(2x^2\right) dx + \int_0^2 \frac{1}{4}\left(\frac{x}{2} + \frac{3x}{4}\right) dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{4}\left[\frac{x^2}{4} + \frac{3x^2}{8}\right]_0^2$$

$$= \frac{4}{3} + \frac{5}{8} = \frac{47}{24}, \text{ and}$$



$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^2\int_0^1 (xy)\frac{x(1+3y^2)}{4}\, dy\, dx = \int_0^2\int_0^1 \frac{x^2y+3x^2y^3}{4}\, dy\, dx$$

$$= \int_0^2 \frac{1}{4}\left[\frac{x^2y^2}{2} + \frac{3x^2y^4}{4}\right]_0^1 dx = \int_0^2 \frac{1}{4}\left(\frac{5x^2}{4}\right) dx = \frac{1}{4}\left[\frac{5x^3}{12}\right]_0^2 = \frac{5}{6}$$

It should be noted that

i) $E(X) + E(Y) = \frac{4}{3} + \frac{5}{8} = \frac{47}{24} = E(X+Y)$, and

ii) $E(X)\ E(Y) = \left(\frac{4}{3}\right)\left(\frac{5}{8}\right) = \frac{5}{6} = E(XY)$

...6.3 **Covariance of Two Random Variables.** The *covariance* of two r.v.'s $X$ and $Y$ is a ...al measure of the extent to which their values tend to increase or decrease together. It is denoted ... or Cov($X$, $Y$), and is defined as the expected value of the product $[X-E(X)]\,[Y-E(Y)]$. That is

$$\begin{aligned}\text{Cov}(X, Y) &= E\{[X-E(X)]\,[Y-E(Y)]\}\\ &= E\,[XY - XE(Y) - YE(X) + E(X)\,E(Y)]\\ &= E(XY) - E(X)\,E(Y) - E(Y)\,E(X) + E(X)\,E(Y)\\ &= E(XY) - E(X)\,E(Y).\end{aligned}$$

...$X$ and $Y$ are independent, then $E(XY) = E(X)\,E(Y)$, and

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y) = 0$$

...is very important to note that covariance is zero when the r.v.'s $X$ and $Y$ are independent but its ...e is not generally true. The covariance of a r.v. with itself is obviously its variance.

...6.4 **Variance of the Sum or Difference of two Random Variables.** Let $X$ an $Y$ be two discrete ...en the variance of the r.v. $X + Y$ is defined as

$$\begin{aligned}\text{Var}(X+Y) &= E[X + Y - E(X+Y)]^2\\ &= E[\{X-E(X)\} + \{Y-E(Y)\}]^2\\ &= E[X-E(X)]^2 + E[Y-E(Y)]^2 + 2E[X-E(X)]\,[Y-E(Y)]\\ &= \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X, Y)\end{aligned}$$

...$X$ and $Y$ are independent, then Cov($X$, $Y$) = 0 and we get

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

...larly, when $X$ and $Y$ are independent, the result for the difference $X - Y$ is

$$\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(Y).$$

...6.5 **Correlation Co-efficient of Random Variables.** Let $X$ and $Y$ be two r.v.'s with non-zero ... $\sigma_X^2$ and $\sigma_Y^2$. Then the *correlation coefficient* which is a measure of linear relationship between ... denoted by $\rho_{XY}$ (the Greek letter rho) or Corr($X$, $Y$), is defined as

$$\rho_{XY} = \frac{E[X-E(X)][Y-E(Y)]}{\sigma_X\sigma_Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

...$X$ and $Y$ are independent r.v.'s, then $\rho_{XY}$ will be zero but zero correlation does not necessarily ...ependence. The correlation coefficient has the following properties:

- Correlation co-efficient is unitless and symmetric in $X$ and $Y$, i.e., $\rho_{XY} = \rho_{YX}$.
- Correlation co-efficient remains unchanged if constants are added to the r.v.'s or if the r.v.'s are multiplied by constants having the same sign.
- Correlation co-efficient is a number between $-1$ and $+1$ inclusive. To prove this property, we standardize the r.v.'s $X$ and $Y$ as

$$Z_1 = \frac{X-E(X)}{\sqrt{Var(X)}} \text{ and } Z_2 = \frac{Y-E(Y)}{\sqrt{Var(Y)}}$$

It is obvious that $E(Z_1) = E(Z_2) = 0$ and $\text{Var}(Z_1) = \text{Var}(Z_2) = 1$.

Now $\quad \text{Var}(Z_1 + Z_2) = \text{Var}(Z_1) + \text{Var}(Z_2) + 2\,\text{Cov}(Z_1, Z_2)$

But $\quad \text{Cov}(Z_1, Z_2) = E(Z_1 Z_2) - E(Z_1)\,E(Z_2)$

$$= E(Z_1 Z_2) \qquad [\because E(Z_1) = E(Z_2) = 0]$$

$$= E\left[\frac{[X - E(X)][Y - E(Y)]}{\sqrt{Var(X)Var(Y)}}\right] = \rho$$

Thus $\quad \text{Var}(Z_1 + Z_2) = 1 + 1 + 2\rho \qquad [\because \text{Var}(Z_1) = \text{Var}(Z_2) = 1]$

$$= 2(1 + \rho)$$

Since $\text{Var}(Z_1 + Z_2)$ must be non-negative, therefore it follows that

$2(1 + \rho) \geq 0$ which implies that $\rho \geq -1$.

Similarly, $\text{Var}(Z_1 - Z_2) = 2(1 - \rho)$, which implies that $\rho \leq 1$.

Hence from these two results, we get

$$-1 \leq \rho \leq 1.$$

**Example 7.22** From the following joint p.d. of $X$ and $Y$, find Var($X$), Var($Y$), Cov($X$, $Y$) a

| $X$ \ $Y$ | 0 | 1 | 2 | 3 | $g(x)$ |
|---|---|---|---|---|---|
| 0 | 0.05 | 0.05 | 0.10 | 0 | 0.20 |
| 1 | 0.05 | 0.10 | 0.25 | 0.10 | 0.50 |
| 2 | 0 | 0.15 | 0.10 | 0.05 | 0.30 |
| $h(y)$ | 0.10 | 0.30 | 0.45 | 0.15 | 1.00 |

Now $\quad E(X) = \sum x_i g(x_i) = 0 \times 0.20 + 1 \times 0.50 + 2 \times 0.30$

$$= 0 + 0.50 + 0.60 = 1.10$$

$$E(Y) = \sum y_j h(y_i) = 0 \times 0.10 + 1 \times 0.30 + 2 \times 0.45 + 3 \times 0.15$$

$$= 0 + 0.30 + 0.90 + 0.45 = 1.65$$

$$E(X^2) = \sum x_i^2 g(x_i) = 0 \times 0.20 + 1 \times 0.50 + 4 \times 0.30 = 1.70$$

$$E(Y^2) = \sum y_j^2 h(y_j) = 0 \times 0.10 + 1 \times 0.30 + 4 \times 0.45 + 9 \times 0.15 = 3.45$$

Thus $\quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 1.70 - (1.10)^2 = 0.49$, and

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 3.45 - (1.65)^2 = 0.7275$

Again $E(XY) = \sum_i \sum_j (x_i y_j) f(x_i, y_j)$

$= 1\text{x}0.10 + 2\text{x}0.15 + 2\text{x}0.25 + 4\text{x}0.10 + 3\text{x}0.10 + 6\text{x}0.05$

$= 0.10 + 0.30 + 0.50 + 0.40 + 0.30 + 0.30 = 1.90$

$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y) = 1.90 - 1.10 \times 1.65 = 0.085$, and

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{0.085}{\sqrt{(0.49)(0.7275)}} = \frac{0.085}{0.595} = 0.14$$

**Example 7.23** If $f(x, y) = x^2 + \frac{xy}{3},\; 0 \le x \le 1, 0 < y \le 2$

$= 0,$ elsewhere,

find Var($X$), Var($Y$) and Corr($X$, $Y$).

The marginal p.d.f.'s are

$$g(x) = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy = 2x^2 + \frac{2}{3}x, \text{ and}$$

$$h(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dy = \frac{1}{3} + \frac{y}{6}$$

Now $$E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx = \int_0^1 x\left(2x^2 + \frac{2x}{3}\right) dx = \frac{13}{18},$$

$$E(Y) = \int_{-\infty}^{\infty} y\, h(y)\, dy = \int_0^2 y\left(\frac{1}{3} + \frac{y}{6}\right) dy = \frac{10}{9}.$$

Thus $\text{Var}(X) = E[X - E(X)]^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2\, g(x)\, dx$

$$= \int_0^1 \left(x - \frac{13}{18}\right)^2 \left(2x^2 + \frac{2x}{3}\right) dx = \frac{73}{1620}$$

$\text{Var}(Y) = E[Y - E(Y)]^2 = \int_{-\infty}^{\infty} (y - \mu_y)^2\, h(y)\, dy$

$$= \int_0^2 \left(y - \frac{10}{9}\right)^2 \left(\frac{1}{3} + \frac{y}{6}\right) dy = \frac{26}{81}, \text{ and}$$

$\text{Cov}(X,Y) = E\{[X - E(X)]\,[Y - E(Y)]\}$

$$= \int_0^1\int_0^2 \left(x - \frac{13}{18}\right)\left(y - \frac{10}{9}\right)\left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \int_0^1 \left( -\frac{2}{9}x^3 + \frac{25}{81}x^2 - \frac{26}{243}x \right) dx = \frac{-1}{162}$$

Hence $\quad Corr(X, Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \dfrac{-1/162}{\sqrt{(73/1620)(26/81)}}$

$$= -0.05$$

## 7.7 MEDIANS AND MODES OF CONTINUOUS RANDOM VARIABLES

If $X$ is a discrete r.v., then a value '$a$' that satisfies the inequalities

$$P(X \le a) \ge \frac{1}{2}, P(X \ge a) \ge \frac{1}{2}$$

is called a *median*. If the r.v. $X$ is continuous, then *median* is a value of $X$ that satisfies the equ $F(x) = \frac{1}{2}$. In other words, the *median* '$a$' is given by

$$\int_{-\infty}^{a} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{2}.$$

Similarly, the two quartiles $Q_1$ and $Q_3$ are defined by the relations

$$\int_{-\infty}^{Q_1} f(x)\,dx = \frac{1}{4} \text{ and } \int_{Q_3}^{\infty} f(x)\,dx = \frac{1}{4}.$$



The *mode* in case of a continuous r.v. $X$ is such a stationary value of $f(x)$ for which the p.d. maximum. That is we get a maximum value when

i) $\dfrac{d}{dx} f(x) = 0$, i.e. $f'(x) = 0$, and (ii) $f''(x) < 0$,

provided that the solution of $f'(x) = 0$ lies within the given range of the r.v. $X$.

It should also be noted that in case of continuous r.v. $X$, the *geometric mean*, G, and the *harmon*ic mean *H*, are defined as

$$\log G = E(\log X) = \int_{-\infty}^{\infty} \log x f(x) dx, \text{ provided the integral exists, and}$$

$$\frac{1}{H} = E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx.$$

**Example 7.24** Let $X$ be a r.v. with p.d.f.

$$f(x) = k(x - x^2), \qquad 0 \le x \le 1,$$
$$= 0, \qquad \text{elsewhere,}$$

where $k$ is a constant. Then find the mean, median, mode, harmonic mean and standard deviation

First of all, we find the value of $k$ which should be such as to make

$$\int_0^1 k(x - x^2)\,dx = 1$$

That is $k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$ or $k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$ or $k = 6$.

Now, the mean, $\mu$ or $E(X)$ is given by

$$\mu = E(X) = \int_0^1 x.6(x - x^2)\,dx$$

$$= 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

The median, $a$, is given by $\int_{-\infty}^{a} f(x)\,dx = \frac{1}{2}$. Thus

$$6\int_0^a (x - x^2)\,dx = \frac{1}{2} \text{ or } 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{1}{2}$$

or $\quad 4a^3 - 6a^2 + 1 = 0$

factorize the equation $4a^3 - 6a^2 + 1 = 0$ as

$$(2a - 1)(2a^2 - 2a - 1) = 0$$

Now either $2a - 1 = 0$ which gives $a = \frac{1}{2}$, or

$2a^2 - 2a - 1 = 0$, which gives $a = \frac{1 \pm \sqrt{3}}{2}$, i.e. $a = -0.366$ or $1.366$.

the values $-0.366$ and $1.366$ are unacceptable since $a$ must lie in the interval $(0, 1)$. The median given by $a = \frac{1}{2}$.

is that value of $x$ for which

$f'(x) = 0$, and $\quad$ (ii) $f''(x) < 0$

$f(x) = 6(x - x^2)$, and $f'(x) = 6(1 - 2x)$

$f'(x) = 0$ when $1 - 2x = 0$ or $x = \frac{1}{2}$

To check that this is maximum, we find

$$f''(x) = 6(-2) = -12 < 0$$

i.e. $f''(x)$ is negative for all values of $x$, there is a maximum at $x = \frac{1}{2}$

Hence the mode $= \frac{1}{2}$.

The harmonic mean, $H$, is given by

$$\frac{1}{H} = \int_0^1 \frac{1}{x} . 6(x - x^2)\, dx$$

$$= 6\int_0^1 (1-x)\, dx = 6\left[x - \frac{x^2}{2}\right]_0^1 = 3$$

$$\therefore \quad H = \frac{1}{3}$$

Again, $\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$= \int_0^1 x^2 . 6(x - x^2)\, dx = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{3}{10}$$

$$\therefore \quad \mu_2 = E(X^2) - [E(X)]^2 = \mu_2' - \mu_1'^2$$

$$= \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20} = 0.05$$

Hence $\sigma = \sqrt{\mu_2} = \sqrt{0.05} = 0.2234$



## 7.8 CHEBYSHEV'S INEQUALITY

If $X$ is a r.v. having mean $\mu$ and variance $\sigma^2 > 0$, and $k$ is any positive constant, then probability that a value of $X$ falls within $k$ standard deviations of the mean is at least $\left(1 - \frac{1}{k^2}\right)$. Th

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2},$$

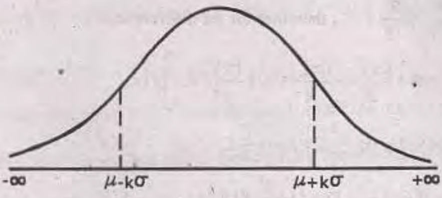
or equivalently $P[|x - \mu| \geq k\sigma] \leq \frac{1}{k^2}$.

oof. By definition, we have

$$\sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx.$$

Dividing the range into 3 disjoint parts $(-\infty, \mu-k\sigma)$, $(\mu-k\sigma, \mu+k\sigma)$ and $(\mu+k\sigma, \infty)$, we have




$$\sigma^2 = \int_{-\infty}^{\mu-k\sigma}(x-\mu)^2 f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma}(x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty}(x-\mu)^2 f(x)dx$$

ing the middle term, we get

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma}(x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty}(x-\mu)^2 f(x)dx$$

$x - \mu| \geq k\sigma$, we have $(x-\mu)^2 \geq k^2\sigma^2$, wherever $x \geq \mu + k\sigma$

$\leq \mu - k\sigma$. It therefore follows that

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma}k^2\sigma^2 f(x)dx + \int_{\mu+k\sigma}^{\infty}k^2\sigma^2 f(x)dx$$

or $$\int_{-\infty}^{\mu-k\sigma}f(x)\,dx + \int_{\mu+k\sigma}^{\infty}f(x)\,dx \leq \frac{1}{k^2}$$

$$P(\mu-k\sigma < X < \mu+k\sigma) = \int_{\mu-k\sigma}^{\mu+k\sigma}f(x)\,dx \geq 1-\frac{1}{k^2}.$$

is inequality is due to Russian mathematician *P.L.* Chebyshev (1821–1894) and it provides a of understanding how the variance measures variability about the mean of a r.v. It holds for all having finite mean and variance. In case of a discrete r.v., the proof is the same with integrals laced by summations.

## OMENT GENERATING FUNCTION

e *moment generating function* (*m.g.f*) usually denoted by $M_0(t)$, of a random variable $X$ about the it exists, is defined as the expected value of the r.v. $e^{tX}$, where $t$ is a real variable lying in a hood of zero. That is

$$M_0(t) = E(e^{tX}) = \sum_{i=1}^{\infty} e^{tx_i} f(x_i), \text{ if } X \text{ is discrete,}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \text{, if } X \text{ is continuous.}$$

Since $e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$, therefore for the discrete case

$$M_0(t) = \sum\left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots + \frac{(tx)^r}{r!} + \ldots\right] f(x)$$

$$= \sum f(x) + t\sum x f(x) + \frac{t^2}{2!}\sum x^2 f(x) + \frac{t^3}{3!}\sum x^3 f(x) + \ldots + \frac{t^r}{r!}\sum x^r f(x) + \ldots$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'.$$

Thus we find that the co-efficient of $\frac{t^r}{r!}$ in the expansion of $M_0(t)$ is just $E(X^r)$ or $\mu_r'$, the $r$th moment about zero. We call the function $M_0(t)$ the *moment generating function* because it generates the moments of the r.v. $X$. It should be noted that a *m.g.f.* would exist only if the sum or integral converges for all values of $t$ in a neighbourhood of zero.



It is more convenient to find the moments by differentiating the *m.g.f.* $r$ times *w.r.* to $t$ and putting $t = 0$. The *m.g.f.* of $X$ is

$$M_0(t) = 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

Differentiating *w.r.* to $t$, we get

$$\frac{dM_0(t)}{dt} = E(X) + tE(X^2) + \frac{t^2}{2!}E(X^3) + \ldots + \frac{t^{r-1}}{(r-1)!}E(X^r) + \ldots$$

and $\frac{dM_0(t)}{dt}\Big|_{t=0} = E(X) = \mu_1'$

Differentiating it again, we obtain

$$\frac{d^2M_0(t)}{dt^2} = E(X^2) + tE(X^3) + \ldots + \frac{t^{r-2}}{(r-2)!}E(X^r) + \ldots$$

and $\frac{d^2M_0(t)}{dt^2}\Big|_{t=0} = E(X^2) = \mu_2'$

ifferentiating $r$ times and equating $t=0$, we get

$$\frac{d^r M_0(t)}{dt^r}\Big|_{t=0} = E(X^r) = \mu_r'$$

e m.g.f. about the value $a$ is defined as the expected value of $e^{t(X-a)}$ and is denoted as

$$M_a(t) = E\left[e^{t(X-a)}\right] = e^{-at}E(e^{tX}) = e^{-at}M_0(t)$$

s the m.g.f. about the value $a$ is equal to $e^{-at}$ times the m.g.f. about the origin.

ilarly, we make a change in the scale, and define the m.g.f. by introducing a new variable u as

$$u_i = \frac{x_i - a}{h} \text{ so that } x_i = a + hu_i.$$

Then $M_0(t) = \sum_i e^{tx_i} f(x_i) = \sum_i e^{t(a+hu_i)} f(x_i) = e^{at}\sum_i e^{thu_i} f(x_i)$

ing $\frac{t}{h}$ for t in both sides, we get $M_u(t) = e^{-at/h} M_0\left(\frac{t}{h}\right)$

The *m.g.f.* has a very important property, namely the *m.g.f.* of the sum of independent random ables is the product of the individual *m.g.f.'s*. Let us consider two independent r.v.'s $X$ and $Y$. Then

$$E\left[e^{t(X+Y)}\right] = E\left[e^{tX}.e^{tY}\right] = E(e^{tX})E(e^{tY}).$$

e variables have identical probability distributions, then we have

$$m.g.f. \text{ of } \sum_{i=1}^{n} X_i = [m.g.f.(X)]^n$$

**7.9.1 Cumulant Generating Function.** The *cumulants* are a set of parameters of a probability ibution defined by the following identity in $t$:

$$\text{Exp}\left[\sum_{r=1}^{\infty}\frac{\kappa_r t^r}{r!}\right] = \sum_{r=0}^{\infty}\mu_r'\frac{t^r}{r!}$$

e $\kappa_r$ is the *rth* cumulant. In other words, the cumulants are given by the co-efficient in the sion of a power series from the natural logarithm of the *m.g.f.* of a random variable, provided such pansion exists. Thus

$$\kappa(t) = \log_e M_0(t)$$

$$= \kappa_1 t + \kappa_2\frac{t^2}{2!} + \kappa_3\frac{t^3}{3!} + \ldots + \kappa_r\frac{t^r}{r!} + \ldots$$

The co-efficient $\kappa_1, \kappa_2, \kappa_3, \ldots$, are the first, the second, the third cumulant, etc. and $\kappa(t)$ is called the *cumulant generating function* (*c.g.f.*). The *c.g.f.* possesses a very important property that the *c.g.f.* of the sum of a number of independent random variables is the sum of their cumulant generating functions.

We differentiate the above relation $r$ times with respect to $t$ and then put $t = 0$ to find the $r$th cumulant. That is

$$\kappa_r = \left[\frac{d^r}{dt^r}\log_e M_0(t)\right]_{t=0}.$$

**7.9.2 Relation between Cumulants and Moments.** By definition, we have

$$\kappa(t) = \log_e M_0(t) = \log_e\left[\sum_{r=0}^{\infty}\mu_r'\frac{t^r}{r!}\right]$$

or

$$\kappa_1 t + \kappa_2\frac{t^2}{r!} + \ldots + \kappa_r\frac{t^r}{r!} + \ldots = \log_e\left[1 + \sum_{r=1}^{\infty}\mu_r'\frac{t^r}{r!}\right]$$

$$= \log_e[1 + z], \text{ where } z = \sum_{r=1}^{\infty}\mu_r'\frac{t^r}{r!}$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$$

$$= \left[\sum_{r=1}^{\infty}\mu_r'\frac{t^r}{r!}\right] - \frac{1}{2}\left[\sum_{r=1}^{\infty}\mu_r'\frac{t^r}{r!}\right]^2 + \frac{1}{3}\left[\sum_{r=1}^{\infty}\mu_r'\frac{t^r}{r!}\right]^3 \ldots$$

$$= \left[\mu_1't + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \mu_4'\frac{t^4}{4!} + \ldots\right] - \frac{1}{2}\left[\mu_1't + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \ldots\right]^2 +$$

$$\frac{1}{3}\left[\mu_1't + \mu_2'\frac{t^2}{2!} + \ldots\right]^3 - \frac{1}{4}[\mu_1't + \ldots]^4 + \ldots$$

$$= \mu_1't + (\mu_2' - \mu_1'^2)\frac{t^2}{2!} + (\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3)\frac{t^3}{3!} + \ldots$$

Comparing the co-efficient of like powers of $t$, we get

$$\kappa_1 = \mu_1',$$

$$\kappa_2 = \mu_2' - \mu_1'^2 = \mu_2,$$

$$\kappa_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = \mu_3$$

$$\kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^2$$

$$= \mu_4 - 3\mu_2^2.$$

**7.9.3 Characteristic Function.** The *m.g.f.* does not exist for many probability distributions. We ...se another function, called the *characteristic function* (*c.f.*). The *characteristic function* of a r.v. $X$, ...ed by $\phi(t)$, is defined as the expected value of the r.v. $e^{itX}$, *i.e.*

$$\phi(t) = E(e^{itX})$$

$$= \sum e^{itx} f(x) \text{ or } \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

...g as $X$ is discrete or continuous, and where t is a real number and $i\left(=\sqrt{-1}\right)$, the imaginary unit. ...aracteristic function always exists because $\left|e^{itX}\right| = 1$ for all real $t$, and hence may be defined for ... probability distribution. The characteristic function has therefore an advantage over the moment ...g function. The *c.f.* may be written as a series

$$\phi(t) = 1 + it\,\mu_1' + \frac{(it)^2}{2!}\mu_2' + \frac{(it)^3}{3!}\mu_3' + \dots + \frac{(it)^k}{k!}\mu_k' + \dots$$

...us the *kth* moment of $X$ about the origin is the co-efficient of $\frac{(it)^k}{k!}$. Applications of these ... are given in the chapters that follow.



## EXERCISES

**...TIVE**

...swer 'True' or 'False'. If the statement is not true then replace the underlined words with words ...at make the statement true:

- A random variable can assume <u>only one</u> value with a given probability.
- A random variable that can assume a finite set of possible values is known as a <u>continuous</u> random variable. 0
- A random variable that can assume any value in a given interval [a, b] is known as <u>discrete</u> random variable.
- The expected value for any discrete random variable is always equal to $\underline{\sum(x-\mu)^2 p(X)}$.
- The variance for any discrete random variable is $\underline{\sum xp(x)}$.
- Discrete random variables may assume only <u>positive values</u>.
- The expected value of a constant is <u>zero</u>.

viii) The expected value of the product of two any random variables is equal to the product of expected values.

ix) Sum of probabilities for any probability distribution is equal to zero.

x) For any random variable having mean $\mu$ and variance $\sigma^2$ and $k > 1$, then the probab a value of $X$ falls within $k$ standard deviation of the mean is less than $\left(1-\frac{1}{k^2}\right)$.

b) **MULTIPLE CHOICE QUESTIONS**

i) A random variable is also known as a

a) Chance variable.
b) Stochastic variable.
c) Variate.
d) All of above.

ii) The distribution function of a random variable $X$, denoted by $F(x)$ is defined as

a) $F(x) = P(X \le x)$
b) $F(x) = P(X \ge x)$
c) $F(x) = P(X = x)$
d) None of above.

iii) A discrete probability distribution may be represented by

a) A table.
b) A mathematical function.
c) A histogram.
d) All of above.



iv) A continuous probability distribution is not represented by

a) A table.
b) A mathematical function.
c) A graph.
d) A density function.

v) If $X$ and $Y$ are two independent random variables, then Var(X – Y) is equal to

a) Var(X) – Var(Y)
b) Var(X) + Var(Y)
c) Var(X) + Var(Y) – 2 COV(X, Y)
d) None of above

If $X$ is a random variable and $a$ and $b$ are constants, then Var(aX + b) is equal to

a) $a^2$ Var(X)
b) Var($a$X)
c) $a^2$ Var(X) + Var($b$)
d) $a$ Var(X)

Given the following distribution for a random viable $X$:

| $x_i$ | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| $f(x_i)$ | 0.10 | 0.20 | 0.20 | 0.25 | 0.15 | 0.10 | 1.00 |

the standard deviation of $X$ is

a) 2.000
b) 1.4654
c) 3.5064
d) 2.1475

Suppose $X$ has a p.d. given by

| $x_i$ | −2 | −1 | 0 | 1 | Total |
|---|---|---|---|---|---|
| $f(x_i)$ | $2a$ | $3a$ | $a$ | $3a$ | 1.00 |

then $a$ is

a) 0.1000
b) 0.1111
c) 0.2000
d) None of above.



If $X$ and $Y$ are two random variables, then $E(X + Y)$ is equal to

a) $E(X) + E(Y)$
b) $E(X) + Y$
c) $E(X) - E(Y)$
d) None of above

If two discrete random variables $X$ and $Y$ are independent, which of the following statements is not true?

a) $P(X = 4) = P(X = 4/Y=2)$
b) $P(X = 4 \text{ and } Y = 2) = P(X = 4)\,P(Y = 2)$
c) $P(X = 4 \text{ and } Y = 2) \neq P(X = 4)\,P(Y = 2)$
d) $P(X = 2) = P(X = 2 \mid Y = 4)$

## SUBJECTIVE

7.1 a) Explain the concept of a random variable. What is a distribution function and what are its properties?

b) Let $X$ be a random variable denoting the number of points appearing in a throw of a die. Determine the distribution function $F(x)$, $x$ is a real number and draw its graph.

Imp 7.2 a) Define a discrete random variable and its probability distribution. What are the two properties of all probability distributions?

b) Suppose $X$ has a p.d. given by

| $x$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $f(x)$ | $3c$ | $3c$ | $6c$ |

(i) Determine $c$. (ii) What is the p.d. of $Y=2X+1$?

c) Determine the p.d. of a r.v. $X$, where $X$ denotes the number of aces in a hand of bridge.

Imp 7.3 a) Define a discrete r.v. Giving illustrations, explain what is meant by a discrete p.d.

b) A bag contains 4 red and 6 black balls. A sample of 4 balls is selected from the bag without replacement. Let $X$ be the number of red balls. Find the p.d. for $X$. (P.U., B.A./B.Sc.

7.4 a) A large store places its last 15 clock radios in a clearance sale. Unknown to any one, 5 radios are defective. If a customer tests 3 different clock radios selected at random, what is the p.d. of $X$ = number of defective radios in the sample?

b) Three balls are drawn from a bag containing 5 white and 3 black balls. If $X$ denotes the number of white balls drawn from the bag, then find the p.d. of $X$.

Imp 7.5 a) Explain the concept of distribution function. Hence or otherwise, differentiate between discrete and continuous random variables.

b) Given the following probability function

$$f(x)=A(4x-2x^2),\ 0\le x\le 2 \text{ and zero otherwise.}$$

Calculate the value of $A$ so as $f(x)$ may be a p.d.f. (P.U., B.A. B.Sc.

Imp 7.6 a) Define a continuous r.v. and its probability density function.

b) A continuous r.v. $X$ has the p.d.f. as follows:

$$f(x)=\begin{cases} x/2 & \text{for } 0<x\le 1 \\ \frac{1}{4}(3-x) & \text{for } 1<x\le 2 \\ \frac{1}{4} & \text{for } 2<x\le 3 \\ \frac{1}{4}(4-x) & \text{for } 3<x\le 4 \\ 0, \text{ elsewhere} & \end{cases}$$

Compute $P(X\ge 3)$, $P(X=2)$, $P(|X|<1.5)$ and $P(1<X<3)$.

a) Explain the concepts of the Probability Function, Probability Density Function and Distribution Function. (P.U., B.A./B.Sc. 1990, 92)

b) A continuous r.v. $X$ has the $p.d.f.$

$$f(x) = A(2-x)(2+x), \ 0 \le x \le 2$$
$$= 0, \quad \text{elsewhere.}$$

Find (i) the value of $A$, (ii) $P(X = \frac{1}{2})$, (iii) $P(X \le 1)$, (iv) $P(X \ge 2)$, (v) $P(1 \le x \le 2)$.

(P.U., B.A./B.Sc. 1993)

Let $X$ be a continuous r.v. with p.d.f.

$$f(x) = 6x(1-x), \ 0 \le x \le 1.$$
$$= 0, \quad \text{otherwise.}$$

i) Check that $f(x)$ is a p.d.f. and sketch it.

ii) Obtain an expression for the distribution function of $X$.

iii) Compute $P\left(\frac{1}{3} < X < \frac{2}{3}\right)$ and $P\left(X \le \frac{1}{2} \mid \frac{1}{3} < X < \frac{2}{3}\right)$.

iv) Determine a number b such that $P(X<b) = 2P(X>b)$. (P.U., M.Sc. 1974)

Suppose that the life length (in hours) of a certain radio tube is continuous random variable $X$ with probability density function

$$f(x) = \frac{100}{x^2}, \ x > 100 \text{ and zero elsewhere.}$$

What is the probability that a tube will last less than 200 hours, if it is known that the tube is still functioning after 150 hours of service?

What is the probability that if 3 such tubes are installed in a set exactly one will have to be replaced after 150 hours of service?

What is the maximum number of tubes that may be inserted into a set so that there is a probability of 0.5 that after 150 hours of service all of them are still functioning?

(P.U., B.A. (Hons.) Part-I, 1970)

Explain the terms: a joint distribution function, a joint probability distribution, marginal and conditional distributions.

Given the joint p.d. of two r.v.'s $X$ and $Y$, whose values $f(x, y)$

are $f(1,1) = \frac{6}{30}$, $f(1,2) = \frac{1}{30}$, $f(1,3) = \frac{1}{30}$, $f(2,1) = \frac{4}{30}$,

$f(2,2) = \frac{5}{30}$, $f(2,3) = \frac{1}{30}$, $f(3,1) = \frac{2}{30}$, $f(3,2) = \frac{4}{30}$,

$f(3,3) = \frac{6}{30}$, find all the marginal and conditional distributions.

Imp 7.11 Suppose that the following table represents the joint p.d. of the discrete r.v. $(X, Y)$:

| $Y \backslash X$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $\frac{1}{12}$ | $\frac{1}{6}$ | 0 |
| 2 | 0 | $\frac{1}{9}$ | $\frac{1}{5}$ |
| 3 | $\frac{1}{18}$ | $\frac{1}{4}$ | $\frac{2}{15}$ |

a) Compute $g(x)$, $h(y)$, $f(x/y)$ and $f(y/x)$.

b) Decide whether $X$ and $Y$ are independent.

Imp 7.12 Let $X$ and $Y$ have the joint probability functions given by

i) $f(x, y) = \dfrac{xy}{66}$, $\quad x = 2, 4, 5; y = 1, 2, 3.$

ii) $f(x, y) = \dfrac{xy^2}{30}$, $\quad x = 1, 2, 3; y = 1, 2.$

Find the marginal probability functions of $X$ and $Y$, and find out if $X$ and $Y$ are independent.

(P.U., B.A./B.Sc.

7.13 a) What is a joint probability density function? How does a marginal probability function differ from a conditional probability function?

b) Given the joint p.d.f. $f(x, y) = 3xy\,(x + y), 0 \le x \le 1, 0 \le y \le 1$ and 0 elsewhere, find marginal and conditional p.f.(s).

7.14 Suppose the joint p.d.f. of $(X, Y)$ is given by

$$f(x, y) = x^2 + \frac{1}{3}xy, 0 \le x \le 1, 0 \le y \le 2 \text{ and 0 elsewhere.}$$

a) Check that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$

b) Compute (i) $P\left(X > \frac{1}{2}\right)$, (ii) $P(Y<X)$, (iii) $P(X+Y>1)$ and (iv) $P\left(Y < \frac{1}{2} \mid X < \frac{1}{2}\right)$

7.15 a) Explain clearly the meaning of marginal and conditional distributions with reference to bivariate density function $f(x, y)$.

b) If two r.v.'s $X$ and $Y$ have the joint p.d.f.

$$f(x, y) = \frac{2}{3}(x + 2y) \qquad for\ 0 < x < 1, 0 < y < 1,$$
$$= 0, \qquad \text{elsewhere;}$$

then find the marginal distributions of $X$ and $Y$, and their conditional distributions and $P\left(X < \frac{1}{2} \mid Y = \frac{1}{2}\right)$.

Given the joint density function of the r.v. $(X, Y)$ as

$$f(x, y) = 3x^2y + 3xy^2, \quad \text{for } 0 \le x \le 1, 0 \le y \le 1,$$
$$= 0, \qquad \text{otherwise.}$$

Find the marginal and conditional density functions. Also find the conditional probability

$P\left[\frac{1}{2} \le x \le \frac{3}{4} \mid \frac{1}{2} \le y \le \frac{2}{3}\right]$. (P.U., B.A. (Hons.) Part-III, 1964, 65, 68, 70)

Explain the idea of joint probability distribution, conditional distribution and marginal distribution. The random variables $X$ and $Y$ are jointly distributed as

$$f(x, y) = 24x^2y(1-x), \quad 0 \le x, y \le 1.$$

Obtain the marginal distribution of $X$, and the conditional distribution of $Y$. Are $X$ and $Y$ independent? (P.U., M.Sc. 1970; B.Sc. Hons. Part-II, 1972)

a) Define $E(X)$, the expected value of a random variable $X$.

b) If $X$ is a r.v. and if $a$ and $b$ are constants, then prove that

$$E(aX + b) = a\,E(X) + b.$$

c) If $X$ and $Y$ are r.v.'s, then show that $E(X+Y) = E(X) + E(Y)$.

d) Show that, under certain conditions to be stated,

$$E(XY) = E(X)\,E(Y).$$

a) Explain random variable and its mathematical expectation.

b) Prove that $E(cX) = cE(X)$, where $c$ is a constant.

c) Two unbiased dice are cast. A payment equal to the sum of the spots on the top sides is given the caster. Compute the expected value of the payment.

a) A committee of size 3 is to be selected at random from 3 women and 5 men. Find the expected number of women on the committee. (P.U., B.A./B.Sc. 1979)

Let $X$ have the possible values

$2, -2, \frac{8}{3}, -4, \ldots, (-1)^{j+1}\frac{2^j}{j}$, and the corresponding probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots (\frac{1}{2})^j, \ldots$

Find the expectation of $X$ if it exists. (P.U., B.A. Hons. 1970)

Define expectation and prove that the expectation of the sum of two random variables is equal to the sum of their expectations. (P.U., B.A./B.Sc. 1979, 81)

b) Verify that $E(X) + E(Y) = E(X + Y)$ by using the random variable $X$ with the p.d. $f(x)$ = ... $x = 1, 2, 3, 4$, and the r.v. $Y$ with the p.d. $f(y) = \binom{3}{y}\left(\frac{1}{2}\right)^y\left(\frac{1}{2}\right)^{3-y}$, $y = 0, 1, 2, 3$.

(Gomal, B.A./B.Sc.

7.22 a) The p.d. of a discrete random variable $X$ is

$$f(x) = \binom{3}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}, \quad x = 0, 1, 2, 3.$$

Find $E(X)$ and $E(X^2)$. (P.U., B.A./B.Sc.

b) Let $X$ be a random variable with probability distribution

| $X$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f(x)$ | 0.125 | 0.50 | 0.20 | 0.05 | 0.125 |

i) Find $E(X)$ and $\text{Var}(X)$.

ii) Find the p.d. of the r.v. $Y = 2X + 1$. Using the p.d. of $Y$ determine $E(Y)$ and Var(Y).

iii) How are $E(X)$ and $E(Y)$, and $\text{Var}(X)$ and $\text{Var}(Y)$ related?



7.23 a) For what value of $A$, the function defined as below will be a p.d.f.?

$$f(x) = Ax^3(1-x), \quad 0 \le x \le 1$$
$$= 0, \quad \text{otherwise}$$

b) Find its mean and variance.

c) Also find $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$ using its distribution function. (P.U., B.A./B.Sc.

7.24 a) Show that $E[X-E(X)]^2 = E(X^2) - [E(X)]^2$.

b) Let $X_1$ and $X_2$ be two independent r.v.'s having variances $k$ and 2 respectively. If $\text{Var}(3X_2 - X_1) = 25$, find $k$.

c) The following table shows the distribution function $F(x)$ of the r.v. $X$:

| $x$: | $x \le 1$ | $x \le 2$ | $x \le 3$ | $x \le 4$ |
|---|---|---|---|---|
| $F(x)$: | 1/8 | 3/8 | 3/4 | 1 |

Find (i) probability distribution of the r.v. $X$, (ii) $E(X)$ and $\text{Var}(X)$. (P.U., B.A./B.Sc.

7.25 a) If $f(x) = \frac{6-|7-x|}{36}$ for $x = 2, 3, \ldots, 12$, then find the mean and variance of the random variable $X$.

b) If $f(x) = \frac{1}{n}$, $(x = 1, 2, \ldots, n)$, then find $E(X)$ and $\text{Var}(X)$.

c) Suppose $X$ can take the values 0, 1, 2, ..., $n$ with frequencies proportional to the binomial co-efficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Show that $E(X) = \frac{n}{2}$ and $Var(X) = \frac{n}{4}$.

(P.U., M.A., 1970; B.A. (Hons.) Part-II, 1970)

Practical:

a) $A$ and $B$ throw with one die for a prize of Rs. 11, which is to be won by the player who first throws 6. If $A$ has the first throw, what are their respective expectations?

b) $A$, $B$, $C$ and $D$ cut a pack of cards successively in the order mentioned. If the person who cuts a spade first, receives £ 175, what are their expectations?

c) A bag contains 2 white and 2 black balls. Three persons $A$, $B$ and $C$ in the order named above draw a ball and do not replace it. The person who draws a white ball first receives Rs. 18. What are their expectations? (P.U., B.A./B.Sc.,1986)

a) A distribution is given by $f(x) = x^2(1-x)$ between $x = 0$ and $x = 1$. Find the mean and standard deviation.

A continuous r.v. $X$ has p.d.f. given by $f(x) = cx$ for $1<x<2$.

(i) Determine the constant $c$. (ii) Find the mean, variance and standard deviation of $X$.

(P.U., B.A./B.Sc. 1980)

A variable $X$ has the p.d.f.

$$f(x) = 0, \quad \text{for } x < 0.$$
$$= \frac{3}{8}(x-2)^2, \text{ for } 0 < x < 2.$$
$$= 0, \quad \text{for } x \geq 2.$$

Find the expected value of $X$ and its standard deviation.

Find the value of $k$ so that the function $f(x)$ defined as follows, may be a density function.

$$f(x) = kx^3(1-x), \qquad 0 < x < 1.$$
$$= 0, \qquad \text{otherwise}$$

Also determine its mean and variance. (P.U., B.A./B.Sc. 1978)

A r.v. $X$ has the p.d.f. as

$$f(x) = Ax(9-x^2), \qquad 0 \leq x \leq 3$$
$$= 0, \qquad \text{otherwise}$$

Find the value of $A$, the mean and the standard deviation of $X$. (P.U., B.A./B.Sc. 1990)

7.29 A r.v. $X$ has the probability density function given by

$$f(x)=\begin{cases}\frac{1}{16}(3+x)^2, & -3<x\le -1\\ \frac{1}{16}(6-2x^2), & -1<x\le 1\\ \frac{1}{16}(3-x)^2, & 1<x\le 3\\ 0, & \text{elsewhere.}\end{cases}$$

Find the mean and the standard deviation of the r.v. $X$. (P.U., B.Sc. Hons. Part-I,

7.30 Find $k$, the mode and the mean for the distribution, the equation of whose p.d.f. is $f(x) = k$ for $0 \le x \le 3$. Also determine its variance. (P.U., B.A./B.Sc.

7.31 A continuous r.v. $X$ has the p.d.f. given by

$$f(x)=k(2-x)(x-5), \qquad 2\le x\le 5$$
$$=0, \qquad \text{elsewhere.}$$

Find the value of $k$, mean and variance. What are the values of the mode and median of distribution of $X$?

7.32 Let $X$ be a.v. with p.d.f.

$$f(x)=6(2-x)(x-1), \qquad 1\le x\le 2$$
$$=0, \qquad \text{elsewhere.}$$

Then show that the geometric mean, $G$, is given by

$6 \log (16\, G) = 19.$ (P.U., B.A. Hons. Part

7.33 Explain what is meant by *skewness of a distribution*, and define a suitable measure of skewness. Use this definition to calculate the skewness of the distribution defined by the density function

$$f(x)=kx^2(1-x)^3, \qquad 0\le x\le 1 \text{ and } 0, \quad \text{otherwise,}$$

where $k$ is a normalizing constant to be determined. It may be assumed that

$$\int_0^1 x^m(1-x)^n\,dx=\frac{m!\,n!}{(m+n+1)!}.$$ tentative

7.34 a) The rectangular distribution is given by $y = k$ between $x = -a$ and $x = a$. Find the variance and mean deviation.

b) The p.d.f. of a rectangular distribution is

$$f(x)=1, \qquad for\ 0\le x\le 1.$$

Obtain the first four mean moments and obtain also the mean deviation of the function. (P.U., B.A.

35 Let $X$ be a continuous random variable with p.d.f. given by

$$f(x)=\begin{cases}\dfrac{x}{2}, & 0\le x\le 1\\ \dfrac{1}{2}, & 1\le x\le 2\\ \dfrac{(3-x)}{2}, & 2\le x\le 3\\ 0, & \text{elsewhere.}\end{cases}$$

Find the mean, the variance and the moment measure of kurtosis of $X$.

Find the first four moments about the mean of the distribution $f(x) = x^2\,(6-x)^2$ between $x = 0$ and $x = 6$, and the kurtosis of the distribution.

Let $X$ and $Y$ have the joint p.d.f. described as follows:

| $(x, y)$ | (1, 1), | (1, 2), | (1, 3), | (2, 1), | (2, 2), | (2, 3) |
|---|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{2}{15}$ | $\frac{4}{15}$ | $\frac{3}{15}$ | $\frac{1}{15}$ | $\frac{1}{15}$ | $\frac{4}{15}$ |

and $f(x, y)$ is equal to zero elsewhere. Find the two marginal p.d.f's and the correlation co-efficient $\rho$.

a) Prove that the correlation co-efficient between two r.v.'s must be a number between $-1$ and $+1$ inclusive.

b) Given $f(x, y) = 2 - x - y$, $0 \le x \le 1$, $0 \le y \le 1$, and 0 elsewhere. Find co-efficient of correlation between $X$ and $Y$.

a) State and prove the Chebyshev's inequality. Explain the significance of this inequality.

b) Show that the variance of the sum (or difference) of two independent r.v.'s is equal to the sum of their variances.

a) Define Moment Generating Function and Characteristic Function of a random variable $X$.

b) Show that the m.g.f. of the sum of two independent r.v.'s is the product of their m.g.f.'s.

(P.U., B.A./B.Sc. 1993)

Two fair dice are rolled. Find the probability distribution of minimum of two numbers.

A bag contains 2 white and 3 black balls. Four persons A, B, C and D in the order named above, each draw a ball and does not replace it. The person who draws a white ball first receives Rs. 10. What are their respective expectations?

(P.U., B.A./B.Sc. 2006)

Define:

i) Probability Function
ii) Probability Density Function
iii) Distribution Function

b) A random variable X has the moment generating function about origin as

$$M_X(t) = (1-3t)^{-4}$$

Obtain the mean and variance of random variable X.

c) The annual gross earnings of a certain pop-singer are a random variable with an expecte value of Rs.40,00,000/- and a standard deviation of Rs.8,00,000/-. The singer's manag receives 15 percent of this amount. Determine the expected value and the standard devia of the amount received by the manager. (P.U., B.A./B.Sc. 200

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# CHAPTER 8

# DISCRETE PROBABILITY DISTRIBUTIONS

# DISCRETE PROBABILITY DISTRIBUTIONS

## 8.1 INTRODUCTION

As discussed in the previous chapter, a discrete probability distribution gives the probability of every possible value of a discrete random variable. We shall introduce here some important discrete probability distributions which are often used in statistical theory and analysis.

## 8.2 BINOMIAL PROBABILITY DISTRIBUTION

Many experiments consist of repeated independent trials, each trial having only two possible complementary outcomes. For example, the two possible outcomes of a trial may be head and tail, success and failure, right and wrong, alive and dead, good and defective, infected and not infected and so forth. If the probability of each outcome remains the *same* throughout the trials then such trials are called the *Bernoulli trials* and the experiment having n Bernoulli trials is called *binomial experiment*. In other words, an experiment is called a binomial probability experiment if it possesses the following four properties:

i) The outcomes of each trial may be classified into one of two categories, conventionally called *Success* (*S*) and *Failure* (*F*). It is to be noted that the outcome of interest is called a success and the other, a *failure*.

ii) The probability of success, denoted by *p*, remains constant for all trials.

iii) The successive trials are all independent.

iv) The experiment is repeated a *fixed* number of times, say n.

When X denotes the number of successes in n trials of a binomial probability experiment, it is called a *binomial random variable* and its p.d. is called the *Binomial Probability Distribution*. The *r.v. X* obviously take on anyone of the (n + 1) integer value 0, 1, 2, …, n.

When the binomial r.v. X assumes a value x, the binomial p.d. is given by:

$$f(x) = P(X = x) = \binom{n}{x} p^{x} q^{n-x}, x = 0, 1, 2, \ldots, n,$$

where $q = 1 - p$; the probability of failure on each trial. The binomial p.d. has two parameters n and p and generally denoted by $b(x\ ;\ n, p)$. The binomial probability distribution is appropriate when a random sample of size n is drawn with replacement from a finite population of size N, or sampling is done from infinite population.

The binomial p.d., which is the most widely used distribution in two-outcome situations, was discovered by the Swiss mathematician Jakob Bernoulli (1654–1704) whose main work on probability, *Ars Conjectandi* (the art of conjecturing) was published posthumously in Basel in 1713.

**8.2.1 Deviation of Binomial Probability Distribution.** To derive a formula that gives the probability of successes in n trials for a binomial experiment, we proceed as follows:

The experiment has n trials, each of which may result in S or F. The sample space has $2^n$ possible sample points or outcomes, each outcome consisting of a sequence $\{a_1, a_2, \ldots, a_n\}$, where each $a_i$, is either S or F. We desire to find the probability of these outcomes according to the number of successes.

First, we consider the probability of zero success, i.e. $P(X = 0)$. In case of zero success, every trial results in F and the event consists of a sequence of n F's, i.e. {FF … F}. Because in each trial, $P(S) = p$

and $P(F) = q$ and trials are independent, so we apply the multiplicative law of probability for independ events and obtain

$$P(FF...F) = P(F)\,P(F)...P(F) \quad \text{[n times]}$$
$$= q^n.$$

Since there is only one sequence of outcomes of n trials resulting in $F$'s, therefore

$$P(X=0) = q^n$$

Next, we consider the probability of one success, i.e. $P(X=1)$. In this case, one trial results in S an the remaining $(n-1)$ trials result in $F$'s. The event consisting of one S and $(n-1)$ $F$'s can occur in sev different sequences. One such sequence is $\{SFF...F\}$ and the probability for this sequence pq^{n-1}$. Another possible sequence is $\{FFSF...F\}$ and the probability for this sequence is the *same* as first sequence. In other words, the probability for any possible sequence consisting of one S and (n–

is $pq^{n-1}$. But the number of mutually exclusive sequences in which one S and $(n-1)$ $F$'s can occur, i

Therefore the probability of exactly one success for all possible sequences combined, is

$$P(X=1) = \binom{n}{1} pq^{n-1}$$

The above argument may be repeated for X = 2, 3, 4, etc.

Finally, we consider the general case, i.e. when $X=x$. The probability of a sequence that has

x successes and (n–x) failures is $p^x q^{n-x}$ and there are $\binom{n}{x}$ different sequences in which x succe

(n–x) failures can occur. Therefore the probability of x success in n trials is

$$P(X=x) = \binom{n}{x} p^x q^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n.$$

Thus we have obtained the formula for the *binomial probability distribution* having n probability p for success. The binomial probability distribution derives its name from the fac

probabilities $\binom{n}{x} p^x q^{n-x}$, for x = 0, 1, 2, ..., n are the successive terms of the *binomial e*

$(q+p)^n$. That is

$$(q+p)^n = \Sigma \binom{n}{x} p^x q^{n-x}, \quad \text{where } x = 0, 1, 2, \ldots, n$$
$$= \binom{n}{0} q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \ldots + \binom{n}{n} q^n$$
$$= b(0; n, p) + b(1; n, p) + b(2; n, p) + \ldots + b(n; n, p).$$

The sum of probabilities, i.e. $\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-r}$ or $\sum_{x=0}^{n} b(x; n, p) = 1$ because $q+p=1$

necessary condition for any probability distribution. It is to be noted that the probability

successes is given by $p(X \le r) = \sum_{x=0}^{r} \binom{n}{x} p^x q^{n-x}$, where $p(X \le r)$ is the *cumulative binomial distribution* of X. There are tables for the cumulative probabilities $P(X \le r) = \sum_{x=0}^{r} b(x;n,p)$ for some values of n, p and r.

**Example 8.1** A fair coin is tossed 5 times. Find the probabilities of obtaining various numbers of heads.

Let us regard the tossing of a coin as an experiment. Then we observe that

i) each toss of a coin (i.e. each trial) has two possible outcomes, heads (*success*) and tails (*failure*);

ii) the probability of a head (*success*) is $p = \frac{1}{2}$ and remains the same for successive tosses;

iii) the successive tosses of the coin are independent; and

iv) the coin is tossed 5 times.

Therefore the *r.v. X* which denotes the number of heads (*successes*) has a binomial probability distribution with $p$=1/2 and n=5. The possible value of X are 0, 1, 2, 3, 4 and 5. Hence



$$P(\text{no head}) = P(X=0) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = 1\times\left(\frac{1}{2}\right)^5 = \frac{1}{32},$$

$$P(\text{1 head}) = P(X=1) = \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{5-1} = 5\times\left(\frac{1}{2}\right)^5 = \frac{5}{32},$$

$$P(\text{2 heads}) = P(X=2) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{5-2} = 10\times\left(\frac{1}{2}\right)^5 = \frac{10}{32},$$

$$P(\text{3 heads}) = P(X=3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3} = 10\times\left(\frac{1}{2}\right)^5 = \frac{10}{32},$$

$$P(\text{4 heads}) = P(X=4) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4} = 5\times\left(\frac{1}{2}\right)^5 = \frac{5}{32}, \text{ and}$$

$$P(\text{5 heads}) = P(X=5) = \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = 1\times\left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

These probabilities can also be obtained by expanding the binomial $\left(\frac{1}{2}+\frac{1}{2}\right)^5$. The binomial probability distribution for the number of heads obtained in 5 tosses of a fair coin is

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{32}$ | $\frac{5}{32}$ | $\frac{10}{32}$ | $\frac{10}{32}$ | $\frac{5}{32}$ | $\frac{1}{32}$ |

**Example 8.2** An event has the probability $p=\frac{3}{8}$. Find the complete binomial distribution

n = 5 trials.

Hence $\quad p=\frac{3}{8}$ so that $q=1-p=\frac{5}{8}$; and n = 5.

Hence the desired probabilities are the successive terms in the binomial expansion of $\left(\frac{5}{8}+\frac{3}{8}\right.$

$$\left[\left(\frac{5}{8}\right)^5+\binom{5}{1}\left(\frac{5}{8}\right)^4\left(\frac{3}{8}\right)+\binom{5}{2}\left(\frac{5}{8}\right)^3\left(\frac{3}{8}\right)^2+\binom{5}{3}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)^3+\binom{5}{4}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^4+\left(\frac{5}{8}\right)^5\right]$$

i.e. $\quad \frac{1}{(8)^5}[(5)^5+5.(5)^5(3)+10.(5)^3(3)^2+10(5)^2(3)^3+5(5)(3)^4+(3)^5]$

i.e. $\quad \frac{1}{32768}[3125+9375+11250+6750+2025+243]$

i.e. $\quad [0.0954+0.2861+0.3433+0.2060+0.0618+0.0074]$

We can now write these probabilities in the form of a probability table as below:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(X=x)$ | 0.0954 | 0.2861 | 0.3433 | 0.2060 | 0.0618 | 0.0074 |



**Example 8.3** Let X have a binomial distribution with n = 4 and $p=\frac{1}{3}$. Find $P(X=1)$, $P$

$P(X=3)$, $P(X=6)$ and $P(X\leq 2)$.

The binomial probability distribution for n = 4 and $p=\frac{1}{3}$, is

$$f(x)=\binom{4}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{4-x} \qquad \text{for } x = 0, 1, 2, 3, 4.$$

Now $\quad P(X=1)=\binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{4-1}=\frac{32}{81};$

$P\left(X=\frac{3}{2}\right)=f\left(\frac{3}{2}\right)=0$; because a r.v. X with a binomial distribution takes only one of the integer values 0, 1, 2, …, n.

$$P(X=3)=f(3)=\binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{4-3}=\frac{8}{81};$$

P(X = 6) = f(6) = 0, because X can take only values 0, 1, 2, 3, 4.

$$P(X\le 2)=\sum_{x=0}^{2}f(x)=f(0)+f(1)+f(2)$$

$$=\binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4+\binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3+\binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$

$$=\frac{16}{81}+\frac{32}{81}+\frac{24}{81}=\frac{72}{81}=\frac{8}{9}.$$

**Example 8.4** A and B play a game in which *A's* probability of winning is 2/3. In a series of 8 games, what is the probability that A will win (i) *exactly* 4 games, (ii) *at least* 4 games, (iii) 6 or *more* games, and (iv) from 3 to 6 games?

We observe that

- there are two possible outcomes, i.e. A will win or will not win the game;
- the probability of A's winning in each game is $p = 2/3$;
- the successive games are independently won or lost; and
- there are 8 games.

Therefore the Binomial probability distribution with $n = 8$ and $p=2/3$ is appropriate.

Let X denote the number of games won by A. Then

$$P(X=4)=\binom{8}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^4=\frac{1120}{6561}=0.1707$$

$P(X\ge 4)=1-P(X<4)$; ($\because$ at least 4 means 4 or more)

$$=1-\sum_{x=0}^{3}\binom{8}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{8-x}$$

$$=1-\left[\left(\frac{1}{3}\right)^8+8\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^7+28\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^6+56\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^5\right]$$

$$=1-\frac{1}{6561}[1+16+112+448]$$

$$=1-\frac{577}{6561}=\frac{5984}{6561}=0.9121$$

iii) $P(X \geq 6) = \sum_{x=6}^{8} \binom{8}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{8-x}$

$$= \binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2 + \binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right) + \binom{8}{8}\left(\frac{2}{3}\right)^8$$

$$= \frac{64}{6561}[28+16+4] = \frac{64 \times 48}{6561} = \frac{1024}{2187} = 0.4682$$

iv) $P(3 \leq X \leq 6) = \sum_{x=3}^{6} \binom{8}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{8-x}$

$$= \binom{8}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^5 + \binom{8}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^4 + \binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3 + \binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2$$

$$= \frac{2^3}{(3)^8}[56+140+224+224]$$

$$= \frac{8 \times 644}{6561} = \frac{5152}{6561} = 0.7852$$

**Example 8.5** The experience of a house-agent indicates that he can provide accommodation for 75 percent of the clients who come to him. If on a particular occasion, approach him independently, calculate the probability that (i) less than 4 clients, (ii) exactly (iii) *at least* 5 clients, will get satisfactory accommodation.



We observe that

a) there are two possible outcomes, *i.e.* each client will get *or* will not get accommodation,

b) on each occasion, probability of getting accommodation is $p = 3/4$,

c) clients approach the house-agent independently, and

d) there are 6 clients.

Therefore the binomial probability distribution with $n$=6 and $p$=3/4 is appropriate to calculate desired probabilities.

Let $X$ denote the number of clients who get satisfactory accommodation. Then we need (i) $P(X<4)$, (ii) $P(X=4)$ and (iii) $P(X \geq 5)$. Hence,

i) $P(X<4) = \sum_{x=0}^{3} \binom{6}{x} \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{6-x}$

$$= \left(\frac{1}{4}\right)^6 + \binom{6}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^5 + \binom{6}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^4 + \binom{6}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3$$

$$= \left(\frac{1}{4}\right)^6 [1+(6)(3)+(15)(9)+(20)(27)]$$

$$= \frac{694}{4096} = 0.169$$

ii) $P(X=4) = \binom{6}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^2 = \frac{15\times 81}{(4)^6} = \frac{1215}{4096} = 0.297$

iii) $P(X \geq 5) = \sum_{x=5}^{6}\binom{6}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{6-x}$

$$= \binom{6}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right) + \binom{6}{6}\left(\frac{3}{4}\right)^6$$

$$= \left(\frac{1}{4}\right)^6 [(6)(3)^5 + (3)^6] = \frac{2187}{4696} = 0.534$$

**8.2.2 Binomial Frequency Distribution.** If the binomial probability distribution is multiplied by the number of experiments or sets, the resulting distribution is known as the *binomial frequency* ...tion. Thus the expected frequency of $x$ successes in $N$ experiments is $N.\binom{n}{x}p^x q^{n-x}$. It should be ...that the n independent trials constitute *one* experiment or one set.



**Example 8.6** Six dice are thrown 729 times. How many times do you expect *at least* three dice to ...ive or a six?

The probability of getting a 5 or a 6 with one die is p=2/6. Since 6 dice are thrown and there are ... the binomial frequency distribution is given by

$$729\left(\frac{2}{3}+\frac{1}{3}\right)^6.$$

...ce the expected number of times *at least* 3 dice showing 5 or 6

$$= 729\left[\sum_{x=3}^{6}\binom{6}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{6-x}\right]$$

$$= 729\left[\binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + \binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^6\right]$$

$$= \frac{729}{(3)^6}[160+60+12+1] = 233.$$

**Example 8.7** A certain event is believed to follow the binomial distribution. In 1024 samples of 5 the result was observed once 405 times and twice 270 times. Find p and q.

The first three term in the expansion of the Binomial Frequency Distribution $N(q+p)^n$ corresponding to x = 0, 1, and 2 are $Nq^n$, $N\binom{n}{1}q^{n-1}p$ and $N\binom{n}{2}q^{n-2}p^2$.

We are given N = 1024, n = 5 and the following information:

$$1024\binom{5}{1}q^{5-1}p = 405,$$

$$1024\binom{5}{2}q^{5-2}p^2 = 270,$$

Dividing the second equation by the first, we get

$$\frac{10q^3p^2}{5q^4p} = \frac{270}{405} \text{ or } \frac{2p}{q} = \frac{2}{3}$$

or $\quad$ 3p = q or 3p = 1 – p or 4p = 1

Hence $p = \frac{1}{4}$ and $q = \frac{3}{4}$



**8.2.3 Properties of the Binomial Probability Distribution.** The properties of the binomial distribution include the mean number of successes, the variance of the number of successes, measures of skewness and kurtosis, etc. and the shape of the distribution. Some of the properties are described below.

1) Let $X$ be a random variable with the binomial distribution $b(x; n, p)$. Then its mean and variance are given by $\mu$ = np and $\sigma^2 = npq$ respectively.

Now $\quad$ Mean, $\mu = E(X)$

$$= \sum_{x=0}^{n} x\binom{n}{x}p^x q^{n-x}, \text{ where } x = 0, 1, 2, \ldots, n.$$

$$= 0.q^n + 1.\binom{n}{1}q^{n-1}p + 2\binom{n}{2}q^{n-2}p^2 + \ldots + np^n$$

$$= np[q^{n-1} + \binom{n-1}{1}q^{n-2}p + \binom{n-1}{2}q^{n-3}p^2 + \ldots + p^{n-1}]$$

$$= np\,(q+p)^{n-1}$$

$= np$, because $q + p = 1$.

**Alternative Method.**

$$\therefore \quad \text{Mean} = E(X) = \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x}$$

But $$x\binom{n}{x} = \frac{x(n)(n-1)!}{x(x-1)!(n-x)!} = n\binom{n-1}{x-1}$$

$E(X) = n\sum_{x=1}^{n}\binom{n-1}{x-1}p^x q^{n-x}$, *for* $x = 1, 2, \ldots, n$ (since the first term in the summation being zero ($x = 0$) is omitted).

$$= np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}q^{(n-1)-(x-1)}$$

Substituting $y = x - 1$ and $m = n - 1$ in the summation, we get

$E(X) = np\sum_{y=0}^{m}\binom{m}{y}p^y q^{m-y}$ (as $x$ ranges from 1 to $n$, so $y$ $(= x - 1)$ must range from 0 to $n - 1$ i.e. $m$)

$= np$ $\quad$ ($\because$ summation is the expansion of $(q+p)^m$)

Hence mean = np. In other words, the mean number of successes is $np$. Similarly the mean number of failures is $nq$.

By definition, the variance $\sigma^2$, is given by

$$\sigma^2 = E[X - \mu]^2 = E(X^2) - [E(X)]^2$$

But $$\begin{aligned} E(X^2) &= E[X(X-1) + X] = E[X(X-1)] + E(X) \\ &= E[X(X-1)] + np \end{aligned}$$

Now $$E[X(X-1)] = \sum_{x=0}^{n} x(x-1)\binom{n}{x}p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\frac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!}p^2 p^{x-2} q^{n-x}$$

$$= n(n-1)p^2\sum_{x=2}^{n}\frac{(n-2)!}{(x-2)!(n-x)!}p^{x-2}q^{n-x}$$

(x starts at 2 since x = 0, 1 add nothing to the sum)

The term (n – x) may be written as [(n – 2) – (x – 2)].

Substituting y = x – 2 and m = n – 2 in the summation, we obtain

$$E[X(X-1)] = n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y q^{m-y}$$

$$= n(n-1)p^2 \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}$$

$$= n(n-1)\, p^2 \qquad (\because \text{ summation is 1})$$

Thus $\sigma^2 = E(X^2) - [E(X)]^2$

$$= E[X(X-1)] + E(X) - [E(X)]^2$$

$$= n(n-1)\, p^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2 = np\,(1-p) = npq, \text{ and}$$

$$\sigma = \sqrt{npq}$$

Hence the variance of the number of successes is $npq$, and the standard deviation is $\sqrt{npq}$

2) **Higher Moments** of the distribution are found as below:

By definition, the moments about the origin are given by the relation

$$\mu_r' = E(X^r)$$

Now $\mu_1' = E(X) = np$

$$\mu_2' = E(X^2) = n(n-1)\, p^2 + np$$

$$\mu_3' = E(X^3)$$

But $X^3 = X(X-1)(X-2) + 3X(X-1) + X$,

$\therefore \quad E(X^3) = E[X(X-1)(X-2)] + 3E[X(X-1)] + E(X)$

Now $\mathrm{E}(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np$,

$$3E[X(X-1)] = 3\sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x}$$

$$= 3n\,(n-1)\,p^2, \text{ and}$$

$$E[X(X-1)(X-2)] = \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x}$$

But $\quad x(x-1)(x-2)\binom{n}{x} = x(x-1)(x-2)\dfrac{n(n-1)(n-2)(n-3)!}{x(x-1)(x-2)(x-3)!(n-x)!}$

$$= n(n-1)(n-2)\binom{n-3}{x-3}$$

$E[X(X-1)(X-2)] = n(n-1)\,(n-2)\,\sum_{x=3}^{n}\binom{n-3}{x-3}p^3\,p^{x-3}\,q^{n-x}$ ($x$ starts at 3 since $x = 0, 1, 2$ add nothing to the summation)

Substituting $y = x - 3$ and $m = n - 3$ in the summation, we have

$$E[X(X-1)(X-2)] = n(n-1)\,(n-2)\,p^3\sum_{y=0}^{m}\binom{m}{y}p^y q^{m-y}$$

$$= n(n-1)\,(n-2)p^3$$

$\mu_3' = n(n-1)\,(n-2)\,p^3 + 3n(n-1)\,p^2 + np$

$\mu_4' = E(X^4)$

$$= \sum_{x=0}^{n} x^4\binom{n}{x}p^x q^{n-x}$$

$x^4$ as $x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x$ and proceeding as above, we get

$$\mu_4' = n(n-1)(n-2)(n-3)\,p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

...re $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$$= [n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np] - 3np[n^2p^2 - np^2 + np] + 2n^3p^3$$

$$= np\,[1 - 3p + 2p^2] = np\,(1-p)\,(1-2p)$$

$$= npq\,(q-p);$$

$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2(\mu_2') - 3(\mu_1')^4$

...stitution and simplification give

$$\mu_4 = n^4[p^4 - q^4] + n^3[-6p^4 + 6p^3 + 6p^3 - 6p^4] +$$

$$n^2[11p^4 - 18p^3 + 7p^2 - 4p^2 + 12p^3 - 8p^4] + n[-6p^4 + 12p^3 - 7p^2 + p]$$

$$= 3n^2p^2\,(1-p)^2 + np\,(1-p)\,(1 - 6p + 6p^2)$$

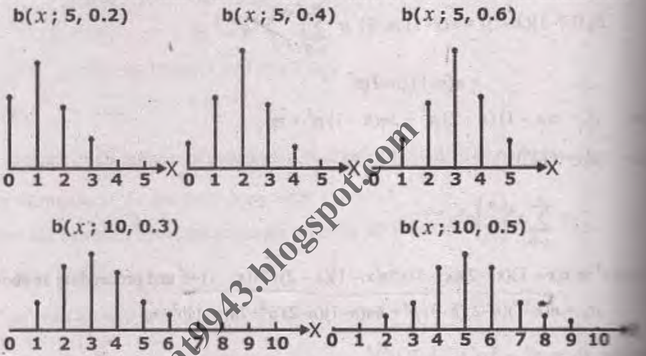
$$= 3n^2p^2q^2 + npq\,(1 - 6pq) = npq\,[1 + 3(n-2)\,pq].$$

Now $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{[npq(q-p)]^2}{(npq)^3} = \dfrac{(q-p)^2}{npq} = \dfrac{(1-2p)^2}{npq}$;

$$\beta_2 = \frac{\mu_4}{\mu_2} = \frac{3n^2p^2q^2 + npq(1-6pq)}{(npq)^2} = 3 + \frac{1-6pq}{npq}$$

Hence moment co-efficient of skewness and kurtosis are respectively $\dfrac{1-2p}{\sqrt{npq}}$ and $3+\dfrac{1-6pq}{npq}$

3) The shape of the binomial probability distribution depends on the values of the two para *p* and *n*. The sketches indicate the influence of *p* and *n* on the shape of the distribution




Thus we observe that, when $p < \dfrac{1}{2}$, the distribution is positively skewed and when $p$ distribution becomes negatively skewed. In general, when $p \neq q$, the distribution will be ske more the difference between *p* and *q*, the greater the skewness will be. When $p = \dfrac{1}{2}$, the dis always symmetrical. As *n*, the number of trials, increases to $\infty$, $\beta_1 \to 0$ and $\beta_2 \to 3$. Hence *n*, the binomial probability distribution is symmetrical and mesokurtic.

**Example 8.8** In the B.A. Examination, 24 candidates offered Statistics. If the prob passing the subject be $\dfrac{1}{3}$, find the mean and the dispersion of the distribution.

Here $n = 24$ and $p = \dfrac{1}{3}$ so that $q = 1 - p = \dfrac{2}{3}$

Hence mean $= np = 24 \times \dfrac{1}{3} = 8$; and

$$\sigma^2 = npq = 24 \times \frac{1}{3} \times \frac{2}{3} = \frac{16}{3} = 5.33.$$

**8.2.4 The Recurrence Formula for the Binomial Distribution.** Beginning with the value of $P(X=0)$, probabilities for other values of $X$, the number of successes can be computed more easily by the *recurrence formula*

$$P(X=x)=\frac{n-x+1}{x}\frac{p}{q}.P(X=x-1)$$

established as follows:

Let $X$ be a random variable with the binomial probability distribution $b(x;\, n,\, p)$. Then

$$P(X=x)=\binom{n}{x}p^x q^{n-x}=\frac{n!}{x!(n-x)!)}p^x q^{n-x}\text{, and}$$

$$P(X=x-1)=\binom{n}{x-1}p^{x-1}q^{n-x+1}=\frac{n!}{(x-1)!(n-x+1)!}p^{x-1}q^{n-x+1},$$

Dividing $P(X=x)$ by $P(X=x-1)$, we get

$$\frac{P(X=x)}{P(X=x-1)}=\frac{n!}{x!(n-x)!}.\frac{(x-1)!(n-x+1)!}{n!}.\frac{p^x q^{n-x}}{p^{x-1}\,q^{n-x+1}}$$

$$=\frac{n-x+1}{x}.\frac{p}{q}$$

Hence $\quad P(X=x)=\dfrac{n-x+1}{x}.\dfrac{p}{q}P(X=x-1)\quad$ for $x = 1, 2, \ldots, n$.

Although computations can be carried out more easily by this method, great care must be taken with accuracy.



**Example 8.9** If $X$ is binomially distributed with mean 3.20 and variance 1.152, find the complete binomial probability distribution.

If $X$ a binomial *r.v.* with parameters $n$ and $p$, then $E(X) = np$ and $\text{Var}(X) = npq$.

Now $\quad E(X) = 3.20$ so $np = 3.20$,

and $\quad \text{Var}(X) = 1.152$ so $npq = 1.152$

Substituting for $np$ in the second equation, we obtain

$$(3.20)\ q = 1.152$$

or $\quad q=\dfrac{1.152}{3.20}=0.36$ so $p = 1 - q = 0.64$

and $\quad n(0.36) = 3.20$ gives $n = 5$.

Therefore $n = 5$ and $p = 0.64$ so that X is $b(x;\, 5,\, 0.64)$.

Now $\quad P(X=x)=\dbinom{5}{x}(0.64)^x\,(0.36)^{5-x},$

and $\quad P(X=0) = (0.36)^5 = 0.006047$

Beginning with the value of $P(X = 0)$, we compute the probabilities of other values of $X$, usi recurrence formula

$$P(X = x) = \frac{n-x+1}{x}\frac{p}{q}.P(X = x-1)$$

$$= \frac{6-x}{x}\left(\frac{0.64}{0.36}\right).P(X = x-1)$$

$$P(X = 1) = \frac{5}{1}\left(\frac{0.64}{0.36}\right)(0.006047) = 0.053751,$$

$$P(X = 2) = \frac{4}{2}\left(\frac{0.64}{0.36}\right)(0.053751) = 0.191115,$$

$$P(X = 3) = \frac{3}{3}\left(\frac{0.64}{0.36}\right)(0.191115) = 0.339760,$$

$$P(X = 4) = \frac{2}{4}\left(\frac{0.64}{0.36}\right)(0.339760) = 0.302009,$$

$$P(X = 5) = \frac{1}{5}\left(\frac{0.64}{0.36}\right)(0.302009) = 0.107381.$$

The sum of probabilities turns out to be 1.000063 instead of 1. Error has been intro rounding process.

**8.2.5 Fitting a Binomial Distribution to Observed Data.** Fitting a binomial distri given frequency distribution consists of (i) estimating the values of $p$ and $n$, the two param completely determine a binomial distribution, and (ii) calculating the probabilities as well as the frequencies of $x = 0, 1, 2, \ldots, n$. Assuming that the given frequency distribution has the chara the fitted theoretical binomial distribution, we calculate the mean of the observed frequency $\bar{x}$ and taking it as the estimate of $\mu$, we equate it to its theoretical value, i.e. $np$, where $n$ is as the largest $x$ value given. Having found the value of $p$, we compute the expected freque procedure is illustrated by the following example.

**Example 8.10** Fit a binomial distribution to the following data, obtained by tossing a b times:

| No. of Heads | 0 | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|---|
| Frequency | 12 | 56 | 74 | 39 | 18 | 1 | 200 |

(P.U., B.A. (H

To fit a binomial distribution, we need to find $n$ and $p$. Hence $n = 5$, the largest $x$-valu we use the relationship $\bar{x} = np$.

Now $\quad \bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{0+56+148+117+72+5}{200} = \dfrac{398}{200} = 1.99$

ing the relation $\bar{x} = np$, we get $5p = 1.99$ or $p = 0.398$.

ng *r.v.* $X$ represent the number of heads, gives the fitted binomial distribution as

$$b(x;5,0.398) = \binom{5}{x}(0.398)^x (0.602)^{5-x}$$

Now the probabilities and frequencies are calculated as below:

| No. of head ($x$) | Probability $f(x)$ | | Expected frequency |
|---|---|---|---|
| 0 | $\binom{5}{0}q^5 = (0.602)^5$ | $= 0.07907$ | 15.8 |
| 1 | $\binom{5}{1}q^4 p = 5.(0.602)^5 (0.398)$ | $= 0.26136$ | 52.5 |
| 2 | $\binom{5}{2}q^3 p^2 = 10.(0.602)^3 (0.398)^2$ | $= 0.34559$ | 69.1 |
| 3 | $\binom{5}{3}q^2 p^3 = 10.(0.602)^2 (0.398)^3$ | $= 0.22847$ | 45.7 |
| 4 | $\binom{5}{4}qp^4 = (0.602)(0.398)^4$ | $= 0.07553$ | 15.1 |
| 5 | $\binom{5}{5}p^5 = (0.398)^5$ | $= 0.00998$ | 2.0 |
| Total | | $= 1.00000$ | 200.0 |



e expected frequencies are obtained by multiplying each of the probability by 200. The es can also be calculated, using the binomial recurrence formula:

$$P(X=x) = \frac{n-x+1}{x}.\frac{p}{q}P(X=x-1)$$

**6 Moment Generating and Cumulant Generating Functions of the Binomial Distribution.** of the binomial probability distribution $b(x; n, p)$ is derived as below:

$$M_0(t) = E(e^{tX}) \qquad \text{(by definition)}$$

$$= \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (q + pe^t)^n.$$

The expansion of this binomial is purely algebraic and need not be interpreted in term probabilities.

We get the moments by differentiating $M_0(t)$ once, twice, etc. with respect to $t$ and putting Thus

$$\mu_1' = E(X) = \left[\frac{d}{dt}(q + pe^t)^n\right]_{t=0}$$

$$= [npe^t (q + pe^t)^{n-1}]_{t=0} = np; \text{ and}$$

$$\mu_2' = E(X^2) = \left[\frac{d^2}{dt^2}(q + pe^t)^n\right]_{t=0}$$

$$= [npe^t (q+pe^t)^{n-1}]_{t=0} + [n(n-1)p^2 e^{2t} (q+pe^t)^{n-2}]_{t=0}$$

$$= np + n(n-1)p^2$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = npq$$

In a similar way, the higher moments are obtained.

The cumulant generating function (*c.f.*) is given by

$$k(t) = \log_e M_0(t)$$

$$= \log_e [(q + pe^t)^n] = n \log_e [q + pe^t]$$

$$= n \log_e \left[q + p\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)\right]$$

$$= n \log_e \left[1 + \left(pt + p\frac{t^2}{2!} + p\frac{t^3}{3!} + \dots\right)\right] (\because q = 1 - p)$$

Expanding the log and comparing powers in $t$, we find the first four cumulants to be

$$k_1 = \mu_1' = np;$$

$$k_2 = \mu_2 = np(1-p) = npq;$$

$$k_3 = \mu_3 = np(1-p)(1-2p) = npq(q-p);$$

$$k_4 = np(1-p)(1-6p+6p^2) = npq(1-6pq).$$

**Example 8.11** Prove for the binomial distribution, the following relation

$$\mu_{r+1} = pq\left(nr\mu_{r-1} + \frac{d\mu_r}{dp}\right)$$

and hence find $\mu_2, \mu_3$ and $\mu_4$. (P.U., M.A. (Stats.), 1969)

By definition, the rth moment about the mean is

$$\mu_r = \sum_{j=0}^{n}(j-np)^r\binom{n}{j}p^j q^{n-j}, \quad \text{where } q = 1-p$$

$$\frac{d\mu_r}{dp} = -rn\sum(j-np)^{r-1}\binom{n}{j}p^j q^{n-j} - \sum(j-np)^r\binom{n}{j}p^j(n-j)q^{n-j-1} + \sum(j-np)^r\binom{n}{j}jp^{j-1}q^{n-j}$$

$$= -rn\mu_{r-1} + \sum(j-np)^r\binom{n}{j}p^j q^{n-j}\left[-(n-j)\frac{1}{q} + j\frac{1}{p}\right]$$

$$= -rn\mu_{r-1} + \frac{1}{qp}\sum(j-np)^{r+1}\binom{n}{j}p^j q^{n-j}$$

$$= -rn\mu_{r-1} + \frac{1}{qp}\mu_{r+1}.$$

Multiplying both sides by $pq$ and transposing, we get

$$\mu_{r+1} = pq\left(nr\mu_{r-1} + \frac{d\mu_r}{dp}\right)$$

Putting r = 1, 2 and 3, we get

$$\mu_2 = pq\left(n\mu_0 + \frac{d\mu_1}{dp}\right) = pq(n+0) = npq, \ (\because \mu_0 = 1)$$

$$\mu_3 = pq\left(2n\mu_1 + \frac{d\mu_2}{dp}\right) = pq\left[2n(0) + \frac{d}{dp}(npq)\right] \ (\because \mu_1 = 0)$$

$$= pq\,[0 + n(1-2p)] = pq[nq - np] = npq\,(q-p),$$

$$\mu_4 = pq\left[3n\mu_2 + \frac{d\mu_3}{dp}\right]$$

$$= pq\left[3n(npq) + \frac{d}{dp}(np - np^2)(1-2p)\right]$$

$$= 3n^2p^2q^2 + npq(1-6pq).$$

## 8.3 HYPERGEOMETRIC PROBABILITY DISTRIBUTION

There are many experiments in which the condition of *independence* is violated and the probability of success does not remain constant for all trials. Such experiments are called *hypergeometric experiments*. In other words, a hypergeometric experiment has the following properties:

i) The outcomes of each trial may be classified into one of two categories, success and failure.

ii) The probability of success *changes* on each trial.

iii) The successive trials are *dependent*.

iv) The experiment is repeated a fixed number of times.

The number of successes, $X$ in a hypergeometric experiment is called a hypergeometric r.v. and its probability distribution is called the *hypergeometric distribution*. When the hypergeometric r.v. assumes a value $x$, the hypergeometric *p.d.* is given by the formula.

$$P(X=x)=\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, \text{ for } x \text{ such that } x=0,1,2,\ldots,n \text{ and } x=0,1,2,\ldots,k.$$

where N = number of units in the set or population, (a positive integer)

$n$ = number of units in the subset or sample (a positive integer) and

$k$ = number of *successes* in the set or population (a non negative integer less than or equal to $N$)

The hypergeometric *p.d.* has three parameters $N$, $n$ and $k$, (or $N$, $n$ an $p=\frac{k}{N}$), and is generally denoted by $h(x; N, n, k)$. The hypergeometric *p.d.* is appropriate when

i) a random sample of size $n$ is drawn without replacement from a finite population of $N$ units.

ii) $k$ of the units are of one kind (classified as *success*) and the remaining $N-k$ of another kind (classified as *failure*).

**8.3.1 Derivation of Hypergeometric Distribution.** Suppose a set contains $N$ elements of which $k$ are classified as *success* and $N-k$ are classified as *failure*; the two classes being mutually exclusive and exhaustive, and we select a subset of $n$ elements $(n \le N)$ from the set without replacement. Then the total number of ways in which a subset of $n$ elements can be chosen from $N$ is $\binom{N}{n}$.

Let $X$ denote the number of successes and let $X=x$ if and only if we choose $x$ successes from $k$ successes and $n-x$ failures from $N-k$ failures in the set. Then the number of ways in which $x$ successes and $n-x$ failures can be chosen, is $\binom{k}{x}\binom{N-k}{n-x}$.

Hence the probability that $x$ of the $n$ elements are successes, is

$$P(X=x)=\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, \text{ for } x \text{ such that } 0\le x\le n \text{ and } 0\le x\le k.$$

is the required formula for the hypergeometric probabilities.

To provide that the sum of the hypergeometric probabilities is one, we use the following useful ematical result (or formula).

$$\sum_{j=0}^{m}\binom{a}{j}\binom{b}{m-j}=\binom{a+b}{m},$$

by expanding both sides of

$$(1+x)^a(1+x)^b=(1+x)^{a+b}$$

equating the coefficients of $x^n$.

$$\sum_{x=0}^{n}P(X=x)=\frac{1}{\binom{N}{n}}\sum_{x=0}^{n}\binom{k}{x}\binom{N-k}{n-x}$$

$$=\frac{1}{\binom{N}{n}}\binom{N-k+k}{n-x+x}=\frac{1}{\binom{N}{n}}\binom{N}{n}=1.$$

The hypergeometric distribution derives its name from the fact that probability generating function be put in the form of a hypergeometric series.



**Example 8.12** An urn contains 4 red balls and 6 black balls. A sample of 4 balls is selected from the urn without replacement. Let $X$ be the number of red balls contained in the sample, then find the probability distribution for $X$.

Here $X$ is a hypergeometric $r.v.$ because

- the results of each draw may be classified as either red (*success*) or black (*failure*),
- the probability of success changes on each draw,
- the successive draws are dependent as the selection is made without replacement,
- the drawing is repeated a fixed number of times ($n = 4$).

Now, $N = 4 + 6 = 10$, $k = 4$, $n = 4$ and the possible value of $X$ are 0, 1, 2, 3, and 4. Therefore the probabilities of these possible outcomes are

$$P(X=0)=h(0;10,4,4)=\frac{\binom{4}{0}\binom{6}{4}}{\binom{10}{4}}=\frac{15}{210}$$

$$P(X=1)=h(1;10,4,4)=\frac{\binom{4}{1}\binom{6}{3}}{\binom{10}{4}}=\frac{80}{210}$$

$$P(X=2)=h(2;10,4,4)=\frac{\binom{4}{2}\binom{6}{2}}{\binom{10}{4}}=\frac{90}{210}$$

$$P(X=3)=h(3;10,4,4)=\frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}}=\frac{24}{210}$$

$$P(X=4)=h(4;10,4,4)=\frac{\binom{4}{4}\binom{6}{0}}{\binom{10}{4}}=\frac{1}{210}$$

Hence the hypergeometric *p.d.* of *X* is as follows:

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $h(x;\ 10, 4, 4)$ | $\frac{15}{210}$ | $\frac{80}{210}$ | $\frac{90}{210}$ | $\frac{24}{210}$ | $\frac{1}{210}$ |



**Example 8.13** The names of 5 men and 5 women are written on slips of paper and placed in a hat. Four names are drawn. What is the probability that 2 are men and 2 are women?

Let $X$ denote the number of men. Then

$N = 5 + 5 = 10$ names to be drawn from;

$n = 4$, and (here possible values of $x$ are 0, 1, 2, 3, 4, i.e. $n$)

$k = 5$.

Hence the hypergeometric distribution is

$$h(x;10,4,5)=\frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}$$

and the required probability *i.e.* $P(X=2)$ is

$$h(2;10,4,5)=\frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}=\frac{10}{21}$$

**Example 8.14** What is the probability that a poker hand will contain exactly 2 aces?

Let us regard the 4 aces as success and the 48 nonaces as failures. Then we have

$N=52$, $n=5$, (number of cards in a poker hand).

$k=4$ and $x=2$. (here possible values of $x$ are 0, 1, 2, 3 and 4, i.e. $k$)

Hence the probability that a poker hand will contain exactly 2 aces is

$$P(X=2)=h(2;52,5,4)=\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}=\frac{2162}{54145}=0.0399\,.$$

**8.3.2 Properties of Hypergeometric Distribution.** The important properties of the hypergeometric probability distribution are given here.

1) The mean and variance of the hypergeometric probability distribution are $\mu=np$ and $\sigma^2=npq\frac{N-n}{N-1}$, where $p=\frac{k}{N}$ an $q=\frac{N-k}{N}$. Let $X$ have the hypergeometric probability distribution given by

$$h(x;N,n,k)=\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}},\ \text{for } x \text{ such that } 0\le x\le n \text{ and } 0\le x\le k\,.$$

Then the mean, $\mu$, is given by

$$\begin{aligned}\mu&=E(X)\\&=\sum_{x=0}^{n}\frac{x.\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}\\&=\sum_{x=0}^{n}\frac{x.k(k-1)!\binom{N-k}{n-x}}{x(x-1)!(k-x)!\binom{N}{n}}=\sum_{x=1}^{n}\frac{k.\binom{k-1}{x-1}\binom{N-k}{n-x}}{\binom{N}{n}}\end{aligned}$$

Let $y = x - 1$, then

$$\mu = \frac{k}{\binom{N}{n}} \sum_{y=0}^{n-1} \binom{k-1}{y}\binom{N-k}{n-1-y} \qquad [\because \text{ when } x=1, y=0, \text{ and when } x=n, y=n-1]$$

$$= \frac{k}{\binom{N}{n}} \binom{N-1}{n-1} \qquad \left[\because \sum_{j=0}^{m} \binom{a}{j}\binom{b}{m-j} = \binom{a+b}{m}\right]$$

$$= \frac{kn(n-1)!(N-n)!}{N(N-1)!} \cdot \frac{(N-1)!}{(n-1)!(N-n)!}$$

$$= \frac{nk}{N} = np, \text{ where } p = \frac{k}{N}.$$

Thus the mean of the hypergeometric distribution is the same as that of the binomial distribution.

By definition, the variance, $\sigma^2$, is given by

$$\sigma^2 = E[X-\mu]^2 = E(X^2) - \mu^2.$$

Now $\quad E(X^2) = E[X(X-1) + X] = E(X) + E[X(X-1)]$

$$= \sum_{x=0}^{n} x.h(x;N,n,k) + \sum_{x=0}^{n} x(x-1)\, h(x;N,n,k)$$

$$= \frac{nk}{N} + \frac{\sum_{x=0}^{n} x(x-1)\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \frac{nk}{N} + \frac{k(k-1)\sum_{x=2}^{n}\binom{k-2}{x-2}\binom{N-k}{n-x}}{\binom{N}{n}}$$

Let $y = x - 2$, then

$$E(X^2) = \frac{nk}{N} + \frac{k(k-1)\sum_{y=0}^{n-2}\binom{k-2}{y}\binom{N-k}{n-2-y}}{\binom{N}{n}} \quad [\because \text{ when } x = 2, y = 0, \text{ and when } x = n, \ldots$$

$$= \frac{nk}{N} + \frac{k(k-1)\binom{N-2}{n-2}}{\binom{N}{n}} = \frac{nk}{N} + \frac{k(k-1).n(n-1)}{N(N-1)}$$

$$\therefore \quad \sigma^2 = E(X^2) - \mu^2$$

$$= \frac{nk}{N} + \frac{k(k-1)n(n-1)}{N(N-1)} - \left(\frac{nk}{N}\right)^2$$

$$= \frac{nk(N-k)}{N^2} . \frac{N-n}{N-1}$$

$$= npq.\frac{N-n}{N-1}, \text{ where } p = \frac{k}{N} \text{ and } q = \frac{N-k}{N}.$$

Proceeding along the same steps as above, we may obtain higher moments, but they are rather complicated.

2) If $N$ becomes indefinitely large, the hypergeometric probability distribution tends to the binomial probability distribution.

Let $p = \frac{k}{N}$. Then $k = Np$ and $N - k = N(1-p) = Nq$.

Substituting these values in the hypergeometric formula, we get

$$h(x;N,n,k) = \frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}}$$

$$= \frac{n!(Np)!(Nq)!(N-n)!}{x!(Np-x)!(n-x)!(Nq-n+x)!N!}$$

$$= \binom{n}{x} \frac{(Np)!(Nq)!(N-n)!}{(Np-x)!(Nq-n+x)!N!}$$

Using Stirling's approximation $\left(n! \cong e^{-n}.n^n\sqrt{2\pi n}\right)$ to all factorial terms and simplifying, we get

$$h(x;N,n,k) \cong \binom{n}{x} \frac{p^{Np+1/2}.q^{Nq+1/2}\left(1-\frac{n}{N}\right)^{N-n+1/2}}{\left(p-\frac{x}{N}\right)^{Np-x+1/2}\left(q-\frac{n-x}{N}\right)^{Nq-n+x+1/2}}$$

Now, if $N$ is allowed to become indefinitely large, then $\frac{x}{N}, \frac{n-x}{N}$ and $\frac{n}{N}$ each approach

Therefore

$$h(x; N, n, k) \cong \binom{n}{x} p^x q^{n-x} = b(x; n, p).$$

## 8.4 POISSON DISTRIBUTION

The *Poisson distribution*, named after the French mathematician Sime'on Denis (1781–1840) who published its derivation in 1837, is used as

a) a limiting approximation of the binomial distribution $b(x; n, p)$, when $p$, the proba success is very small but $n$, the number of trials is so large that the product $np$ = moderate size;

b) a distribution in its own right by considering a *Poisson process* where events occur over a specified interval of time or space or length. Such random events might be of deaths by horse-kicks per year; the number of telephone calls received per minute switchboard; the number of taxicab arrivals at an intersection per day; the number born blind per year in a large city; the number of typing errors per page in a book; of red blood cells in a specimen of blood; the number of radioactive particles given period; the number of flaws per unit length of some material, the number of claims made to a company in a given time; etc.

Generally, most statisticians use Poisson approximation when $p$ is 0.05 or less and n is but in fact, the larger $n$ is and the smaller $p$ is, the better will be the approximation. If we goes to infinity and $p$ approaches zero in such a way that $\mu = np$ remains constant, then the of the binomial probability distribution is

$$\lim_{\substack{n\to\infty \\ p\to 0}} b(x; n, p) = \frac{\mu^x e^{-\mu}}{x!}, \quad x = 0, 1, 2, \ldots, \infty$$

where $e = 2.71828$. The Poisson distribution has only one parameter $\mu > 0$, and is denoted The parameter $\mu$ may be interpreted as the mean rate of occurrence of events. It is relevant this is a probability distribution as the function is obviously non-negative, i.e. $p(x; \mu) \geq 0$ unity, *i.e.*

$$\sum_{x=0}^{\infty} p(x; \mu) = \sum_{x=0}^{\infty} P(X = x)$$

$$= \sum_{x=0}^{\infty} \frac{\mu^x . e^{-\mu}}{x!} = e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!}$$

$$= e^{-\mu}\left[1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \ldots\right] = e^{-\mu}.e^{\mu} = 1.$$

e Poisson probability distribution is also called the *law of small numbers* or the rare events distribution. has found wide application in the field of Biology, Physics, Operation Research and Management ences. The Poisson distribution is appropriate when the number of *possible* occurrence is very large e number of *actual* occurrence is very small in a fixed period of time.

**8.4.1 Derivation of Poisson Approximation to the Binomial.** To derive an approximation la to the binomial distribution $b(x; n, p)$ when $n \to \infty, p \to 0$, and the product $np$ remains constant, oceed as below:

The binomial distribution $b(x; n, p)$ may be written as

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x}, \qquad \text{for } x = 0, 1, \ldots, n.$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!} p^x q^{n-x}$$

Let $np = \mu$. Then $p = \frac{\mu}{n}$ and $q = 1 - p = 1 - \frac{\mu}{n}$.

g all terms involving $p$, we get

$$p) = \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!}\left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x}$$

$$= \frac{\mu^x}{x!} \cdot \frac{n(n-1)(n-2)\ldots(n-x+1)}{n^x}\left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \frac{\mu^x}{x!}\left[1 \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\right]\left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

g $n \to \infty$ and $p \to 0$ so that $np = \mu$ remains constant, we observe that each of the terms $-\frac{2}{n}\Big), \ldots, \left(1-\frac{x-1}{n}\right)$ and $\left(1-\frac{\mu}{n}\right)^{-x}$ approaches unity. The term $\left(1-\frac{\mu}{n}\right)^n$ may be written as

$$\left(1-\frac{\mu}{n}\right)^n = \left[\left(1-\frac{\mu}{n}\right)^{n/\mu}\right]^\mu$$

g $\frac{n}{\mu} = k$, the expression becomes $\left[\left(1-\frac{1}{k}\right)^k\right]^\mu$

If $n$ increases indefinitely, so does $k$. Therefore $\left(1-\frac{1}{k}\right)^k$ tends to $e^{-1}$, where $e$ =2.71828. Th

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = e^{-\mu}$$

Thus the limiting value of $P(X=x)$ is given by the expression

$$\lim_{n\to\infty} b(x;n,p) = \frac{\mu^x}{x!}.1.1...1.e^{-\mu} = \frac{\mu^x.e^{-\mu}}{x!}, \qquad \text{for } x = 0, 1, \ldots, \infty$$

In other words, if $X$ is a binomial $r.v.$ such that

$$P(X=x) = \binom{n}{x} p^x q^{n-x}, \text{ then}$$

$$\lim_{\substack{n\to\infty \\ np=\mu}} P(X=x) = \frac{\mu^x.e^{-\mu}}{x!}, \qquad x = 0, 1, 2, \ldots, \infty$$

It is denoted by p(x; $\mu$ ). Hence a r.v. $X$ having $p.d.$ p(x; $\mu$ ) is said to have a Poisson d
with parameter $\mu$ .



**Example 8.15** If $X$ is a Poisson random variable with parameter $\mu$ =2, find the proba
$x$ = 0, 1, 2, 3 or more. (P.U., B.A./B.Sc

Here the Poisson distribution is

$$p(x;2) = \frac{e^{-2}(2)^x}{2!} \qquad (x = 0, 1, 2, \ldots)$$

The desired probabilities for $x$ = 0, 1, 2, 3 or more are computed as below:

$P(X=0) = p(0; 2) = e^{-2} = 0.135335$

$$P(X=1) = p(1;2) = \frac{e^{-2}.2}{1!} = 2(0.135335) = 0.27067$$

$$P(X=2) = p(2;2) = \frac{e^{-2}.(2)^2}{2!} = \frac{4}{2}(0.135335) = 0.27067,$$

$$P(X \geq 3) = 1 - P(X < 3)$$

$$= 1 - [P(x=0) + P(x=1) + P(X=2)]$$

$$= 1 - [0.135335 + 0.27067 + 0.27067]$$

$$= 1 - 0.676675 = 0.353325$$

**Example 8.16** Two hundred passengers have made reservations for an airplane flight. If the probability that a passenger who has a reservation will not show up is 0.01, what is the probability that exactly three will not show up?

Let us regard a "no show" as success. Then this is essentially a binomial experiment with $n = 200$ and $p = 0.01$. Since $p$ is very small and $n$ is considerably large, we shall apply the Poisson distribution, using $\mu = (200)(0.01) = 2$.

Therefore, if $X$ represents the number of successes (*not showing up*), we have

$$P(X = 3) = p(3;2) = \frac{(2)^3.e^{-2}}{3!}$$

$$= \frac{(8)(0.1353)}{3\times 2\times 1} \qquad \left(\because\ e^{-2} = \frac{1}{(2.71828)^2} = 0.1353\right)$$

$$= 0.1804.$$

**Example 8.17** The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that of 12 such men at least 11 will reach their fifty-first birthday?

Here p = 0.01125 and $n = 12$. We compute the desire probability by means of Poisson distribution because the probability of death is very small.



Therefore $\mu = np = 12 \times (0.01125) = 0.135$, and the Poisson distribution is

$$p(x;0.135) = \frac{e^{-0.135}(0.135)^x}{x!}$$

Now the probability that no person will die, *i.e.* all the 12 persons will survive, is

$$p(0;\,0.135) = e^{-0.135}$$

$$= 1 - 0.135 + \frac{(0.135)^2}{2!} - \frac{(0.135)^3}{3!} + \ldots;$$

$$= 0.8737,$$

the probability that 1 person will die, *i.e.* 11 persons will survive, is

$$p(1;\,0.135) = \frac{e^{-0.135}(0.135)^1}{1!}$$

$$= (0.8737)(0.135) = 0.1179$$

the probability that at least 11 persons will survive

$$= p(0;\,0.135) + p(1;\,0.135)$$

$$= 0.8737 + 0.1179 = 0.9916.$$

**Example 8.18** A sampling plan calls for taking 100 items from a very large lot which is one percent defective. Let $X$ be the number of defectives found in the sample of 100. Construct a table of probabilities $P(X=x)$, for $x = 0, 1, 2, 3, 4$, using first the binomial probability distribution and then the Poisson approximation.

Let $p$ denote the probability that an item is defective. Then the binomial probabilities with $n$ = $p = 0.01$ and $q = 0.99$ are

$$P(X=x)=\binom{100}{x}(0.01)^x(0.99)^{100-x}, \qquad x=0,1,2,3 \text{ and } 4.$$

These probabilities by means of the Poisson approximation, using $\mu = np = (100)(0.01) = 1$, are

$$P(X=x)=p(x;1)=\frac{1.e^{-1}}{x!}, \qquad x=0,1,2,3 \text{ and } 4.$$

The desired probabilities, on simplification, are given in the following table:

| $x$ | $P(X=x)$ | |
|---|---|---|
| | **Binomial** | **Poisson** |
| 0 | 0.3660 | 0.3679 |
| 1 | 0.3697 | 0.3679 |
| 2 | 0.1849 | 0.1839 |
| 3 | 0.0610 | 0.0613 |
| 4 | 0.0149 | 0.0153 |

These results indicate that the approximation is very good.

**8.4.2 Poisson Frequency Distribution.** When the Poisson distribution is multiplied by $N$, the number of sets of experiments, each of $n$ trials, the resulting distribution is known as the *Poisson frequency distribution*, and is denoted by

$$f(x)=N.\frac{e^{-\mu}..\mu^x}{x!}, \qquad \text{where } x=0,1,2,\ldots,\infty$$

**Example 8.19** For a machine making parts, there is a small probability of 0.002 for a part to be defective. The parts are supplied in bundles of 10. Calculate approximately the number of bundles containing no defective, one defective or two defectives in a consignment of 10,000 bundles, given that $e^{-0.02} = 0.9802$.

Let $p$ be the probability of part being defective. Then $p = 0.002$ and $n = 10$.

Since $p$ is extremely small, we apply the Poisson approximation, using $\mu = np = 10 \times .002 = 0.02$

Hence the approximate number of bundles containing no defective, one defective or two defectives are the terms for $x = 0$, 1, and 2 in

$$N.p(x;\mu)=10{,}000.\frac{(0.02)^x.e^{-0.02}}{x!}$$

Putting $x = 0$, we get

$$10000 \times e^{-0.02} = 10000 \times 0.9802 = 9{,}802.$$

Putting $x = 1$, we get

$$10000 \times e^{-0.02}(0.02) = 10000 \times 0.9802 \times 0.02 = 196.$$

Putting $x = 2$, we obtain

$$10000.\frac{e^{-0.02}.(0.02)^2}{2!} = \frac{10000 \times 0.9802 \times (0.02)^2}{2} = 2 \text{ approx.}$$

**8.4.3 Properties of the Poisson Distribution.** Some of the main properties of the Poisson ribution are given below:

1. If the random variable $X$ has a Poisson distribution with parameter $\mu$, then its mean and variance are given by $E(X) = \mu$ and $\text{Var}(X) = \mu$.

By definition,

$$\begin{aligned}
\text{Mean} &= E(X) \\
&= \sum_{x=0}^{\infty} x.p(x;\mu), \text{ where } p(x;\mu) = \frac{e^{-\mu}.\mu^x}{x!} \\
&= 0.e^{-\mu} + 1.\mu e^{-\mu} + 2.\frac{\mu^2}{2!}e^{-\mu} + 3.\frac{\mu^3}{3!}e^{-\mu} + \ldots \\
&= \mu e^{-\mu}\left[1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \ldots\right] \\
&= \mu e^{-\mu}.e^{\mu} = \mu
\end{aligned}$$

**ive Method**

$$\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x.\frac{e^{-\mu}.\mu^x}{x!} \\
&= e^{-\mu}\sum_{x=0}^{\infty} \frac{x.\mu^x}{x(x-1)!} \\
&= \mu.e^{-\mu}\sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!} \quad \text{(since the first term in the summation being zero is omitted)}
\end{aligned}$$

$y = x - 1$, then

$$E(X) = \mu e^{-\mu}\sum_{y=0}^{\infty} \frac{\mu^y}{y!} \qquad (y = 0, 1, 2, \ldots \infty)$$

$$= \mu e^{-\mu}.e^{\mu} = \mu$$

$= \mu$, the parameter of the distribution.

$\text{Var}(X) = E(X^2) - [E(X)]^2$, where

$$E(X^2) = E[X(X-1) + X] = E(X) + E[X(X-1)]$$

$$= \sum_{x=0}^{\infty} x.\frac{e^{-\mu}.\mu^x}{x!} + \sum_{x=0}^{\infty} x(x-1)e^{-\mu}.\frac{\mu^x}{x!}$$

$$= \mu + e^{-\mu}\sum_{x=0}^{\infty} x(x-1).\frac{\mu^x}{x(x-1)(x-2)!}$$

$$= \mu + \mu^2 e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^{x-2}}{(x-2)!}$$ (*x* starts at 2, as the first two terms in the summ... are ...

Let $y = x - 2$, then

$$E(X^2) = \mu + \mu^2 e^{e-\mu}\sum_{y=0}^{\infty}\frac{\mu^y}{y!} \qquad (y = 0, 1, 2, \ldots..)$$

$$= \mu + \mu^2 e^{-\mu} e^{-\mu} = \mu + \mu^2$$

Hence $\mathrm{Var}(X) = \mu + \mu^2 - \mu^2 = \mu$

We observe that the Poisson distribution has an interesting property that its mean equals its varian...

2. Higher moments of the distribution are found as below:

By definition, $\mu_3' = E(X^3) = \sum_{x=0}^{\infty} x^3 . p(x;\mu)$

Writing $x^3$ in the factorial form, *i.e.*

$x^3 = x(x-1)(x-2) + 3x(x-1) + x$, we have

$$\mu_3' = e^{-\mu}\sum_{x=0}^{\infty}[x(x-1)(x-2) + 3x(x-1) + x].\frac{\mu^x}{x!}$$

$$= e^{-\mu}.\mu^3\sum_{x=3}^{\infty}\frac{\mu^{x-3}}{(x-3)!} + 3\mu^2.e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^{x-2}}{(x-2)!} + \mu.e^{-\mu}\sum_{x=1}^{\infty}\frac{\mu^{x-1}}{(x-1)!}$$

$$= e^{-\mu}.\mu^3\sum_{x=0}^{\infty}\frac{\mu^y}{y!} + 3\mu^2.e^{-\mu}\sum_{y=0}^{\infty}\frac{\mu^y}{y!} + \mu.e^{-\mu}\sum_{y=0}^{\infty}\frac{\mu^y}{y!}$$

$= \mu^3 + 3\mu^2 + \mu$, and

$$\mu_4' = E(X^4) = \sum_{x=0}^{\infty} x^4 . p(x;\mu)$$

Writing $x^4$ in the factorial form as

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x. \text{ we get}$$

$$\mu_4' = e^{-\mu}\sum_{x=0}^{\infty}[x(x-1)(x-2)+6x(x-1)(x-2)+7x(x-1)+x]\frac{\mu^x}{x!}$$

$$= \mu^4.e^{-\mu}\sum_{x=4}^{\infty}\frac{\mu^{x-4}}{(x-4)!}+6\mu^3.e^{-\mu}\sum_{x=3}^{\infty}\frac{\mu^{x-3}}{(x-3)!}+7\mu^2.e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^{x-2}}{(x-2)!}+\mu.e^{-\mu}\sum_{x=1}^{\infty}\frac{\mu^{x-1}}{(x-1)!}$$

$$= \mu^4 + 6\mu^3 + 7\mu^2 + \mu.$$

Hence $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$$= (\mu^3 + 3\mu^2 + \mu) - 3\mu(\mu^2 + \mu) + 2\mu^3 = \mu;$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4,$$

$$= (\mu^4 + 6\mu^3 + 7\mu^2 + \mu) - 4\mu(\mu^3 + 3\mu^2 + \mu) + 6\mu^2(\mu^2 + \mu) - 3\mu^4$$

$$= 3\mu^2 + \mu.$$

Thus $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\mu^2}{\mu^3} = \frac{1}{\mu}$, and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\mu^2 + \mu}{\mu^2} = 3 + \frac{1}{\mu}.$$

3. The *shape* of the Poisson distribution depends on the value of its parameter $\mu$. As the distribution takes on an infinite number of $x$ values (*theoretically*), the distribution will be positively skewed. The distribution tends to be symmetrical as $\mu$ becomes larger and larger.

4. **Reproductive Property.** If two independent r.v.'s $X$ and $Y$ have Poisson distributions with parameter $\mu$ and $v$, then their sum $X+Y$ has also a Poisson distribution with parameter $\mu + v$.

**Proof.** Here $X$ is $p(x;\ \mu)$ and $Y$ is $p(y;\ v)$; and we desire to find $P(X+Y=k)$ for $k = 0, 1, 2, \ldots$

For $k = 0$, $P(X+Y=0) = P(X=0)\ P(Y=0)$ ($\because$ $X$ and $Y$ are independent)

$$= e^{-\mu}.e^{-v} = e^{-(\mu+v)}$$

For $k=1$, $P(X+Y=1) = P(X=0).\ P(Y=1) + P(X=1).\ P(Y=0)$

$$= e^{-\mu}.ve^{-v} + \mu e^{-\mu}.e^{-v}$$

$$= e^{-(\mu+v)}(\mu + v)$$

Similarly for $k = 2$, $P(X+Y=2) = \dfrac{e^{-(\mu+\nu)}}{2!}(\mu+\nu)^2$.

And, in general, for $X + Y = k$ when $X = i$ and $Y = k - i$ *for* $i = 0, 1, 2, \ldots, k$, we have

$$P(X+Y=k) = P(X=0)\, P(Y=k)+P(X=1).P(Y=k-1)+\ldots+P(X=k).P(Y=0)$$

$$= \frac{e^{-\mu}}{0!}.\frac{\nu^k e^{-\nu}}{k!}+\frac{\mu e^{-\mu}}{1!}.\frac{\nu^{k-1}e^{-\nu}}{(k-1)!}+\ldots+\frac{\mu^k e^{-\mu}}{k!}.\frac{e^{-\nu}}{0!}$$

$$= e^{-(\mu+\nu)}\left[\frac{\nu^k}{k!}+\mu\frac{\nu^{k-1}}{(k-1)!}+\ldots+\frac{\mu^k}{k!}\right]$$

Multiplying the r.h.s. by $\dfrac{k!}{k!}$, we get

$$P(X+Y=k) = \frac{e^{-(\mu+\nu)}}{k!}\left[\binom{k}{0}\nu^k+\binom{k}{1}\nu^{k-1}\mu+\ldots+\binom{k}{k}\mu^k\right]$$

$$= \frac{e^{-(\mu+\nu)}}{k!}(\mu+\nu)^k,$$

which is a Poisson distribution with parameter $\mu+\nu$. This result can easily be generalized.

The converse of this result, that is, *if X and Y are independent, and X+Y has a Poisson distri then each of the r.v.'s X and Y has a Poisson distribution,* is also true was proved by Raikov.

**8.4.4 The Recurrence Formula for the Poisson Distribution.** Let $X$ be a Poisson variable with parameter $\mu$. Then

$$P(X=x) = e^{-\mu}\frac{\mu^x}{x!}, \text{ and}$$

$$P(X=x-1) = e^{-\mu}\frac{\mu^{x-1}}{(x-1)!}$$

. Dividing, we get

$$\frac{P(X=x)}{P(X=x-1)} = \frac{e^{-\mu}.\mu^x}{x!}.\frac{(x-1)!}{e^{-\mu}.\mu^{x-1}} = \frac{\mu}{x}$$

Therefore $P(X=x) = \dfrac{\mu}{x}.P(X=x-1)$ for $x = 1, 2, 3, \ldots$ is the recurrence formula for the Poisson distribution. Using this recurrence relationship, the Poisson probabilities can be obtained more ea

**8.4.5 Fitting a Poisson Distribution to Observed Data.** The Poisson distribution can be fitted (or derived) if we know the value of its mean which is usually obtained by equating $\mu$ to the mean estimate from the observed frequency distribution, provided that the probability of occurrence is very small. Using this value of mean, the *expected* or theoretical frequencies are computed. The following classical example will illustrate the procedure.

**Example 8.20** Bortkiewicz (1868–1931) collected data on the number of deaths from horse-kicks in Prussian Army Corps over a period of 20 years. This distribution of deaths was as follows

| No. of deaths | 0 | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|---|
| Frequency | 109 | 65 | 22 | 3 | 1 | 0 | 200 |

Fit a Poisson distribution to these data and compute the theoretical frequencies.

To fit a Poisson distribution, we need to compute (estimate) the value of mean of the given distribution and equate it to $\mu$, the mean of the Poisson distribution.

$$\text{Now} \quad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0+65+44+9+4}{200}$$

$$= \frac{122}{200} = 0.61 \text{, which is an estimate of } \mu .$$

Hence the fitted Poisson distribution is given by

$$P(X=x) = p(x;0.61) = \frac{e^{-0.61}(0.61)^x}{x!} \text{, where } x = 0, 1, 2, 3, \ldots$$

The theoretical or expected frequencies of $x$ deaths are computed by multiplying the probabilities by $n$ which is 200 here. The probabilities are computed by using the *Poisson recurrence formula,* which takes the form

$$P(X=x) = \frac{0.61}{x} . P(X=x-1) \text{, for } x = 0, 1, 2, 3, 4, 5.$$

In case the table values for $e^{-\mu}$ are not available, they are computed by use of logarithms as

Let $\quad y = e^{-0.61}$

Then $\quad \log y = -0.61 \log e = (-0.61)(0.4343)$

$$= -0.2649 = \bar{1}.7351 \text{ so that } y = 0.5434.$$

The probabilities and the frequencies are then computed as below:

| No. of death ($x$) | Probability $p(x; 0.61)$ | | Expected frequency 200 x prob. |
|---|---|---|---|
| 0 | $e^{-0.61}$ | $= 0.5434$ | 108.68 |
| 1 | $e^{-0.61}\dfrac{(.61)}{1} = \dfrac{0.61}{1}(0.5434)$ | $= 0.3315$ | 66.30 |
| 2 | $e^{-0.61}\dfrac{(.61)^2}{2!} = \dfrac{0.61}{2}(0.3315)$ | $= 0.1011$ | 20.22 |
| 3 | $e^{-0.61}\dfrac{(.61)^3}{3!} = \dfrac{0.61}{3}(0.1011)$ | $= 0.0206$ | 4.12 |
| 4 | $e^{-0.61}\dfrac{(.61)^4}{4!} = \dfrac{0.61}{4}(0.0206)$ | $= 0.0031$ | 0.62 |
| 5 | $e^{-0.61}\dfrac{(.61)^5}{5!} = \dfrac{0.61}{5}(0.0031)$ | $= 0.0004$ | 0.08 |
| Total | | $= 1.0001$ | $= 200.02$ |

**8.4.6 Poisson Process.** In an earlier section, the Poisson distribution was obtained as a lim approximation to the binomial distribution. The terms of the Poisson distribution can also be deriv a *Poisson process* which may be defined as a physical process governed at least in part by some mechanism. The occurrence of traffic deaths per month in a city is an example of a Poisson pro Poisson process has the following properties:

i) The probability that an event occurs in a very short time interval $\Delta$, is proportiona length of the time interval, i.e. is approximately $\lambda\Delta$, which $\lambda$ is a positive quantity a be interpreted as the average number of occurrences per unit of time.

ii) The probability that two or more events occur in such a short interval is so small that neglected.

iii) Events occurring in non-overlapping intervals of time are statistically independent.

These properties are assumed to hold for events occurring randomly in *regions of space*, et these properties, it can be shown that the probability for the number of occurrences of a random an interval of stated length $t$ is given by the Poisson distribution with the parameter $\lambda t$. Th Poisson process formula is

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!},$$

where $t$ = number of units of time,

$x$ = number of occurrences in $t$ units of time, and

$\lambda$ = average number of occurrences per unit of time.

The derivation of the Poisson process formula is beyond the scope of this book.

**Example 8.21** Telephone calls are being placed through a certain exchange at random times on the of four per minute. Assuming a Poisson process, determine the probability that in a 15-second there are 3 or more calls. (P.U., B.A./B.Sc. 1979)

Taking a minute as the unit of time, we have .

$\lambda$ = 4 calls per minute

= 4 calls per 60-seconds.

As 15 second is $\frac{15}{60} = \frac{1}{4}$ units of time, so $t = \frac{1}{4}$ and therefore the average number of calls per interval i.e. $\lambda t = 4 \times \frac{1}{4} = 1$.

Hence, using the Poisson process formula $p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$, we have $P$(3 or more calls in nds interval).

$$= 1 - P(0, 1 \text{ or } 2 \text{ calls in 15-second interval})$$

$$= 1 - \sum_{x=0}^{2} p(x;\lambda t) = 1 - \sum_{x=0}^{2} \frac{e^{-1}(1)^x}{x!}$$

$$= 1 - \sum_{x=0}^{2} \frac{(0.3679)(1)^x}{x!} \qquad (\because e^{-1} = 0.3679)$$

$$= 1 - (0.91975) = 0.08025$$

**Example 8.22** Flaws in a certain type of drapery material appear on the average of one in 150 If we assume the Poisson distribution, find the probability of *at most* one flaw in 225 square

Taking 150 square feet as the unit of area, we have

$\lambda$ = 1 flaw per 150 square feet.

225 square feet are $\frac{225}{150} = 1.5$ units of area, so $t$ = 1.5 and therefore the average number of 225 square feet, *i.e.* $\lambda t = 1 \times 1.5 = 1.5$. Assuming the flaws are a Poisson process, we have

most one flaw in 225 square feet) $= \sum_{x=0}^{1} p(x;\lambda t)$

$$= \sum_{x=0}^{1} \frac{e^{-1.5}(1.5)^x}{x!}, \text{ where } e^{-1.5} = 0.2231$$

$$= 0.2231 + 0.3347 = 0.5578$$

**8.4.7 Moment Generating and Cumulant Generating Functions of the Poisson Distribution.** The *m.g.f.* for the Poisson distribution with respect to the origin, is found as below:

$$M_0(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tX}\, p(x;\, \mu)$$

$$= \sum_{x=0}^{\infty} e^{tX} \cdot \frac{\mu^x e^{-\mu}}{x!} = e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$$

$$= e^{-\mu}\, e^{\mu e^t} = e^{\mu(e^t - 1)}.$$

The mean and variance are obtained as:

$$\mu_1' = E(X) = \left[\frac{d}{dt}\left\{e^{\mu(e^t-1)}\right\}\right]_{t=0}$$

$$= \left[\mu e^t \cdot e^{\mu(e^t-1)}\right]_{t=0} = \mu;$$

and $$\mu_2' = E(X^2) = \left[\frac{d^2}{dt^2}\left\{e^{\mu(e^t-1)}\right\}\right]_{t=0}$$

$$= \left[\mu e^t \cdot e^{\mu(e^t-1)} + \mu^2 e^{2t} \cdot e^{\mu(e^t-1)}\right]_{t=0}$$

$$= \mu + \mu^2$$

Hence the variance is

$$\sigma^2 = \mu_2' - \mu_1'^2 = \mu + \mu^2 - \mu^2 = \mu.$$

The cumulant generating function (*c.g.f.*) of the Poisson distribution with respect to origin is given by

$$\kappa(t) = \log_e M_0(t)$$

$$= \mu\ (e^t - 1)$$

$$= \mu\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^r}{r!} + \ldots\right)$$

$$\kappa_r = \text{co-efficient of } \frac{t^r}{r!} \text{ in } k(t) = \mu, \text{ for all } r.$$

Hence all the cumulants of the Poisson distribution are equal to $\mu$.

**Example 8.23** Prove the following recurrence formula for a Poisson distribution $p(x; m)$:

$$\mu_{r+1} = rm\mu_{r-1} + m\frac{d\mu_r}{dm}$$

ion. $$\mu_r = \sum_{x=0}^{\infty} p(x;m)(x-m)^r = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!}(x-m)^r$$

ating with respect to $m$, we get

$$\frac{d\mu_r}{dm} = \sum_{x=0}^{\infty}\left[-r\frac{e^{-m}m^{x}}{x!}(x-m)^{r-1} + x\frac{e^{-m}m^{x-1}}{x!}(x-m)^r - \frac{e^{-m}m^x}{x!}(x-m)^r\right]$$

g both sides by $m$ and simplifying, we get

$$m\frac{d\mu_r}{dm} = -rm\mu_{r-1} + \sum_{x=0}^{\infty}\frac{e^{-m}.m^x}{x!}(x-m)^r.(x-m)$$

$$= -rm\mu_{r-1} + \mu_{r+1}$$

$$\mu_{r+1} = rm\mu_{r-1} + m\frac{d\mu_r}{dm}$$

## ATIVE BINOMIAL DISTRIBUTION

binomial experiments, the number of successes varies and the number of trials is fixed. But periments in which the number of successes is fixed and the number of trials varies to produce mber of successes. Such experiments are called *negative binomial experiments*. In other gative binomial experiment possesses the following four properties:

The outcomes of each trial may be classified into one of two categories: success ($S$) and failure ($F$).

The probability of success, denoted by $p$, remains constant for all trials.

The successive trials are all independent.

The experiment is repeated a *variable number* of times to obtain a *fixed* number of successes.

$X$ denotes the number of trials to produce $k$ successes in a negative binomial experiment, it is *ve binomial variable*, and its p.d., is called the *negative binomial distribution*. When the al r.v. $X$ assumes a value $x$, on which the $k$th success occurs, the negative binomial given by

$$P(X=x) = \binom{x-1}{k-1}p^k q^{x-k}, \qquad x = k, k+1, k+2, \ldots$$

binomial p.d. has two parameters $k$ and $p>0$, and is generally denoted by $b^*(x; k, p)$. To the negative binomial distribution, the probability function sums to one, we proceed as

$$b^*(x;k,p) = \sum_{x=k}^{\infty}\binom{x-1}{k-1}p^k q^{x-k}, \qquad (x = k, k+1, k+2, \ldots)$$

Let $y = x - k$. Then

$$\text{Sum of prob} = \sum_{y=0}^{\infty}\binom{y+k-1}{k-1}p^k q^y, \qquad (y = 0, 1, 2, \ldots)$$

$$= p^k \sum_{x=k}^{\infty}\binom{y+k-1}{k-1}q^y$$

$$= p^k\left[\binom{k-1}{k-1}q^0 + \binom{k}{k-1}q + \binom{k+1}{k-1}q^2 + \ldots\right]$$

$$= p^k\left[1 + kq + \frac{k(k+1)}{2!}q^2 + \ldots\right]$$

$$= p^k[1-q]^{-k} = p^k p^{-k} = 1.$$

The distribution takes its name from the fact that $\binom{x-1}{k-1}p^k q^{x-k}$ is a term in the exp $p^k(1-q)^{-k}$, a binomial with negative index. Thus the probabilities at $k$th, $(k+1)$ *th*, $(k+2)$*th*, … tri

$$p^k\left[1,\ kq,\ \frac{k(k+1)}{2}q^2, \ldots\right]$$



This distribution is sometimes also called the *Pascal distribution*, after the French mathema Pascal (1623–1662). The distribution is found to occur in many biological situations and sampling from a binomial population.

**8.5.1 Derivation of the Negative Binomial Distribution.** The negative binomial dis be derived in various ways. The following derivation is based on the Bernoulli trials.

To find an expression for the probability that $x$ trials are made to achieve $k$ succe condition that the last trial must be success, we proceed as follows:

A sequence containing $k$ successes in exactly $x$ independent trials with the condition a success, can be obtained as

$$\underbrace{SS\ldots S}_{k-1\ times}\ \underbrace{FF\ldots FS}_{x-k\ times}.$$

Thus the probability of a success on the $x$th trial preceded by $(k-1)$ successes and $(x-k)$

$$p^{k-1}q^{x-k}p = p^k q^{x-k}$$

Since the last trial must be a success, therefore the total number of mutually exc which $(k-1)$ successes and $(x-k)$ failures preceding the last success can occur *in any order*,

$$\binom{x-1}{k-1}$$

Hence the formula for the probability that *kth* success occurs on the *xth* trial is

$$P(X=x)=\binom{x-1}{k-1}p^k q^{x-k}\text{, where } x=k,\ k+1,\ k+2,\ldots$$

The negative binomial distribution can also be obtained when two or more Poisson *r.v.*'s are ...ed term by term.

**Example 8.24** A person throws a pair of fair dice. What is the probability that he will get a total ... the second time on the eighth throw?

The probability that he gets a total of 7 is $\frac{6}{36}$, *i.e.* $p=\frac{1}{6}$.

...ce the number of successes is fixed, therefore the negative binomial distribution with $k=2$ ... success) and $x=8$ is used.

Hence $$P(X=8)=\binom{8-1}{2-1}\left(\frac{1}{6}\right)^2\cdot\left(\frac{5}{6}\right)^{8-2}$$

$$=7\cdot\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6=0.0651$$



**...ample 8.25** Three people each toss a coin and the odd man pays for the coffee. If the coins all ...s or all show tails, they are tossed again. What is the probability that a decision is reached in ... or fewer?

... reach a decision on any trial, the coins must result in either '2 heads and 1 tail' or '2 tails and 1 ... The probability of these events is computed by the binomial distribution because

- each coin has two possible results, a head or a tail.
- the probability of getting a head is $p=\frac{1}{2}$ and remains the same for each coin.
- three people toss the coin independently.
- three coins are tossed in each case. (i.e. $n=3$)

$$P(\text{2 heads and 1 tail})=\binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)=\frac{3}{8},$$

$$P(\text{1 head and 2 tails})=\binom{3}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2=\frac{3}{8}.$$

...obability of reaching a decision is $\frac{3}{8}+\frac{3}{8}=\frac{6}{8}=0.75$

... we find the probability that a decision is reached in 5 tosses or fewer. We observe that

- ... reach or do not reach a decision with each set,
- ... probability of reaching a decision is $p=0.75$ and remains the same for all sets,

iii) the results of all sets are independent,

iv) a *variable* number of sets is required to produce 1 decision

We therefore compute the required probability by means of the negative binomial distribution where

$k = 1, p = 0.75$ and $x \le 5$ (the number of trials).

Thus $$P(X \le 5) = \sum_{x=1}^{5} \binom{x-1}{k-1} (0.75)^5 (0.25)^{x-k}, \; k = 1$$

$$= 0.75 + 0.1875 + 0.0469 + 0.0117 + 0.0029$$

$$= 0.9990$$

**8.5.2 Properties of the Negative Binomial Distribution.** The important properties of the negative binomial distribution are given below:

1. The mean of the negative binomial distribution is less than its variance.

We find the mean and variance by deriving the *m.g.f.* of the negative binomial distribution.

The *m.g.f.* about the origin is

$$M_0(t) = E(e^{tX})$$

$$= p^k \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} q^x . e^{tx}$$

$$= p^k \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} (qe^t)^x$$

$$= p^k (1-qe^t)^{-k}$$



Now $$\text{Mean} = E(X) = \left[\frac{dM_0(t)}{dt}\right]_{t=0}$$

$$= \left[q^k . kqe^t (1-qe^t)^{-k-1}\right]_{t=0}$$

$$= p^k . kq (1-q)^{-k-1}$$

$$= kqp^k p^{-k-1} = \frac{kq}{p}.$$

and $$\sigma^2 = E(X^2) - [E(x)]^2, \text{ where}$$

$$E(X^2) = \left[\frac{d^2 M_0(t)}{dt^2}\right]_{t=0}$$

$$= [p^k . kqe^t (1-qe^t)^{-k-1} + p^k . k(k+1) q^2 e^{2t} (1-qe^t)^{-k-2}]_{t=0}$$

$$= p^k kq (1-q)^{-k-1} + k(k+1) q^2 p^k (1-q)^{-k-2}$$

$$= \frac{kq}{p} + \frac{k(k+1)\, q^2}{p^2}$$

Thus $$\sigma^2 = \frac{kq}{p} + \frac{k(k+1)\,q^2}{p^2} - \left(\frac{kq}{p}\right)^2$$

$$= \frac{kqp + k^2q^2 + kq^2 - k^2q^2}{p^2} = \frac{kq(p+q)}{p^2} = \frac{kq}{p^2}$$

The variance will be greater than mean, if

$$\frac{kq}{p^2} > \frac{kq}{p} \quad \text{or if} \quad \frac{1}{p^2} > \frac{1}{p}$$

if $1 > p$, which is obviously true.

Hence we observe that the variance of the negative binomial distribution is greater than its mean. This is an important feature of this distribution.

2. The negative binomial distribution is always positively skewed.

## 8.6 GEOMETRIC DISTRIBUTION

When an experiment consists of independent trials with probability $p$ of success and the trials are repeated until the *first* success occurs, it is called a *geometric experiment.* In other words, a geometric experiment has the following four properties:

i) The outcomes of each trial may be classified into one of two categories, success and failure.
ii) The probability of success $p$ remains constant for all trials.
iii) The successive trials are all independent.
iv) The experiment is repeated a *variable* number of times until the *first success* is obtained.



If $X$ represents the number of trials needed for the first success, then $X$ is called a geometric *r.v.* and its distribution is called the *geometric distribution*. It has only one parameter $p$ and is denoted by $g(x;\, p)$. The geometric distribution derives its name from the fact that its successive terms constitute a geometric progression. Since a geometric *r.v.* represents how long one has to wait for a success, it is also called a waiting *time r.v.* It is interesting to note that a geometric distribution is a special case of a negative binomial distribution when $k = 1$.

**8.6.1 Derivation of the Geometric Distribution.** Let the random variable $X$ denote the number of trials required upto and including the first success of an event. Then $X$ takes the values $1, 2, 3 \ldots, \infty$. Since $X=x$ if and only if the first $(x-1)$ trials result in failures and the $xth$ trial yields a success in the Bernoullian sequence of trials, we therefore have the probability distribution of $X$, as

$$P(X=x) = q^{x-1}\, p, \qquad x = 1, 2, 3, \ldots, \infty\,.$$

This is obviously a probability distribution as (i) $P(X=x) \geq 0$ and (ii) the sum of the probabil unity, *i.e.*

$$\sum_{x=1}^{\infty} P(X=x) = p + qp + q^2p + q^3p + \ldots$$

$$= p\,[1 + q + q^2 + q^3 + \ldots]$$

$$= p\,[1-q]^{-1} = pp^{-1} = 1.$$

**Example 8.26** If the probability that a person will believe a rumour about the retireme certain politician is 0.25, what is the probability that

a) the sixth person to hear the rumour will be first to believe it;

b) the twelfth person to hear the rumour will be the fourth to believe it?

Let $X$ denote the number of a person who hears the rumour. Then the number of a perso believes it, will be considered a success.

a) Since the sixth person is the *first* to believe the rumour, *i.e.* the *first success* occurs on the trial, therefore the geometric distribution with $p = 0.25$ and $x = 6$ is appropriate.

Hence, using the geometric distribution $P(X=x) = pq^{x-1}$, we get

$$P(X=6) = (0.25)\,(0.75)^5 = 0.059.$$

b) Since the twelfth person hearing the rumour will be the *fourth to believe* implies that success occurs on the $12^{th}$ trial, therefore the negative binomial distribution with $p=0.2$ and $x=12$ is appropriate.

Hence, using the negative binomial distribution

$$b^*(x;\,k,\,p) = \binom{x-1}{k-1} p^k q^{x-k}\text{, we get}$$

$$b^*(12;\,4,\,0.25) = \binom{12-1}{4-1}(0.25)^4\,(0.75)^{12-4}$$

$$= \binom{11}{3}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^8 = \frac{165\times 6561}{265\times 65536}$$

$$= 0.0645.$$

**8.6.2 Properties of the Geometric Distribution:**

1. The mean and variance of the geometric distribution are $\mu = 1/p$ and $\sigma^2 = q/p^2$.

Let the *r.v.* $X$ have a geometric distribution $g(x;\,p) = pq^{x-1}$. Then

$$\mu = E(X) = \sum x\,.\,g(x;\,p)$$

$$= \sum x\,.\,q^{x-1}p\text{, where } x = 1, 2, 3, \ldots, \infty$$

$$= p + 2qp + 3q^2p + 4q^3p + \ldots$$

$$= p[1 + 2q + 3q^2 + 4q^3 + \ldots]$$

$$= p[1-q]^{-2} = p \cdot p^{-2} = \frac{1}{p}\text{; and}$$

$\sigma^2 = E(X^2) - [E(X)]^2$, where

$$E(X^2) = \sum_{x=1}^{\infty} x^2 \cdot q^{x-1} p$$

$$= p + 2^2qp + 3^2q^2p + 4^2q^3p + \ldots$$

$$= p\,[1 + 4q + 9q^2 + 16q^3 + \ldots]$$

$$= p\,[(1+3q+6q^2+10q^3+\ldots) + (q+3q^2+6q^3+\ldots)]$$

$$= p\,[(1-q)^{-3} + q(1-q)^{-3}]$$

$$= \frac{1}{p^2} + \frac{q}{p^2}$$

Thus $\sigma^2 = \frac{1}{p^2} + \frac{q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}.$

The distribution is positively skewed.

**Moment Generating Function of the Geometric Distribution.** The *m.g.f.* of the geometric distribution is derived as below:

$$M_0(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= pe^t \sum_{x=1}^{\infty} (e^t q)^{x-1}$$

$$= pe^t\,[1 + qe^t + (qe^t)^2 + \ldots]$$

$$= pe^t\,[1 - qe^t]^{-1} = \frac{pe^t}{1-qe^t}\text{, where } qe^t < 1.$$

To differentiate the *m.g.f.*, we write it as

$$M_0(t) = \frac{p}{e^{-t} - q} = p(e^{-t} - q)^{-1}$$

Thus $M_0'(t) = pe^{-t}\,(e^{-t} - q)^{-2}$, and

$$M_0''(t) = 2pe^{-2t}\,(e^{-t} - q)^{-3} - pe^{-t}(e^{-t} - q)^{-2}.$$

Hence $E(X) = p(1-q)^{-2} = \dfrac{1}{p}$,

$$E(X^2) = 2p(1-q)^{-3} - p(1-q)^{-2}$$

$$= \frac{2}{p^2} - \frac{1}{p}; \text{ and}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}$$

## 8.7 MULTINOMIAL DISTRIBUTION

A binomial experiment becomes a *multinomial experiment* when there are more than t
outcomes of each trial. For example, manufactured items may be classified as good, average,
or a road accident may result in no injury, minor injuries, severe injuries, or fatal injuries. A m
experiment has the following properties:

i) The outcomes of each trial may be classified into one of $k$ mutually exclusive ca $C_2, \ldots, C_k$.

ii) The probability of the *ith* outcome is $p_i$ which remains constant and $\sum p_i = 1$.

iii) The successive trials are all independent.

iv) The experiment is repeated a *fixed* number of times, say, $n$.

**8.7.1 Derivation of the Multinomial Distribution.** If $n$ independent trials be ma *specified order* in which $C_1$ occurs $x_1$ times, $C_2$ occurs $x_2$ times, …, $C_k$ occurs $x_k$ $x_1+x_2+\ldots+x_k=n$, is

$$\underbrace{C_1 \ldots C_1}_{x_1 \text{ times}}\ \underbrace{C_2 \ldots C_2}_{x_2 \text{ times}} \ldots \underbrace{C_k \ldots C_k}_{x_k \text{ times}}$$

The probability of this happening by multiplicative law is

$$\underbrace{p_1 \ldots p_1}_{x_1 \text{ times}}\ \underbrace{p_2 \ldots p_2}_{x_2 \text{ times}} \ldots \underbrace{p_k \ldots p_k}_{x_k \text{ times}} = p_1^{x_1}\ p_2^{x_2} \ldots p_k^{x_k}$$

But we are interested in events occurring in any order. Therefore the total number
exclusive orders in which this can happen, is

$$\binom{n}{x_1, x_2, \ldots, x_k} = \frac{n!}{x_1!x_2!, \ldots, x_k!}$$

Hence the required multinomial probability is

$$P(X_1=x_1, X_2 = x_2, \ldots, X_k = x_k)$$

$$= \frac{n!}{x_1!\,x_2!,\ldots,x_k!}(p_1)^{x_1}.(p_2)^{x_2}\ldots(p_k)^{x_k}$$

$$= \frac{n!}{\prod_{i=1}^{k} x_i!}\prod_{i=1}^{k}(p_i)^{x_i}\text{, where } \sum_{i=1}^{k} x_i = n \;\&\; \sum_{i=1}^{k} p_i = 1.$$

The multinomial distribution takes its name from the fact that the above probabilities correspond to erms in the multinomial expansion of $(p_1 + p_2 + \ldots + p_k)^n$. The parameters of this distribution are $n$, $p_2, \ldots, p_k$.

The mean and variance of the multinomial distribution are

$$E(X_i) = np_i \text{ and } \mathrm{Var}(X_i) = np_iq_i$$

When $k = 2$, the multinomial distribution reduces to the binomial probability distribution. Thus ial distribution is a special case of the multinomial distribution.

**Example 8.27** A box contains 5 red, 4 white and 3 blue marbles. A sample of six marbles is drawn replacement, i.e., each marble is replaced before the next one is drawn. Find the probability that out marbles selected, 3 are red, 2 are white and one is blue.

Let $X_1$, $X_2$ and $X_3$ denote the red, white and blue marbles. Then

$$p_1 = P(X_1 = 3) = \frac{5}{12}$$

$$p_2 = P(X_2 = 2) = \frac{4}{12}$$

$$p_3 = P(X_3 = 1) = \frac{3}{12}$$

$$P(X_1 = 3, X_2 = 2, X_3 = 1) = \frac{6!}{3!\,2!\,1!}\left(\frac{5}{12}\right)^3\left(\frac{4}{12}\right)^2\left(\frac{3}{12}\right)^1 = \frac{625}{5184}$$



## EXERCISES

### CTIVE

Answer 'True' or 'False'. If the statement is not true then replace the underlined words with words that make the statement true:

- A discrete probability distribution is graphically represented by a <u>curve</u>.
- A binomial experiment always has <u>three or more</u> possible outcomes to each trial
- Discrete random variables may assume <u>any</u> values.
- In a binomial experiment, the individual trials are <u>independent</u> of each other.

v) In a binomial distribution the mean is equal to its variance.

vi) In a hypergeometric probability distribution trials are dependent.

vii) The mean and variance of the Poison distribution are not equal.

viii) The Poisson distribution is defined as a limiting approximation of the binomial distrib when p is large, but n is small so that $\mu = np$ is of moderate size.

ix) Another name for Poisson probability distribution is the rare events distribution.

x) Poisson probability distribution has two parameters.

xi) In a negative binomial distribution, the successive trials are all dependent.

xii) The mean of the negative binomial distribution is greater than its variance.

xiii) The geometric probability distribution is symmetrical.

xiv) A binomial distribution is a special case of a multinomial distribution when each trial ca assume two possible outcomes.

xv) The outcomes of each trial for binomial distribution may be classified into one of mutually exclusive categories.



b) **MULTIPLE CHOICE QUESTIONS**

i) Which of the following is not a property of a binomial experiment?

a) The successive trials are all independent.
b) The experiment is repeated a fixed number of times say n.
c) The probability of success, denoted by p, remains constant for all trials.
d) There are three or more possible outcomes for each trial.

ii) The standard deviation of the binomial distribution is:

a) $\sqrt{np}$
b) $\sqrt{npq}$
c) $npq$
d) $pq$

iii) For a binomial distribution, the mean and variance are related by:

a) $\mu < \sigma^2$
b) $\mu = \sigma^2$
c) $\mu > \sigma^2$
d) $\mu < \sqrt{\sigma^2}$

For a binomial distribution $P(X=x)={}^{12}C_x(0.5)^x(0.5)^{12-x}$ x = 0, 1, 2, …, 12, the mean is:

a) 3
b) 6
c) $\sqrt{6}$
d) $\sqrt{3}$

For a Poisson distribution, the mean and variance are related by:

a) $\mu=\sigma^2$
b) $\mu<\sigma^2$
c) $\mu>\sigma^2$
d) None of above

A binomial distribution may be approximated by a Poisson distribution when

a) n is large and p is small
b) n is small and p is large
c) n is small and p is small
d) n is large and p is large

Which of the following is not a property of a hypergeometric experiment?

a) The probability of success changes on each trial.
b) The successive trials are independent.
c) The experiment is repeated a fixed number of times.
d) The outcomes or each trial may be classified into one of two categories, success and failure.

For a negative binomial distribution, the mean and variance are related by:

a) $\mu=\sigma^2$
b) $\mu<\sigma^2$
c) $\mu>\sigma^2$
d) None of above

Which of the following is not a property of a multinomial experiment?

a) The successive trials are all independent.
b) The experiment is repeated a fixed number of times.
c) The outcomes of each trial may be classified into one of k categories ($k\geq 2$).
d) The probability of success changes on each trial.

Binomial
Q 1





**SUBJECTIVE**

8-1

8.1 a) What is a binomial experiment and what are its properties?

b) Derive the binomial distribution and find its mean and variance

(P.U., B.A./B.Sc.

8.2 a) The probability of an event occurring on any one occasion is $p$. Prove that the prob its occurring on exactly $x$ of $n$ occasions is $\binom{n}{x}p^x q^{n-x}$, where $q=1-p$.

(P.U., B.A./B.Sc. 1984

b) Let $X$ have a binomial distribution with $n=3$ and $p=0.4$ $P\left(X=\frac{3}{2}\right)$, $P(X=2)$, $P(X\le 2)$, $P(X=-2)$ and $P(X\ge 2)$.

(B.Z.U., B.A./B.Sc.

8.3 a) A die is rolled five times and a 5 or 6 is considered a success. Find the probability success, (ii) at least 2 successes, (iii) at least one but not more than 3 successes.

(B.Z.U., B.A./B.Sc.

b) Using the binomial distribution, find the probability of

i) 3 successes in 8 trials when $p = 0.4$,

ii) 2 failures in 6 trials when $p = 0.6$,

iii) 2 or fewer successes in 9 trials when $p = 0.4$. (P.U., M.A. Ec

8.4 a) Find the probability of getting (i) exactly 4 heads and (ii) not more than 4 head coins are tossed.

b) Find the probability of (i) 2 or more heads, (ii) fewer than 4 heads in a single toss coins.

8.5 a) If the probability of getting caught copying someone else's exam is 0.2, find the of not getting caught in 3 attempts. Assume independence.

b) If 60% of the voters in a large district prefer candidate $A$, what is the probability sample of 12 voters exactly 7 will prefer $A$?

c) The probability that a patient recovers from a delicate heart operation is 0.9. probability that exactly five of the next 7 patients having this operation survive?

8.6 a) The incidence of occupational disease in an industry is such that the workmen chance of suffering from it. What is the probability that out of 6 workmen (i) not 2, and (ii) 4 or more will catch the disease? (P.U., B.A./B.S

b) If on the average rain falls on twelve days in every thirty, find the probability that three days of a given week will be fine and the remaining wet, (ii) rain will fall days of a given week.

8.7 An insurance salesman sells policies to 5 men, all of identical age and in good health. the actuarial tables, the probability that a man of this particular age will be alive 30 years 2/3. Find the probability that in 30 years (i) all men, (ii) *at least* 3 men, (iii) only two *most* one man will be alive.

a) A multiple-choice quiz has 15 questions, each with 4 possible answers of which only 1 is the correct answer. What is the probability that sheer guess work yields from 5 to 10 correct answers?

b) A commuter drives to work each morning. The route she takes each day includes ten stoplights. Assume the probability each stoplight is red when she gets to it, is 0.2 and that these stoplights (trials) are independent. What is the distribution for $X$, the number of times she must stop for a red light on her way to work? Evaluate $P(X=0)$ and $P(X \leq 5)$.

a) Find the successive terms of the binomial frequency distribution $600\,(0.3+0.7)^6$.

b) Five dice are tossed 96 times. Find the expected frequencies when throwing of a 4, 5 or 6 is regarded as a success.

a) A perfect cubic die is thrown a large number of times in sets of 8. The occurrence of a 5 or a 6 is called a success. In what proportion of the sets would you expect 3 successes?

b) An irregular six-faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times, 10,000 sets of 10 throws would you expect it to give no even number?

(P.U., B.A./B.Sc. 1975)

Four dice are thrown and the number of sixes in each throw are recorded. This is repeated 108 times. Write down the theoretical frequencies of 0, 1, 2, 3 and 4 sixes. Calculate the mean number of sixes in a single throw.

a) Find the mean and standard deviation of the binomial distribution $(q+p)^3$.

b) Find the mean and variance of the binomial $(q+p)^n$. (P.U., B.A./B.Sc. 1962, 69)

c) A r.v. $X$ is binomially distributed with mean 3 and variance 2, compute $P(X=7)$.

a) In a binomial distribution, the mean and the standard deviation were found to be 36 and 4.8 respectively. Find $p$ and $n$. (P.U., B.A./B.Sc. (Hons), 1966)

b) A random variable is binomially distributed with mean 12.38 and variance 8.64. Find $n$ and $p$. (I.U., M.A. Econ., 1989)

c) Is it possible to have a binomial distribution with mean = 5 and $s.d.$ = 3?

(P.U., B.A./B.Sc. (Hons.) 1969)

Obtain the first four moments of a binomial distribution.

Show that in a binomial distribution where $p$ is the probability of success, the moments about the mean are given by $\mu_2 = npq;\ \mu_3 = npq(q-p);\ \mu_4 = npq[1+3(n-2)pq]$. (P.U., D. St. 1962)

Show that for the binomial distribution $b(x;\, n, p)$

$$\beta_1 = \frac{(1-2p)^2}{npq} \text{ and } \beta_2 = 3 + \frac{1-6pq}{npq}$$

a) Let $X$ be a random variable having a binomial distribution with parameters $n$=25 and $p$=0.2. Evaluate $P[X < \mu - 2\sigma]$.

b) Given binomial probability function $P(X=x)=\binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x}$. Find the medi... mode of the distribution. (P.U., B.A./B.Sc.

8.18 A biased coin is tossed 4 times and the number of heads noted. The experiment is perfor... times in all. The results obtained are shown in the table:

| No. of heads | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency | 12 | 50 | 151 | 200 | 87 |

a) Find the probability of obtaining a head when the coin is tossed.

b) Calculate the theoretical frequencies of 0, 1, 2, 3, 4 heads, using the associated theo... binomial distribution.

8.19 The incidence of defective items in 200 samples of 6 is shown in the following table:

| No. of defectives per sample | 0 | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|---|
| No. of samples | 36 | 70 | 61 | 25 | 7 | 1 | 0 | 200 |

Assuming these results follow a binomial distribution, compute the theoretical ... probabilities and frequencies.

8.20 Fit a binomial distribution to the following data:

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f$ | 30 | 62 | 46 | 10 | 2 |

(P.U., B.A./B.Sc.

8.21 Following data give the number of questions correctly answered out of 10 questions for 1... questions.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f$ | 0 | 1 | 3 | 8 | 16 | 28 | 18 | 13 | 9 | 4 | 0 |

Examine whether the distribution is binomial and find its mean and standard deviation.

8.22 Show that, if two symmetrical binomial distributions ($p = q = ½$) of degree $n$ (and of the ... number of observations) are so superposed, the *rth* term of the one coincides with the $(r-1)$... of the other, the distribution formed by adding superposed terms is symmetrical binomial ... $(n+1)$.

B.D

8.23 a) If $X$ has a binomial distribution $b(x; n, p)$, then show that $E(X) = np$, $\text{Var}(X) =$ ... $M_0(t) = (q+pe^t)^n$.

b) If the m.g.f. of $X$ is $M_0(t) = [1/4+(3/4)\,e^t]^{12}$, find $E(X)$, $\text{Var}(X)$ and $P(X \geq 10)$.

c) Derive the m.g.f. of the binomial distribution. Use it to find the mean and variance of the binomial distribution. (P.U., B.A./B.Sc. 1992)

24 Show that for the binomial distribution $(q+p)^n$, where $p=1-q$,

$$k_{r+1} = pq\frac{dk_r}{dp}, \quad r \geq 1; \text{ and}$$

hence find out the first four cumulants. (P.U., M.Sc. (Stat.) 1969)

25 a) What is a hypergeometric experiment and what are its properties?

b) Derive the hypergeometric probability distribution.

26 a) Find the mean and variance of the hypergeometric distribution. (P.U., B.A./B.Sc. 1980, 82, 84, 91, 94)

b) Determine the probability distribution for the number of white beads among 5 beads drawn at random from a bowl containing 4 white and 7 black beads. Use this to compute the mean and variance and check the results by using the formulas.

27 a) A committee of size 3 is selected from 4 men and 2 women. Find the probability distribution for the number of men on the committee.

b) A homeowner plants 6 bulbs selected at random from a box containing 4 tulip bulbs and 4 daffodil bulbs. What is the probability that he planted 2 daffodil bulbs and 4 tulip bulbs?

28 Ten vegetable cans, all the same size, have lost their labels. It is known that 5 contain tomatoes and 5 contain corn. If five are selected at random, what is the probability that all contain tomatoes? What is the probability that 3 or more contain tomatoes? (B.Z.U., B.A./B.Sc. 1991)

29 a) Determine the probability that the Income Tax Authorities will catch 3 income tax returns with illegitimate deductions, if it randomly selects 6 returns from among 20 income tax returns of which 8 contain illegitimate deductions.

b) To avoid detection at customs, a traveller has placed six narcotic tablets in a bottle containing nine vitamin pills that are similar in appearance. If the customs official selects 3 of the tablets at random for analysis, what is the probability that the traveller will be arrested for illegal possession of narcotics?

a) Discuss the difference in conditions that must exist in a problem situation for application of the hypergeometric and the binomial distributions.

b) Show that the hypergeometric distribution $h(x; N, n, k)$ can be approximated by the binomial distribution for large $N$ and $k$. (P.U., B.A./B.Sc. 1994)

A random sampling of 4 members of a 150 members club has shown that 3 prefer no smoking in the clubhouse dining room. What is the probability that this will occur if in fact only 20% of members prefer no smoking in the dining room. Find this probability assuming that the sample was obtained under

a) sampling without replacement, and

b) sampling with replacement.

Compare the two answers.

Poisson 339-340



8.32 a) Describe a Poisson distribution.

b) Derive the Poisson distribution as the limiting form of the binomial distribution, st clearly the assumptions you make. (P.U., B.A./B.Sc. 1975, 80

c) If $X$ is a Poisson random variable with $\mu = 1.6$, find $P(X=0)$, $P(X=1)$, $P(X=2)$ and $P(X>$

8.33 a) A Poisson distribution is given by $P(X = x) = \dfrac{e^{-3}3^x}{x!}$. Find the probabilities for x = 0,

3 and 4. (P.U., B.A./B.Sc. 1974

b) Suppose that $X$ has a Poisson distributions if $P(X=1)=0.3$ and $P(X=2)=0.2$, calculate $P($ and $P(X=3)$.

c) A random variable $X$ has a Poisson distribution such that $P(X=2) = 3P(X=4)$. Find $P($ and $P(X \le 3)$ upto three places of decimals. (P.U., B.A./B.Sc.

8.34 a) Prove that, if $n \to \infty$ and $p \to 0$ such that $np = \mu$, then

$$\binom{n}{k} p^k (1-p)^{n-k} \to \frac{e^{-\mu}\mu^k}{k!}$$

where $e = \lim_{n\to\infty} (1 + 1/n)^n = 2.7183$ approximately. (P.U., B.A./B.Sc.

b) Past experience in the production of a certain component has shown that the propor defectives is 0.03. Components leave the factory in boxes of 500. What is the probabil

i) a box contains 3 or more defectives;

ii) two successive boxes contain 6 or more defectives between them?

(P.U., B.A./B.Sc. 197

8.35 a) Define the Poisson distribution and derive its mean and variance.

b) Ten percent of the tools produced in a certain manufacturing process turn out to be de Find the probability that in a sample of 10 tools chosen at random, exactly two defective by using (i) the binomial distribution and (ii) the Poisson approximatio binomial distribution. (P.U., B.A./B.Sc. 197

8.36 a) Suppose that the number of insurance claims closely approximates a Poisson distri with $\mu = 0.05$. Find the probability of (i) no claim and (ii) 1 or fewer claims.

b) Assume that the probability of being killed in an accident in a coal mine during a $\frac{1}{1400}$. Use the Poisson distribution to calculate the probability that in the mine em 350 miners, there will be at least one fatal accident in a year.

8.37 a) A secretary makes 2 errors per page on the average. What is the probability that on the page she makes (i) 4 or more errors? (ii) no error?

b) A car hire firm has 2 cars, which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with parameter 1.5. Calculate the proportion of days on which neither car is used, and the proportion of days on which some demand is refused. [$e^{-1.5}$=0.2231]. (B.Z.U., B.A./B.Sc. 1990)

A manufacturer of cotter pins knows that 5 percent of his product is defective. If he sells cotter pins in boxes of 100, and guarantees that not more than 4 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality? ($e^{-5}$=.0067)
(B.Z.U., B.A./B.Sc. 1988)

a) Find the mean and the variance of the Poisson distribution. State the relation between binomial and Poisson distributions. (P.U., B.A./B.Sc. 1970)

b) Given that $X$ has a Poisson distribution with variance 1, calculate $P(X=2)$.

a) Criticise the following statement:

"The mean of a Poisson distribution is 5 while its standard deviation is 4."

b) In a Poisson distribution the first two frequencies were 250 and 160. Find the frequencies of the next two values of the variable.

a) Show that the mean and the variance of a Poisson distribution are equal.
(P.U., B.A./B.Sc. 1986)

b) Find the first four moments of the Poisson distribution $p(x;\mu)$, and hence prove that $\beta_1 = \frac{1}{\mu}$ and $\beta_2 = 3+\frac{1}{\mu}$. (P.U., B.A./B.Sc. 1983, 88)

Show that the discontinuous (Poisson) distribution whose probabilities corresponding to the values 0, 1, 2, …, j, …, are

$$e^{-\lambda}, e^{-\lambda}\lambda, \frac{e^{-\lambda}\lambda^2}{2!}, \ldots, \frac{e^{-\lambda}\lambda^j}{j!}, \ldots$$

has the second, third, fourth and fifth central moments given by $\mu_2 = \mu_3 = \lambda$, $\mu_4 = \lambda(1+3\lambda)$, $\mu_5 = \lambda(1+10\lambda)$. (P.U., M.A. Stats. 1969)

Show that, if $X_1$ and $X_2$ are independent Poisson variables with parameters $\lambda_1$ and $\lambda_2$ respectively, then $Y=X_1+X_2$ is a Poisson variable with parameters $\lambda_1+\lambda_2$.

Does the difference of two Poisson variables follow Poisson distribution?

Fit a Poisson distribution to "Student's" yeast cell data.

| Yeast cell count | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Frequency | 213 | 128 | 37 | 18 | 3 | 1 |

poissoned 44-46

8.45 a) Fit (i) a Poisson (ii) a binomial to the following distribution:

| $x$ | 0 | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|---|
| $f$ | 531 | 354 | 99 | 15 | 1 | 1000 |

b) The frequency of accidents per shift in a factory is shown in the following table:

| Accidents per shift | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Frequency | 300 | 96 | 34 | 9 | 1 | 0 |

Use the Poison distribution to estimate the probability of .

i) no accidents in a shift. ii) more than one accident in a shift.

(P.U., B.A./B.Sc. (H

8.46 A skilled typist, on routine work, kept a record of mistakes made per day during 300 work

| Mistakes per day | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| No. of days | 143 | 90 | 42 | 12 | 9 | 3 | 1 |

Compute the frequencies of the Poisson distribution which has the same total frequency as the above distribution.

8.47 The number of road accidents notified to a certain police station per day is shown in the frequency table relating to a period of 300 successive days:

| Accidents per day | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Frequency | 90 | 113 | 64 | 21 | 7 | 3 | 1 | 1 |

Poisson process

Calculate the mean number of accidents a day. Use the Poisson distribution, with th calculate the expected frequencies. Assuming that this distribution continues to app probability of 4 or more accidents being notified on any one day.

Poissoned

8.48 a) Define a Poisson process. What are its properties or assumptions?

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b) Suppose that customers enter a certain shop at the rate of 30 persons an hou Poisson distribution, calculate the probability that in a 3-minutes interval, no enter the shop. (P.U., B.A

8.49 a) Flaws in plywood occur at random with an average of one flaw per 50 square the probability that a 4 feet x 8 feet sheet will have no flaws? At most one flaw?

b) A doctor receives an average of 3 telephone calls from 9 p.m. until 9 a.m. the Assuming arrivals of calls are a Poisson process, what is the probability that not be disturbed by a call if she goes to bed at midnight and rises at 6 a.m.?

(P.U., B.A

8.50 A computer system in a company has a breakdown once in 25 days, on the average breakdowns are a Poisson process, what is the probability of (i) exactly one breakd 10 days? (ii) more than one breakdown in the next 10 days?

The number of cars passing over a toll bridge during the time interval 10 to 11 a.m. is 300. The cars pass individually and collectively at random. Find the probability that

i) not more than 4 cars will pass during 1-minutes interval 10:45 to 10:46.

ii) 5 or more cars will pass during the same interval.

Find the m.f.g. about the mean for a Poisson distribution and use it to deduce the moment ratios $\beta_1$ and $\beta_2$ for the distribution.

a) Use m.g.f. to prove that the sum of two independent Poisson variables is a Poisson variable.

b) Suppose that the probability of an insect laying $n$ eggs is a Poisson distribution with mean $m$, and that the probability of an egg developing is $p$. Assuming natural independence of the eggs, show that the probability of a total of $k$ survivors is given by the Poisson distribution with parameter $mp$. (P.U., M.A. Stats 1969)

What is a negative binomial experiment and what are its properties? Derive the negative binomial distribution.

The probability that a swimmer will succeed in swimming across a lake is 0.4. What is the probability that the tenth swimmer is the fourth one to cross the lake?

If $X$ has a negative binomial distribution, then show that $E(X)=kq/p$ and $\text{Var}(X)=kq/p^2$.

If $E(X)=10$ and $\sigma=3$, can $X$ have a negative binomial distribution?

Find the probability that a person flipping a coin gets the third head on the seventh flip.

In each of a succession of independent trials, the probability of a certain event $A$ is $p$. Trials are continued until the event $A$ has been observed exactly $k$ times. If $X$ be the number of trials, show that the distribution of $X$ is

$$P(X=x)=\binom{x-1}{k-1}p^k q^{x-k}, \qquad x=k, k+1, \ldots$$

Find the expected value and standard deviation of $X$.

The probability that a person will install a black telephone in a residence is estimated to be 0.3. Find the probability that the 10th phone installed in a new sub-division is the 5th black phone.

Describe the negative binomial distribution and show that its variance is greater than its mean.

Calculate the first four cumulants for the negative binomial distribution. (P.U., M.A. Stat. 1969)

What is a geometric experiment and what are its properties? Derive the geometric probability distribution.

Show that the mean and variance of the geometric distribution are $\mu = 1/p$ and $\sigma^2 = q/p^2$.

When flipping an unbiased coin, determine the probability that the first head occurs on the third trial.

8.59 a) What is a multinomial experiment and what are its properties? Derive the formula for multinomial distribution.

b) Find the probability of being dealt a bridge hand of 13 cards containing 5 spades, 2 hearts, diamonds and 3 clubs.

8.60 a) The painted light bulbs produced by a company are 50% red, 30% blue and 20% green. sample of 5 bulbs, find the probability that 2 are red, 1 is green and 2 are blue.

b) A box contains 5 red, 3 white and 2 blue marbles. A sample of 6 marbles is drawn replacement. Find the probability that (i) 3 are red, 2 are white and 1 is blue, (ii) 2 are are white and 1 is blue, (iii) 2 of each colour appears.

8.61 a) Derive the mean of hypergeometric distribution.

b) The reception office at a building receives an average of 4.9 phone calls per half hour. the probabilities of receiving exactly 6 phone calls at this office during

i) half hour  ii) an hour

c) The painted light bulbs produced by a company are 50% red, 30% blue and 20% green. sample of 5 bulbs, find the probability that 2 red, 1 is green and 2 are blue.

(P.U., B.A.

8.62 a) Derive the Poisson distribution as the limiting form of the binomial distribution clearly the assumptions you make. Also derive the moment generating function Poisson distribution.

b) During a promotional campaign of a new drink, a soft drink company places prize caps on one of every ten bottles. Hoping to win a prize, a child decides to buy a bottle new cola each day for one full week. What is the probability that the child win prize

i) at least one day?  ii) first two days?  iii) all days?

(P.U., B.A.

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# CHAPTER 9

# CONTINUOUS PROBABILITY DISTRIBUTIONS

# CONTINUOUS PROBABILITY DISTRIBUTIONS

## NTRODUCTION

In this chapter, we shall consider some important continuous probability distributions or density ons which are met in practice. Of all the continuous probability distributions, the *normal ution* is perhaps the most important distribution which is used extensively in solving problems both ability and in statistical inference.

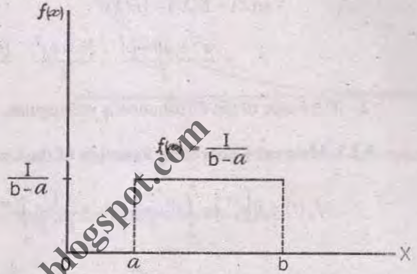
## NIFORM DISTRIBUTION

The density function of a continuous r.v. $X$ is called a *uniform distribution* when between the enc any two subintervals of the same length containing $X$, have the same probability.

Alternatively, a r.v. $X$ is said to be *distributed* if its density function is d as

$$= \frac{1}{b-a}, \quad a \le x \le b$$

$$= 0, \qquad \text{elsewhere.}$$




ibution derives its name from the fact ensity is constant or uniform over the $[a, b]$ and is 0 elsewhere.

called the *rectangular distribution* because its total probability is confined to a rectangular h base equal to $(b-a)$ and height $1/(b-a)$. The parameters of this distribution are $a$ and $b$. Since ed that $X$ with this distribution is random variable, therefore we must have that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}[x]_a^b = \frac{b-a}{b-a} = 1$$

bution arises in the study of rounding off errors, etc. Its distribution function will be

$$F(x) = \begin{cases} 0, & \text{for } x<a, \\ \dfrac{x-a}{b-a}, & \text{for } a \le x \le b, \\ 1 & \text{for } x>b, \end{cases}$$

### Properties of the Uniform Distribution.

Let $X$ have the uniform distribution over $[a, b]$. Then its mean is $\frac{a+b}{2}$ and variance is $\frac{(b-a)^2}{12}$.

$$E(X) = \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}, \text{ the midpoint of the interval.}$$

And $\quad \text{Var}(X) = E(X^2) - [E(X)]^2$, where

Now $\quad E(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^2\, dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + a^2}{3}$$

$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$$

2. The shape of the distribution is rectangular.

**9.2.2 Moment Generating Function of the Uniform Distribution.** The *m.g.f.* is obtained a

$$M_o(t) = E\left[e^{tX}\right] = \int_a^b e^{tx} \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b e^{tx}\, dx = \frac{1}{b-a}\left[\frac{1}{t} e^{tx}\right]_a^b$$

$$= \frac{e^{bt} - e^{at}}{(b-a)t}.$$

Putting $a = 0$ and $b = 1$, we obtain a r.v. with a uniform distribution in the interval [0, 1]. The *k*th moment about the origin is

$$\mu_k' = \int_0^1 x^k\, dx = \left[\frac{x^{k+1}}{k+1}\right]_0^1 = \frac{1}{k+1}$$



## 9.3 EXPONENTIAL DISTRIBUTION

A random variable $X$ is said to have an *exponential distribution* with parameter $\lambda$, if
defined by

$$f(x) = \lambda e^{-\lambda x} \qquad \text{for } x > 0,$$
$$= 0, \qquad \text{elsewhere,}$$

where $\lambda > 0$. The *p.d.f.* may also be written as

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \qquad \text{for } x > 0,$$
$$= 0, \qquad \text{elsewhere,}$$

The function $f(x)$ is a proper *p.d.f.* since

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = \left[-e^{\lambda x}\right]_0^{\infty} = 1$$

The *distribution function* of the exponential r.v. $X$ is given by

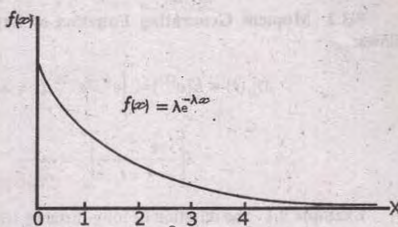
$F(x) = P(X \le x)$

$= \int_0^x \lambda e^{-\lambda x}\,dt = \left[-e^{-\lambda t}\right]_0^x$

$= 1 - e^{-\lambda x}$, for $x > 0$,

$= 0$, elsewhere,

and hence $P(X > x) = e^{-\lambda x}$.




**9.3.1 Properties.** The following are some of the important properties of the exponential distribution with parameter $\lambda$.

1. The mean and standard deviation of the exponential distribution are equal.

Now $E(X) = \mu = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^{\infty} x.\lambda e^{-\lambda x}\,dx$

To integrate it by parts, we use the formula $\int u\,dv = uv - \int v\,du$ and make the substitution, $\lambda e^{-\lambda x}dx = dv$ and $x = u$, so that $v = -e^{-\lambda x}$ and $du = dx$. Then we have

$$\int_0^{\infty} x.\lambda e^{-\lambda x}\,dx = \left[-xe^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= 0 + \left[\frac{-e^{-\lambda x}}{\lambda}\right]_0^{\infty} = \frac{1}{\lambda}$$

Thus $E(X)$ or $\mu = \dfrac{1}{\lambda}$.

Again $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$, where

$$E(X^2) = \int_0^{\infty} x^2\,\lambda.e^{-\lambda x}\,dx$$

$$= \lambda\left[x^2\left(-\frac{1}{\lambda}e^{-\lambda x}\right)\right]_0^{\infty} + \int_0^{\infty} 2x.e^{-\lambda x}\,dx \quad \text{(integrating by parts)}$$

$$= 0 + \frac{2}{\lambda}E(X) = \frac{2}{\lambda^2}$$

$$\therefore \quad \mathrm{Var}(X) = \sigma^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

and hence $\sigma = \dfrac{1}{\lambda}$.

2. The distribution is extremely skewed and thus there does not exist any mode.

**9.3.2 Moment Generating Function of Exponential Distribution.** The *m.g.f.* is obtained as follows:

$$M_0(t) = E[e^{tX}] = \int_0^\infty e^{tx}\,\lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-(\lambda - t)x}\,dx$$

$$= \lambda\left[\frac{-e^{-(\lambda-t)x}}{\lambda - t}\right]_0^\infty = \frac{\lambda}{\lambda - t} \qquad \text{for } t < \lambda.$$

**Example 9.1** The duration of long-distance telephone calls is found to be exponentially distributed with a mean of 3 minutes. What is the probability that a call will last (i) more than 3 minutes, (ii) more than 5 minutes?

Let $X$ be the exponential *r.v.* with parameter $\lambda$. Then the mean, *i.e.* $\dfrac{1}{\lambda} = 3$, so that $\lambda = \dfrac{1}{3}$.

Now (i) the probability that a call will last more than 3 minutes, is given by

$$P(X > 3) = \int_3^\infty \left(\frac{1}{3}\right) e^{-x/3}\,dx = \left[-e^{-x/3}\right]_3^\infty = e^{-1} = 0.3679$$

and (ii) the probability that a call will last more than 5 minutes, is given by

$$P(X > 5) = \int_5^\infty \left(\frac{1}{3}\right) e^{-x/3}\,dx = \left[-e^{-x/3}\right]_5^\infty = e^{-1.7} = 0.1827$$



## 9.4 GAMMA AND BETA DISTRIBUTIONS

The *gamma* and *beta* distributions derive their names from the well known gamma and beta functions which are very important in many areas of probability theory and mathematics. Therefore, before proceeding to these distributions, it is appropriate to review the gamma and beta functions and some of their main properties.

**9.4.1 Gamma Function.** The *gamma function* for any number $n > 0$, denote by $\Gamma(n)$, is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

We can give an alternative meaning to $\Gamma(n)$ by proving that $\Gamma(n+1) = n!$.

Now by definition, $\Gamma(n+1)=\int_0^\infty x^n.e^{-x}\,dx, \qquad n>0$

To integrate the above integral by parts, we use the formula $\int_{-\infty}^{\infty} u\,dv = uv - \int_{-\infty}^{\infty} v$ du and make the substitutions $u=x^n$, $dv=e^{-x}dx$ so that $du=nx^{n-1}$ and $v=-e^{-x}$. Then we obtain

$$\Gamma(n+1)=\left[-e^{-x}x^n\right]_0^\infty+\int_0^\infty nx^{n-1}.e^{-x}\,dx$$

$$=n\int_0^\infty x^{n-1}e^{-x}\,dx$$

$=n\Gamma(n)$ by the definition of $\Gamma$ –function.

Again $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx$

$$=\left[-e^{-x}x^{n-1}\right]_0^\infty+\int_0^\infty (n-1)e^{-x}x^{n-2}\,dx$$

$$=(n-1)\Gamma(n-1)$$

Let $n$ be a positive integer. Then repeating the application of $\Gamma(n)=(n-1)\Gamma(n-1)$, we get

$$\Gamma(n+1)=n\,.\,(n-1)\Gamma(n-1)$$
$$=n(n-1)(n-2)\ \ldots\ 3.2.\Gamma(1)$$

Putting $n=0$, we find that

$$\Gamma(1)=\int_0^\infty e^{-x}\,dx=\left[-e^{-x}\right]_0^\infty=-e^{-\infty}+1=1.$$

Hence $\Gamma(n+1)=n(n-1)(n-2)\ \ldots\ 3.2.1=n!$

Writing $x=y^2$ in $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx$, we have

$$\Gamma(n)=2\int_0^\infty e^{-y^2}.\,y^{2n-1}\,dy,$$

which is another form of $\Gamma$ -function.

Putting $n=\dfrac{1}{2}$ in this form, we obtain

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-y^2}\,dy=\int_0^\infty e^{-y^2}\,dy=\sqrt{\pi}\,.$$

**9.4.2 Beta Function.** The *beta function* for any two positive numbers *m, n,* denoted by $B(m, n)$, is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \qquad \text{for } m > 0, n > 0.$$

When $m = n = 1$, $B(1, 1) = \int_0^1 x^0(1-x)^0\,dx = 1$

Writing $z = 1 - x$, we find that

$$B(m, n) = -\int_1^0 (1-z)^{m-1} z^{n-1}\,dz = \int_0^1 (1-z)^{m-1} z^{n-1}\,dz$$

$$= B(n, m)$$

Hence the *B*-function is symmetrical about *m* and *n*.

Now let $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$. Then the substitution gives

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Putting $m = n = \frac{1}{2}$, we find that $B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi$

Again let $x = \dfrac{1}{1+z}$ so that $dx = \dfrac{-1}{(1+z)^2}dz$. Then

$$B(m, n) = \int_0^\infty \frac{z^{n-1}}{(1+z)^{m+n}}dz, \text{ which is also } B\text{-function.}$$

It is interesting to note that the gamma and beta functions are related according to the following formula:

$$B(m, n) = \frac{\Gamma(m)\,.\,\Gamma(n)}{\Gamma(m+n)}, \qquad \text{for } m > 0, n > 0.$$

Putting $m = n = \frac{1}{2}$, we have

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right).\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

But $\quad B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

Hence $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \;\; or \;\; \int_0^{\infty} e^{-x} x^{-1/2}\, dx = \sqrt{\pi}$

This is a very important result.

**9.4.3 Gamma Distribution.** A continuous r.v. $X$ is said to have a *gamma distribution* with parameter $m > 0$, if its p.d.f. is defined by

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(m)} x^{m-1} . e^{-x} & for\ 0 \le x < \infty \\ 0, & elsewhere \end{cases}$$

A gamma variable with parameter $m$ is usually denoted by $\gamma(m)$. A straightforward integration shows that $\int_0^{\infty} f(x)\, dx = 1$ and hence it represents a p.d.f.

The distribution function $F(x)$ is

$$F(x) = \int_0^{x} \frac{1}{\Gamma(m)} x^{m-1} . e^{-x}\, dx, \qquad x \ge 0$$

$$= 0, \qquad x < 0$$

which is also called the *Incomplete Gamma function* and has been tabulated by Karl Pearson.



**9.4.4 Properties of Gamma Distribution.** The important properties of the gamma distribution are given as follows:

1. The mean and variance of the gamma distribution are equal to its parameter $m$.

Now, $\quad \mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^{\infty} x . \frac{1}{\Gamma(m)} x^{m-1} . e^{-x}\, dx$

$$= \int_0^{\infty} \frac{1}{\Gamma(m)} x^{m} . e^{-x}\, dx = \frac{\Gamma(m+1)}{\Gamma(m)} = \frac{m\Gamma(m)}{\Gamma(m)} = m$$

and $\quad \mathrm{Var}(X) = E(X^2) - [E(X)]^2$, where

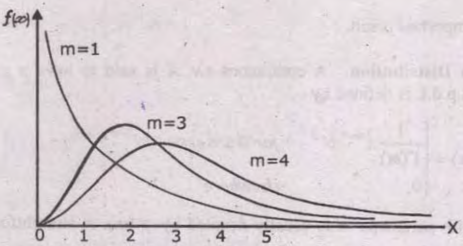
$$E(X^2) = \int_0^{\infty} x^2 \frac{1}{\Gamma(m)} x^{m-1} . e^{-x}\, dx$$

$$= \frac{\Gamma(m+2)}{\Gamma(m)} = \frac{(m+1)(m)\Gamma(m)}{\Gamma(m)} = m(m+1)$$

$$\therefore \quad \sigma^2 = m(m+1) - m^2 = m.$$

Hence mean and variance are each equal to $m$.

2. The curve of $f(x)$ is asymptotic to the $X$-axis. If $m > 1$, the curve has a mode at $x = m-1$. If $m > 2$, it touches the $X$-axis at the origin




3. The curve becomes asymptotic to both axes when $m$ lies between zero and one.
4. **Reproductive Property.** The sum of two independent Gamma distributions with parameters $m$ and $n$ is a Gamma distribution with parameter $(m + n)$.

**9.4.5 Moment Generating Function of Gamma Distribution.** The *m.g.f.* of $X$ with respect to origin is

$$M_0(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{1}{\Gamma(m)} x^{m-1} e^{-x}\, dx$$

$$= \int_0^\infty \frac{1}{\Gamma(m)} x^{m-1} e^{-x(1-t)}\, dx$$

Let $u = x(1-t)$, so that $dx = \frac{du}{1-t}$. Then substituting these values, we get

$$M_0(t) = \frac{1}{\Gamma(m)} \int_0^\infty e^{-u} \left(\frac{u}{1-t}\right)^{m-1} \frac{du}{1-t}$$

$$= \frac{1}{(1-t)^m} \cdot \int_0^\infty \frac{1}{\Gamma(m)} . u^{m-1} . e^{-u}\, du$$

$$= (1-t)^{-m}, \text{ provided that } |t| < 1.$$

On differentiating $M_0(t)$ $r$ times with respect to $t$ and putting $t = 0$, we find.

$$\mu'_r = m(m+1)\ldots(m+r-1)$$

Thus $\quad \mu'_1 = m, \;\; \mu'_2 = m(m+1)$ and hence $\mu_2 = \mu'_2 - \mu'^2_1 = m$, etc.

The cumulant generating function for this distribution with respect to origin is:

$$\kappa(t) = \log_e M_0(t) = -m \log(1-t)$$

$$= m\left[t + \frac{t^2}{2} + \frac{t^3}{3} + \dots\right] = m\sum_{r=1}^{\infty} \frac{t^r}{r!}(r-1)!$$

Thus the $r$th cumulant is given as

$$\kappa_r = m(r-1)! = m\,\Gamma(r).$$

**9.4.6 Beta Distribution of the First Kind.** A continuous r.v. $X$ is said to have a beta distribution with two parameters $m$ and $n$, if its *p.d.f.* is defined by

$$f(x) = \begin{cases} \dfrac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}, & 0 \le x \le 1;\ m,n>0 \\ 0, & \text{elsewhere.} \end{cases}$$

This distribution is known as a *beta distribution of the first kind* and a beta variable of the first kind is briefly referred to as $\beta_1(m,n)$. It is a proper probability density function since the area under the curve is one.

The distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ \displaystyle\int_0^x \frac{1}{B(m,n)} . x^{m-1}(1-x)^{n-1}\,dx & \text{for } 0 \le x \le 1 \\ 0, & \text{for } x > 1 \end{cases}$$

This is also called the *incomplete beta function*, and it has been extensively tabulated.



**9.4.7 Properties of $\beta_1(m, n)$.** The main properties of this distribution are given below:

1. The mean and variance of this distribution are $\dfrac{m}{m+n}$ and $\dfrac{mn}{(m+n)^2\,(m+n+1)}$ respectively.

   They are computed as below:

$$\mu = E(X) = \int_0^1 \frac{1}{B(m,n)}.\, x^{m-1}(1-x)^{n-1}.\, x\,dx$$

$$= \int_0^1 \frac{1}{B(m,n)}.\, x^m (1-x)^{n-1}\,dx = \frac{B(m+1,n)}{B(m,n)}$$

$$= \frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} = \frac{m}{m+n}$$

$$\text{Var}(X) = \sigma^2 = E(X^2) - [E(X)]^2, \text{ where}$$

$$E(X^2) = \int x^2 f(x)dx$$

$$= \int_0^1 \frac{1}{B(m, n)} \cdot x^{m+1} (1-x)^{n-1}\, dx$$

$$= \frac{B(m+2, n)}{B(m,n)} = \frac{m(m+1)}{(m+n)(m+n+1)}$$

$$\therefore \quad \sigma^2 = \frac{m(m+1)}{(m+n)(m+n+1)} - \left(\frac{m}{m+n}\right)^2 = \frac{mn}{(m+n)^2 (m+n+1)}$$

2. **Higher Moments.** The *rth* moment about the origin 0 is given by

$$\mu'_r = E(X^r) = \frac{1}{B(m, n)} \int_0^1 x^r \cdot x^{m-1} (1-x)^{n-1}\, dx$$

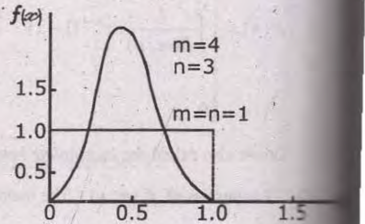
$$= \frac{B(m+r, n)}{B(m, n)} = \frac{\Gamma(m+r)}{\Gamma(m)} \cdot \frac{\Gamma(m+n)}{\Gamma(m+r+n)}$$

$$= \frac{m(m+1) \ldots (m+r-1)}{(m+n)(m+n+1) \ldots (m+n+r-1)}$$

It should be noted that the *m.g.f.* for this distribution does not have a simple form.

3. The shape of the beta distribution for $m=4$, $n=3$ is indicated in the figure.

4. If $m$ and $n$ are both greater than 1, it has a *modal* value at $x = \frac{m-1}{m+n-2}$. If $m = n = 1$, it reduces to the uniform distribution over the unit interval. The curve of $f(x)$ touches the $X$-axis at $x=0$, when $m>2$.






**9.4.8 Beta Distribution of Second Kind.** A continuous r.v. $X$ is said to have a *beta dis*[...] *the second kind* with parameters $m$ and $n$, if its p.d.f. is defined by

$$f(x) = \begin{cases} \frac{1}{B(m, n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}}, & \text{for } 0 \le x < \infty,\ m, n > 0, \\ 0, & \text{otherwise.} \end{cases}$$

A beta variable of the second kind is generally denoted by $\beta_2(m, n)$. To check that the function ...presents a proper probability distribution, we observe that

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{B(m, n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Let $1+x=\dfrac{1}{y}$ so that $x=\dfrac{1-y}{y}$ and $dx=\dfrac{-dy}{y^2}$. Then substitution gives

$$\int_0^\infty f(x)\,dx = \int_0^1 \frac{1}{B(m, n)} \cdot y^{n-1}(1-y)^{m-1}\,dy = 1.$$

...4.9 **Properties of $\beta_2$ (m, n).** The important properties of beta distribution of the second kind are ...below:

1. Moments about the origin are easily calculated as follows:

$$\mu_r' = E(X^r) = \int_0^\infty \frac{1}{B(m, n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}} \cdot x^r\,dx$$

$$= \frac{1}{B(m, n)} \int_0^1 y^{n-r-1}(1-y)^{m+r-1}\,dy, \text{ on putting } 1+x=\frac{1}{y}$$

$$= \frac{1}{B(m, n)} \int_0^1 z^{m+r-1}(1-z)^{n+r-1}\,dz, \text{ where } z = 1-y$$

$$= \frac{B(m+r, n-r)}{B(m, n)} = \frac{m(m+1)\ldots(m+r-1)}{(n-1)(n-2)\ldots(n-r)},\ r<n,$$

2. If $m>1$, the distribution is unimodal with a mode at $x=\dfrac{m-1}{n+1}$. If $m = 1$, the distribution is J-shaped.

3. The curve of $f(x)$ is asymptotic to the $X$-axis and it touches it at the origin if $m>2$.

4. The curve touches the $Y$-axis at the origin if $1<m<2$, and the curve becomes asymptotic to both axes when $m$ lies between 0 and 1.

## ...RMAL DISTRIBUTION

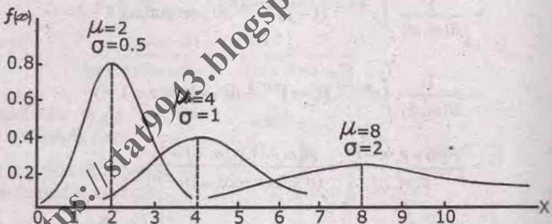
...e *normal probability distribution*, which is considered the cornerstone of the modern statistical ... was discovered by Abraham de Moivre (1667–1754) as the limiting form of the binomial ... by increasing $n$, the number of trials, to a very large number for a fixed value of $p$. But his ...mained unnoticed until 1924 when it was found in a library by Karl Pearson (1857–1936). The ... Pierre S. Laplace (1749–1827) is also associated with the derivation of the normal distribution. ...rmal distribution is also called the Gaussian distribution in honour of the great German

mathematician Carl *F.* Gauss (1777–1855), who also derived its equation mathematically as probability distribution of the errors of measurements. It was Karl Pearson who in 1893 called it *normal* distribution and is best known by this name today.

A normal distribution is defined by the p.d.f.

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-[(x-\mu)^2/2\sigma^2]},\text{ for }-\infty<x<\infty,\text{ and }\sigma>0$$

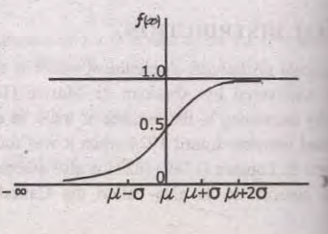
where $\mu$ is the mean, $\sigma$ is the standard deviation and $\pi$ (=3.1416) and e(=2.7183) are constants. Obviously a normal distribution is characterized by two parameters $\mu$ and $\sigma$, its mean and standard deviation. Since $\int_{-\infty}^{\infty} f(x)\,dx=1$ (see properties), the function f(x) is a proper probability density function. The normal distribution having mean $\mu$ and variance $\sigma^2$ is usually denoted by N($\mu,\sigma^2$). The '$X$ ~ N($\mu,\sigma^2$)' means that a r.v. *X* is normally distributed with mean $\mu$ and variance $\sigma^2$. The graph of normal distribution, which is symmetrical bell-shaped curve, is called the *normal curve*. The location and shape of the normal curve are determined by $\mu$ and $\sigma$. To put it differently, $\mu$ changes the position of the normal curve along horizontal axis while $\sigma$ determines the horizontal spread. A sketch for different values of $\mu$ and $\sigma$ is given below:




The *distribution function* of the normal probability distribution is given by

$$F(x)=P(X<x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-[(t-\mu)^2/2\sigma^2]}dt,$$

which is sketched below:




This curve is the *ogive* of the normal curve.

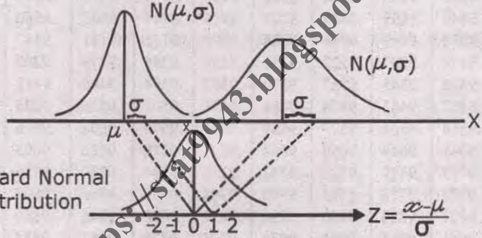
**9.5.1 Standardized Normal Distribution.** A normal probability distribution depends on the es of the parameters $\mu$ and $\sigma^2$ and the various possible values for these two parameters will result in limited number of different normal distributions. The r.v. $Z = \dfrac{X-\mu}{\sigma}$, as we have seen, has zero and unit variance. Every normally distributed r.v. $X$ with mean = $\mu$ and variance = $\sigma^2$ is therefore niently transformed into a new normal r.v. $Z$ with zero mean and unit variance by using the ing expression

$$Z = \frac{X-\mu}{\sigma}$$

Then the p.d.f. of $Z$, denoted by $\phi(z)$ ($\phi$ is pronounced *phi*) becomes

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad \text{for } -\infty < z < \infty .$$

he following figures illustrate the transformation of the original normal distribution into the d normal distribution.




probability distribution of $Z$ which has zero mean and unit variance, is called the d *normal distribution or unit normal distribution* and is denoted by $N(0, 1)$. The *distribution* he standard normal distribution, usually denoted by $\Phi(z)$ ($\Phi$ is capital $\phi$) is

$$\Phi(z) = P(Z < z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt ,$$

een tabulated for positive values of $z$. (Table 9.1 contains these values). The values of $\Phi(z)$ values of $z$ are obtained form the identify $\Phi(-z) = 1 - \Phi(z)$. It should be noted that $\dfrac{x-\mu}{\sigma}\Big)$ and for any $a$ and $b$ (positive or negative) $P(a<Z<b) = \Phi(b) - \Phi(a)$.

**Table 9.1. Cumulative Standard Normal Probabilities**

*Values of* $\Phi(z)$

| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | 6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | 6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7703 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |
| 3.5 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 |

| | | | | | |
|---|---|---|---|---|---|
| $z$ | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 |
| $\phi(z)$ | 0.90 | 0.95 | 0.975 | 0.99 | 0.995 |
| $\phi(-z) = 1-\phi(z)$ | 0.10 | 0.05 | 0.025 | 0.01 | 0.005 |

**9.5.2 Properties of Normal Distribution.** The main properties of the normal distribution are given below:

1. The function $f(x)$ defining the normal distribution is a proper p.d.f., i.e. $f(x) \geq 0$ and the total area under the normal curve is unity.

**Proof.** Cleary $f(x)$ is always non-negative, and the total probability (area) is

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Let $z = \dfrac{x-\mu}{\sigma}$. Then $\sigma\ dz = dx$. Substituting these values, we get

$$\text{Area} = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma\ dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{0} e^{-z^2/2}\,dz + \int_{0}^{\infty} e^{-z^2/2}\,dz \right]$$

The function $\int_{-\infty}^{0} e^{-z^2/2}\,dz$ being an even function of $z$, can be by letting $w = -z$, written as

$$\int_{-\infty}^{0} e^{-z^2/2}\,dz = \int_{0}^{\infty} e^{-w^2/2}\,dw. \text{ Then}$$

$$\text{Area} = 2.\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2}\,dz$$

Let $v = \dfrac{1}{2}z^2$, so that $dv = z dz$. Then

$$\text{Area} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-v} \frac{dv}{\sqrt{2v}} = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-v}\, v^{-1/2}\,dv$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} . \sqrt{\pi} = 1.$$

Thus the total area (probability) under the normal curve is unity and hence the function $f(x)$ defines a proper *p.d.f.*

2. The mean and variance of the normal distribution are $\mu$ and $\sigma^2$ respectively.

**Proof.** By definition, $\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \,.\, e^{-[(x-\mu)^2/2\sigma^2]}\, dx$$

Let $\quad z = \frac{x-\mu}{\sigma}$. Then $x = \mu + z\,\sigma$ and $dx = \sigma\, dz$.

{Limits:
when $x = \infty$, $z = \infty$;
when $x = -\infty$, $z = -\infty$ }

Therefore $\quad E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + z\sigma) e^{-z^2/2}\, dz$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2}\, dz$$

The first integral represents $\mu$ times the area under a normal curve with zero mean and unit variance and hence is equal to $\mu$. The second integral being an odd function, equals zero. Thus

$E(X) = \mu$, i.e. $\mu$ is the mean of the normal distribution.



And $\quad \mathrm{Var}(X) = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-[(x-\mu)^2/2\sigma^2]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\sigma^2 \,.\, e^{-z^2/2}\, dz, \qquad \text{on putting } z = \frac{x-\mu}{\sigma}$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \,.\, z\, e^{-z^2/2}\, dz,$$

To integrate by parts we use the formula $\int_{-\infty}^{\infty} u\,dv = uv - \int_{-\infty}^{\infty} v\,du$ and make the substitution $dv = z e^{-z^2/2}\, dz$ and $u = z$ so that $v = -e^{-z^2/2}$ and $du = dz$. Then

$$\mathrm{Var}(X) = \frac{\sigma^2}{\sqrt{2\pi}} \left[ -z \,.\, e^{-z^2/2} \right]_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz = 0 + \sigma^2 = \sigma^2$$

Hence $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

3. The median and the mode of the normal distribution are each equal to $\mu$, the mean of the distribution.

Now, the median, $a$, is given by $\int_{-\infty}^{a} f(x)\, dx = \frac{1}{2}$. Therefore

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} e^{-[(x-\mu)^2/2\sigma^2]}\, dx = \frac{1}{2}$$

or

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a-\mu}{\sigma}} e^{-z^2/2}\, dz = \frac{1}{2}, \text{ on putting } z = \frac{x-\mu}{\sigma}.$$

But we know from the symmetry of the standard normal distribution that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2}\, dz = \frac{1}{2}$$

$\frac{a-\mu}{\sigma} = 0$ or $a = \mu$, *i.e.* $\mu$ is the median of the distribution.

Again the *mode,* if any, is that value of $x$ for which $f'(x) = 0$ and $f''(x) < 0$.

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[(x-\mu)^2/2\sigma^2]} \left(\frac{x-\mu}{\sigma^2}\right)\left(-\frac{2}{2}\right)$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}} (x-\mu) e^{-(x-\mu)^2/2\sigma^2}$$

Putting $f'(x) = 0$, we see that $x = \mu$.

Differentiating, we obtain

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} \left[ e^{-[(x-\mu)^2/2\sigma^2]} - e^{-[(x-\mu)^2/2\sigma^2]} \frac{(x-\mu)^2}{\sigma^2} \right]$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \left[ 1 - \frac{(x-\mu)^2}{\sigma^2} \right]$$

Putting $x = \mu$ in $f''(x)$, we see that $f''(x) < 0$. Thus $x = \mu$ is the mode of the normal distribution. Hence median and mode are both equal to $\mu$. Since mean = median = mode, the normal distribution is symmetrical and unimodal.

4. The mean deviation of the normal distribution is approximately $\frac{4}{5}$ of its standard deviation.

**Proof.** Now the mean deviation of the normal random variable $X$ from the mean, $\mu$, is given by

$$M.D. = E[|X-\mu|] = \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-[(x-\mu)^2/2\sigma^2]} dx$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-z^2/2} dz, \qquad \text{where } z = \frac{x-\mu}{\sigma}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} -ze^{-z^2/2} dz + \int_{0}^{\infty} ze^{-z^2/2} dz \right\}$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} ze^{-z^2/2} dz = \sigma\sqrt{2/\pi} \left[ -e^{-z^2/2} \right]_0^{\infty}$$

$$= \sigma\sqrt{2/\pi} = 0.7979\sigma = \frac{4}{5}\sigma, \text{ approximately.}$$

5. The normal curve has points of inflection which are equidistant from the mean.

**Proof:** The *point of inflexion* by which we mean a point at which the concavity changes, is obtained by solving the equation $f''(x) = 0$.



Differentiating the function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[(x-\mu)^2/2\sigma^2]}$, we get

$$f'(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} (x-\mu) e^{-(x-\mu)/2\sigma^2}$$

Equating $f'(x)$ to zero, we see that $x = \mu$. We also observe that $f'(x) > 0$ for $x < \mu$ and for $x > \mu$, $f'(x) < 0$. Thus the *maximum of the function* $f(x)$ *is at* $x = \mu$ and its value is $\frac{1}{\sigma\sqrt{2\pi}}$.

To find the points of inflection, we take the second derivative. Thus

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} \left[ -\frac{(x-\mu)^2}{\sigma^2} e^{-(x-\mu)^2/2\sigma^2} + e^{-(x-\mu)^2/2\sigma^2} \right]$$

$$= \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \left[ 1 - \frac{(x-\mu)^2}{\sigma^2} \right]$$

Equating $f''(x)$ to zero, we are left with the following equation

$$1 - \frac{(x-\mu)^2}{\sigma^2} = 0$$

which gives $(x-\mu)^2 = \sigma^2$

or $\quad x-\mu = \pm\sigma$ or $x = \mu+\sigma$ or $\mu-\sigma$.

At these two points, the values of the function $f(x)$ is $\dfrac{1}{\sigma\sqrt{2\pi e}}$.

Hence the two points of inflection of normal curve are $\left[\mu-\sigma, \dfrac{1}{\sigma\sqrt{2\pi e}}\right]$ and $\left[\mu+\sigma, \dfrac{1}{\sigma\sqrt{2\pi e}}\right]$. In other words, the points of inflection occur on the right and on the left of the mean at a distance equal to standard deviation and thus the graph of the normal curve is bell-shaped.

6. For the normal distribution, the odd order moments about the mean are all zero and the even order moments are given by

$$\mu_{2n} = (2n-1)(2n-3)\ldots 5.3.1\ \sigma^{2n}.$$

**Proof.** The *odd order* moments about the mean are given by

$$\mu_{2n+1} = E(X-\mu)^{2n+1}$$

$$= \int_{-\infty}^{\infty}(x-\mu)^{2n+1}\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \int_{-\infty}^{\infty}(\sigma z)^{2n+1}\frac{1}{\sigma\sqrt{2\pi}}e^{-z^2/2}\,\sigma\,dz, \text{ where } z=\frac{x-\mu}{\sigma}$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^{2n+1}\cdot e^{-z^2/2}\,dz,$$

$= 0$, because the integral is an odd function of $z$.

Thus $\quad \mu_3 = 0 = \mu_5 = \ldots\ldots$

The *even order* moments about the mean are obtained as below:

$$\mu_{2n} = E(X-\mu)^{2n}$$

$$= \int_{-\infty}^{\infty}(x-\mu)^{2n}\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \int_{-\infty}^{\infty}(z\sigma)^{2n}\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-z^2/2}\,\sigma dz, \text{ on putting } z=\frac{x-\mu}{\sigma}$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} . e^{-z^2/2}\, dz,$$

$$= \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_{0}^{\infty} z^{2n} . e^{-z^2/2}\, dz,$$

Let $y = \frac{z^2}{2}$. Then $dy = z\,dz$, and therefore

$$\mu_{2n} = \frac{\sigma^{2n}}{\sqrt{2\pi}} 2^{n+1/2} \int_0^\infty y^{n-1/2} e^{-y}\, dy$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty y^{(n+1/2)-1} e^{-y} . dy$$

$$= \frac{2^n . \sigma^{2n}\, \Gamma(n+1/2)}{\sqrt{\pi}}$$

$$= \frac{2^n . \sigma^{2n}}{\sqrt{\pi}} . \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right) \ldots \left(\frac{5}{3}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$= (2n-1)(2n-3) \ldots 5.3.1\ \sigma^{2n}$$

Putting $n = 1$ and 2, we get $\mu_2 = \sigma^2$ and $\mu_4 = 3\sigma^4$.

Hence $\beta_1 = 0$ meaning that skewness is zero, and $\beta_2 = \frac{\mu_4}{\mu_2^{\ 2}} = \frac{3\sigma^4}{\sigma^4} = 3$, *i.e.* the normal curve ha kurtosis. This is, in fact, the origin of the choice of the apparently arbitrary value 3 in the defin platy-, and lepto-kurtosis.



7. If $X$ is $N(\mu, \sigma^2)$ and if $Y = a + bX$, then $Y$ is $N(a + b\mu, b^2\sigma^2)$.

**Proof.** Finding expectation and variance of $Y$, we get

$$E(Y) = E(a + bX) = a + b\mu, \ (\because\ a \text{ and } b \text{ are constants})$$

and $$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = b^2\sigma^2$$

Now the function $X = a + bX$ may be either a decreasing or increasing function accord negative or positive. If $Y$ is normally distributed, then its *p.d.f.* is given by

$$h(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{y-a}{b} - \mu\right)^2 / 2\sigma^2} . \left|\frac{1}{b}\right| \qquad \left\{\because h(y) = f(x)\left|\frac{dx}{dy}\right|\right\}$$

$$= \frac{1}{|b|\sigma\sqrt{2\pi}} e^{-[y-(a+b\mu)]^2/2b^2\sigma^2}$$

which represents the *p.d.f.* of a *r.v.* that is $N(a+b\mu,\, b^2\sigma^2)$

It follows that, if $X$ is $N(\mu, \sigma^2)$ and if $Z = \dfrac{X-\mu}{\sigma}$, $Z$ is $N(0, 1)$.

8. The sum of independent normal variables is a normal variable. Stated differently, if $X_1$ is $N(\mu_1, \sigma_1^2)$ and $X_2$ is $N(\mu_2, \sigma_2^2)$, then for independent $X_1$ and $X_2$, $X_1 + X_2$ is $N(\mu_1+\mu_2,\, \sigma_1^2+\sigma_2^2)$.

9. No matter what the values of $\mu$ and $\sigma$ are, areas under normal curve remain in certain fixed proportions within a specified number of standard deviations on either side of $\mu$. For example, the interval.

   i) $\mu \pm \sigma$ will always contain 68.26%,

   ii) $\mu \pm 2\sigma$ will always contain 95.44%,

   iii) $\mu \pm 3\sigma$ will always contain 99.73%.

Practically all of the area is between $\mu - 3\sigma$ and $\mu + 3\sigma$, the range of the distribution is therefore approximately 6 standard deviations (theoretically curve goes from $-\infty$ to $+\infty$), and we usually truncate the graph at these points. This is a very important property of the normal distribution as most of *tests of significance for large samples are based on it.*

10. The Quartile Deviation, $Q$, is found as

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-Q}^{\mu+Q} e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{2}.$$

or $$\frac{1}{\sqrt{2\pi}} \int_{0}^{Q/\sigma} e^{-z^2/2}\, dz = \frac{1}{4}, \text{ where } z = \frac{x-\mu}{\sigma}$$

we find, from area table, that $\dfrac{Q}{\sigma} = 0.6745$ or $Q = 0.6745\sigma$ which is also called the *Probable Error*, a term not used nowadays.

This also gives the values of the quartiles which are:

$$Q_1 = \mu - 0.6745\sigma \text{ and } Q_3 = \mu + 0.6745\sigma.$$

11. The normal curve approaches, but never really touches, the horizontal axis on either side of the mean towards plus and minus infinity, that is the curve is asymptotic to the horizontal axis as $x \to \pm\infty$.

**Example 9.2** For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. Find its mean and standard deviation.

For a normal distribution, the first moment about an arbitrary mean $a$ is given by

$$\mu_1' = \int_{-\infty}^{\infty}(x-a)\,f(x)\,dx, \text{ where } f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

Now $$40 = \int_{-\infty}^{\infty}(x-10)\,f(x)\,dx = \int_{-\infty}^{\infty}xf(x)\,dx - 10\int_{-\infty}^{\infty}f(x)\,dx$$

$$= \mu - 10, \text{ where } \mu \text{ denotes mean.}$$

$$\therefore \quad \mu = 40+10 = 50$$

The fourth moment about 50 (which is mean) implies that $\mu_4$=48. But for a normal distribution fourth moment about the mean is $3\sigma^4$. Therefore, we have

$$3\sigma^4 = 48, \text{ which gives } \sigma = 2$$

Hence mean = 50 and standard deviation = 2.

**9.5.3 Moment Generating and Cumulant Generating Functions of the Normal Distribution**

Let the *r.v.* $X$ be $N(\mu, \sigma^2)$. Then the *m.g.f.* of $X$ with respect to origin is given by

$$M_0(t) = E(e^{tX}) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx}\,e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Let $z=\dfrac{x-\mu}{\sigma}$ so that $dx = \sigma dz$. Then

$$M_0(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{t(\mu+z\sigma)}\,e^{-z^2/2}\,.\sigma dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tz\sigma - z^2/2}\,dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(z^2-2tz\sigma+t^2\sigma^2-t^2\sigma^2)}\,dz$$

$$= e^{\mu t+\frac{1}{2}t^2\sigma^2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(z-t\sigma)^2}\,dz$$

$$= e^{\mu t+\frac{1}{2}t^2\sigma^2}$$

The *m.g.f.* with respect to mean, $\mu$, is

$$M_\mu(t) = E[e^{t(X-\mu)}] = e^{-\mu t}\,E(e^{tX})$$

$$= e^{-\mu t} \cdot e^{\mu t + \frac{1}{2}t^2\sigma^2} = e^{\frac{1}{2}t^2\sigma^2}$$

$$= 1 + \frac{\left(\frac{1}{2}t^2\sigma^2\right)}{1!} + \frac{\left(\frac{1}{2}t^2\sigma^2\right)^2}{2!} + \ldots + \frac{\left(\frac{1}{2}t^2\sigma^2\right)^n}{n!}$$

ence $\mu_{2n+1} = 0$; and

$\mu_{2n}$ = coefficient of $\dfrac{t^{2n}}{(2n)!}$

$$= \left(\frac{1}{2}\sigma^2\right)^n \frac{(2n)!}{n!} = \frac{\sigma^{2n}}{2^n} \cdot \frac{(2n)!}{n!}$$

$= 1.3.5\ldots(2n-1)\,\sigma^{2n}$, for all $n \geq 0$.

The cumulant generating function is given by

$$\kappa(t) = \log_e M_0(t) = \mu t + \frac{1}{2}t^2\sigma^2$$

Thus $\kappa_1 = \mu$, $\kappa_2 = \sigma^2$ and $\kappa_r = 0$ for $r \geq 3$.

ence all the cumulants after the second are equal to zero.

**Example 9.3** If the *p.d.f. of the r.v. X* is

$$f(x) = \frac{1}{\sqrt{32\pi}} e^{-(x+7)^2/32} \qquad -\infty < x < \infty$$

d its mean, variance and moment generating function:

The function *f(x)* may be written as

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-[x-(-7)]^2/2(4)^2}$$



Comparing with the general form of the normal distribution, we find that $\mu = -7$ and $\sigma^2 = 16$.
e $X$ is $N(-7, 16)$.

Substituting the values of mean and variance in the relation *m.g.f.* $= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, we get

$$m.g.f. = e^{-7t + \frac{1}{2}(16)t^2} = e^{-7t+8t^2}$$

**Example 9.4** Let $X = N(0, 1)$ mean that $X$ has a normal distribution with zero mean and unit ance. What will be the distribution of $2X-3; \frac{7}{8}X+5$ and $4X$?

Given that $E(X) = 0$ and $\text{Var}(X) = 1$.

Let $Y = 2X - 3$. Then $E(Y) = E(2X-3) = 2E(X) - 3 = -3$, and

$\text{Var}(Y) = \text{Var}(2X - 3) = 2^2\ \text{Var}(X) = 4(1) = 4.$

Hence the distribution of $2X - 3$ is $N(-3, 4)$.

Similarly, we find that $\frac{7}{8}X + 5$ is $N\left(5, \frac{49}{64}\right)$ and $4X$ is $N(0, 16)$.

**Example 9.5** If $X$ is $N(0, 1)$, then find the distribution of $X^2$.

The distribution of $X$ is

$$dF = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx, \qquad -\infty < x < \infty$$

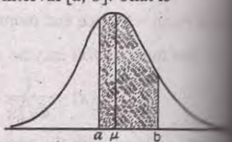
Let $y = x^2$ so that $x = \sqrt{y}$ and $dx = \frac{1}{2\sqrt{y}} dy$. Then the distribution of $Y$ is

$$\begin{aligned} dF &= \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{dy}{\sqrt{y}}, \qquad 0 \le y < \infty \\ &= \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot y^{-1/2}\, dy \\ &= \frac{1}{\Gamma(1/2)} e^{-m} . m^{1/2-1}\ dm, \text{ where } m = \frac{y}{2} \text{ and } 0 \le m < \infty . \end{aligned}$$

This is a Gamma variable with parameter $\frac{1}{2}$.

**9.5.4 Tabulated Area of Normal Distribution.** The area under the normal curve between any two ordinates at $X$=a and $X$=b equals the probability that the r.v. $X$ lies in the interval $[a, b]$. That is

$$P(a \le X \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx,$$




which is represented by the area of the shaded region. (see figure)

But integrals of this type cannot be solved by ordinary means. They are, however, evaluated by methods of numerical integration, and numerical approximations for some function have been tabulated for quick reference.

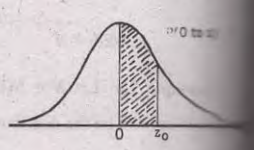



**Table 9.2** on page (365) gives the areas (probabilities) for the standard normal distribution from the mean, $z = 0$ to a specified positive value of $z$, say $z_0$. Since normal curves are symmetrical, therefore $P(0 \text{ to } z) = P(0 \text{ to } -z)$. That is why the areas for negative values of $z$ are not tabulated. It is important to note that this single table for the standard normal distribution suffices for the calculation of probabilities for any normal distribution. Hence, to *use* the table of areas for the standard normal distribution, the values of the r.v. $X$ in any problem are changed to the values of the standard normal variable Z and the desired probabilities are

btained from Table 9.2. Thus, to find $P(a<X<b)$, we would change $X$ into $Z$ as follows:

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right),$$

$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right),$$

here $\frac{a-\mu}{\sigma}$ and $\frac{b-\mu}{\sigma}$ are the $z$-values of the standard normal variable $Z$. In practice, a normal curve etch for the given problem, showing under the $X$-scale, a scale for the corresponding values of $z$ will p in solving the problem.

**able 9.2 Areas under the Unit Normal Curve**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .0000 | .0040 | .0080 | .0120 | .0159 | .0199 | .0239 | .0279 | .0319 | .0359 |
| 0.1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0753 |
| 0.2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |
| 0.3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |
| 0.4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |
| 0.5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2083 | .2123 | .2157 | .2190 | .2224 |
| 0.6 | .2257 | .2291 | .2324 | .2357 | .2380 | .2422 | .2454 | .2486 | .2518 | .2549 |
| 0.7 | .2580 | .2611 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |
| 0.8 | .2881 | .2910 | .2939 | .2967 | .2995 | .3023 | .3051 | .3078 | .3106 | .3133 |
| 0.9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |
| 1,1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3880 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3990 | .3997 | .4015 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4430 | .4441 |
| 1.6 | .4452 | .4463 | .4474 | .4485 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4690 | .4706 |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4758 | .4762 | .4767 |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 |
| 2.2 | .4861 | .4865 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4977 | .4978 | .4979 | .4980 | .4980 | .4981 |
| 2.9 | .4981 | .4982 | .4983 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 |
| 3.0 | .49865 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 |
| 3.1 | .49903 | .4991 | .4991 | .4991 | .4992 | .4992 | .4992 | .4992 | .4993 | .4993 |

**Example 9.6** Let the r.v. $Z$ have the standard normal distribution. Find

i) $P(0 \leq Z \leq 1.20)$,

ii) $P(-1.65 \leq Z \leq 0)$,

iii) $P(0.6 \leq Z \leq 1.67)$,

iv) $P(-1.30 \leq Z \leq 2.18)$,

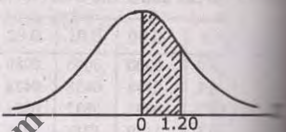
v) $P(-1.96 \leq Z \leq -0.84)$,

vi) $P(Z \geq 1.96)$, and

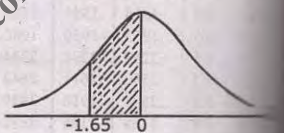
vii) $P(Z \leq -2.15)$.

First we draw the normal curve sketch, shading the desired area (probability) for each part.

i) To find $P(0 \leq Z \leq 1.20)$, in Table 9.2 page (365) we move downward the column marked $Z$ until 1.2 is reached, and then move across that row to the column headed 0.00 to find entry 0.3849. Therefore $P(0 \leq Z \leq 1.20) = 0.3849$.




ii) Since the normal curve is symmetrical about the mean, therefore area between z = 0 and positive value of $z$ is equal to the area between $z = 0$ and a negative value of $z$ of the same magnitude.



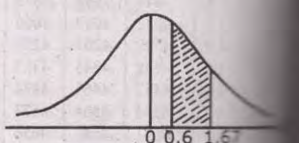

Hence, using **Table 9.2** page 365, we have $P(-1.65 \leq Z \leq 0) = P(0 \leq Z \leq 1.65) = 0.4505$

iii) $P(0.06 \leq Z \leq 1.67)$

$= P(0 \leq Z \leq 1.67) - P(0 \leq Z \leq 0.6)$

$= 0.4525 - 0.2257$

$= 0.2268$. (From area tables)

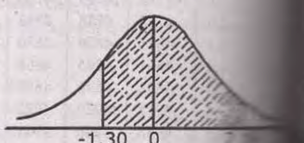



iv) $P(-1.30 \leq Z \leq 2.18)$

$= P(-1.30 \leq Z \leq 0) + P(0 \leq Z \leq 2.18)$

$= 0.4032 + 0.4854$

$= 0.8886$ (From area tables)

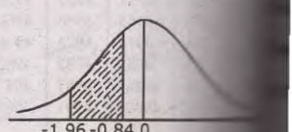



v) $P(-1.96 \leq Z \leq -0.84)$

$= P(-1.96 \leq Z \leq 0) - P(-0.84 \leq Z \leq 0)$

$= 0.4750 - 0.2995$

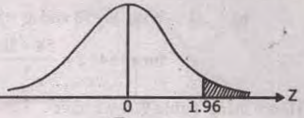
$= 0.1755$ (From area tables)



i) $P(Z \geq 1.96) = 0.5 - P(0 \leq Z \leq 1.96)$

$= 0.5 - 0.4750$

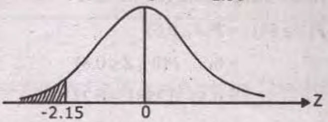
$= 0.0250$




i) $P(Z \leq -2.15) = 0.5 - P(-2.15 \leq Z \leq 0)$

$= 0.5 - 0.4842$

$= 0.0158$




**Example 9.7** A random variable $X$ is normally distributed with $\mu = 50$ and $\sigma^2 = 25$. Find the ility (a) that it will fall between (i) 0 and 40, (ii) 55 and 100; (b) that it will be (i) large than 54, ller than 57.

e draw the normal curve sketch showing $x$ and $z$ values, and the desired area for each part. With nd $\sigma = 5$, we have

$$z = \frac{x-50}{5}$$

a) (i) At $x = 0$, we computer $z = \dfrac{0-50}{5} = -10$, and

at $x=40$, we find $z = \dfrac{40-50}{5} = -2.0$.

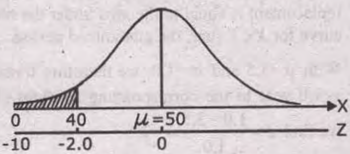
using **Table 9.2** page (365), we have

$P(0 \leq X \leq 40)$

$= P(-10 \leq Z \leq -2)$

$= P(-10 \leq Z \leq 0) - P(-2 \leq Z \leq 0)$

$= 0.5 - 0.4772 = 0.0228.$

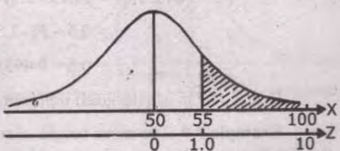





ii) We have for $x = 55$,

$$z = \frac{55-50}{5} = +1.0.$$

For $x = 100$, $z = \dfrac{100-50}{5} = 10.0$




es and the corresponding $z$ values are the figure.

using **Table 9.2**, we have

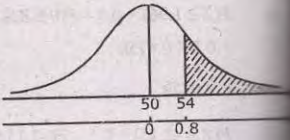
$\leq 100) = P(1.0 \leq Z \leq 10.0)$

$= P(0 \leq Z \leq 10.0) - P(0 \leq Z \leq 1.0)$

$= 0.5 - 0.3413 = 0.1587$

b) i) With $\mu=50$ and $\sigma=5$, we have

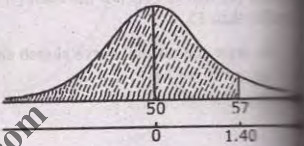
$$\text{for } x=54,\ z=\frac{54-50}{5}=0.80.$$




Hence using **Table 9.2**, we have

$$\begin{aligned} P(X\geq 54) &= P(Z\geq 0.8)\\ &= 0.5-P(0\leq Z\leq 0.8)\\ &= 0.5-0.2881=0.2119. \end{aligned}$$

ii) $$\text{At } x=57,\ z=\frac{57-50}{5}=1.40.$$

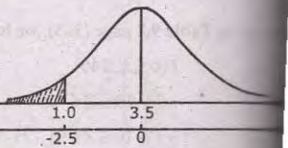



Therefore using **Table 9.2,** we have

$$\begin{aligned} P(X<57) &= P(Z<1.40)\\ &= 0.5+P(0\leq Z\leq 1.40)\\ &= 0.5+0.4192\\ &= 0.9192. \end{aligned}$$

**Example 9.8** The length of life for an automatic dishwasher is approximately normally d with a mean of 3.5 years and a standard deviation of 1.0 years. If this type of dishwasher is gua 12 months, what fraction of the sales will require replacement?

The fraction of sales requiring replacement is equal to the area under the normal curve for $X\leq 1$ year, the guaranteed period.




With $\mu=3.5$ and $\sigma=1.0$, we therefore transform $x=1.0$ year to the corresponding $z$ values. Thus, we find $z=\dfrac{1.0-3.5}{1.0}=-2.5.$

The $x$ value and the corresponding $z$ values are indicated in the figure. From Table 9.2, we

$$\begin{aligned} P(X\leq 1.0) &= P(Z\leq -2.5)\\ &= 0.5-P(-2.5\leq Z\leq 0)\\ &= 0.5-0.4938=0.0062. \end{aligned}$$

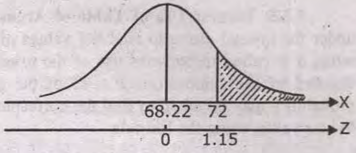
Hence 0.62% of sales need replacement before 12 months.

**Example 9.9** The mean height of soldiers is 68.22 inches with a variance of 10.8 (in.)$^2$. the distribution of heights to be normal, how many soldiers in a regiment of 1000 would you over 6 feet tall?

With $\mu=68.22$ and $\sigma^2=10.8$ (in.)$^2$, we first compute the $z$ value.

At $x=72$ (6 ft.), we have

$$z=\frac{72-68.22}{\sqrt{10.8}}=\frac{3.78}{3.29}=1.15$$




The $x$ values and the corresponding $z$ values are shown in normal curve sketch, and we find the *right tail* area which is shaded.

Therefore using **Table 9.2** page (365), we find

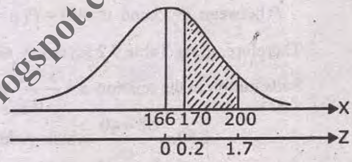
$$P(X \geq 72) = P(Z \geq 1.15) = 0.5 - P(0 \leq Z \leq 1.15)$$
$$= 0.5 - 0.3749 = 0.1251$$

If there are 1,000 soldiers in the regiment, then number expected to be over 6 feet (or 72 inches) is 1000 x 0.1251 = 125.

**Example 9.10** If the moment generating function of $X$ is $M(t) = e^{166t+200t^2}$, find (i) $P(170<X<200)$, (ii) $P(148 \leq X \leq 172)$.

Comparing $M(t) = e^{166t+200t^2}$ with the *m.g.f.* of $N(\mu, \sigma^2)$, we find $\mu = 166$ and $\sigma^2 = 400$.

To find the desired probabilities, we transform $x$ values to $z$ values, using $z = \dfrac{X-166}{20}$. Therefore




At $x = 170$, we get

$$z = \frac{170-166}{20} = 0.2 \text{, and}$$

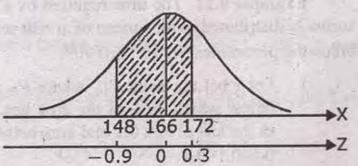
at $x = 200$ we find

$$z = \frac{200-166}{20} = 1.7 .$$

Hence using **Table 9.2** page (365), we find

$$P(170<X<200) = P(0.2<Z<1.7) = P(0<Z<1.7) - P(0<Z<0.2)$$
$$= 0.4554 - 0.0793 = 0.3761$$

The figure illustrates the problem.




At $x = 148$, we compute

$$z = \frac{148-166}{20} = -0.9 \text{, and}$$

for $x = 172$, we find

$$z = \frac{172-166}{20} = +0.3 .$$

Hence using **Table 9.2** page (365), we get

$$P(148 \leq X \leq 170) = P(-0.9<Z \leq 0.3)$$
$$= P(-0.9<Z \leq 0) + P(0 \leq Z \leq 0.3)$$
$$= 0.3159 + 0.1179 = 0.4338$$

**9.5.5 Inverse Use of Table of Areas under Normal Curve.** When we use the Table of ar under the normal curve to find the values of $z$ corresponding to a given probability in the body of t table, it is called the *inverse* use of the *area Table*. The value of $z$ given that $P(0 \le Z \le z_0) = k$ m denoted by the symbol $(z \mid P = k)$. If the given probability does not appear in the table, the clo probability may be taken to find the corresponding value of $z$. The $z$ value thus obtained is then cha to an $x$ value, using the formula

$$z = \frac{x-\mu}{\sigma}$$

which may be written as $x = \mu + z\sigma$.

The value of $z$ is positive when $x$ lies to the right of $\mu$ and it is negative when $x$ lies to the le $\mu$. A sketch of the standard normal curve showing the given probability and the location of $x$ o correct side of $\mu$ helps in solving the given problem.

**Example 9.11** In a normal distribution $\mu = 40$ and $P(25 \le X \le 55) = 0.8662$. Find $P(20 \le X \le$

Given $\mu = 40$ and $P(25 \le X \le 55) = 0.8662$, where $X$ is a normal random variable.

We find that 25 and 55 lie on either side of the mean $\mu = 40$ at equal distance, so that

$P(\text{between } X=25 \text{ and } \mu=40) = P(\mu=40 \text{ and } X=55) = 0.4331$

Therefore, using Table 9.2 inversely, we find $(z \mid P = 0.4331) = 1.50$

Substitution in the relation $z = \frac{x-\mu}{\sigma}$ gives

$$1.50 = \frac{55-40}{\sigma} \text{ which yields } \sigma = 10$$

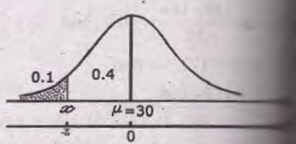
$$\text{Now } P(20 \le X \le 60) = P\left(\frac{20-40}{10} \le \frac{X-40}{10} \le \frac{60-40}{10}\right) = P(-2 \le Z \le 2)$$

$$= P(-2 \le Z \le 0) + P(0 \le Z \le 2)$$

$$= 0.4772 + 0.4772 = 0.9544$$

**Example 9.12** The time required by a nurse to inject a shot of penicillin has been obser normally distributed, with a mean of $\mu = 30$ seconds and a standard deviation of $\sigma = 10$ seconds. following percentiles: (i) $10^{th}$, (ii) $90^{th}$.

i) Let $x$ bet the point $P_{10}$, where $P_{10}$ is a point at or below which 10% of the area lies. Then the area to the left of $x$ is 0.1 and area between $\mu$ and $x$ is 0.5–0.1=0.4.

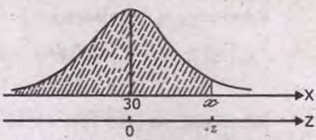



Looking at **Table 9.2** page (365) we find that a probability of 0.4 does not appear in the table, so we take the closest probability to 0.4, which is 0.3997.

Thus, using Table 9.2 inversely we find $(z/P = 0.3997) = 1.28$

Since $x$ lies to the left of $\mu$, therefore $z$ is negative at this point.

Hence $x = \mu + z\sigma = 30 + (-1.28)(10) = 17.2$ seconds.

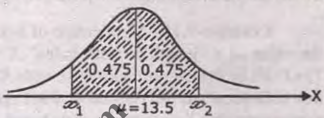



ii) Let $x$ be the point $P_{90}$, where $P_{90}$ is a point at or below which 90% of the area lies. Then the area to the left of $x$ is 0.9 and the area between $\mu$ and $x$ is $0.9 - 0.5 = 0.4$.

From **Table 9.2** page (365), we find $(z/P = 0.4) = 1.28$. Since $x$ lies to the right of $\mu$, therefore $z$ is ive at this point.

nce $x = \mu + z\sigma = 30 + (1.28)(10) = 42.8$ seconds.

**Example 9.13** In a normal distribution with $\mu = 13.5$ and $\sigma = 3.6$, find two points such that a single rvation has 95% chance for falling between them.




Let $x_1$ and $x_2$ be the two points between which the bility of an observation falling is 0.95. As the curve is etrical, so half of 0.95, i.e. 0.475 is the area lying on side of $\mu$.

Using **Table 9.2** inversely, we find

$$(z/P = 0.475) = 1.96$$

The point $x_1$ is to the left of $\mu$, so $z = -1.96$ and at $x_2$ which is to the right of $\mu$, $z = 1.96$. Therefore

$$x_1 = \mu + z\sigma = 13.5 + (-1.96)(3.6) = 6.4, \text{ and}$$

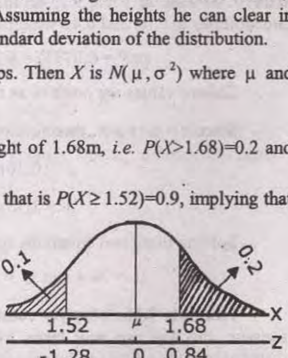
$$x_2 = \mu + z\sigma = 13.5 + (1.96)(3.6) = 20.6$$

**Example 9.14** An athlete finds that in a high jump he can clear a height of 1.68m once in five s and a height of 1.52m nine times out of ten attempts. Assuming the heights he can clear in jumps form a normal distribution, estimate the mean and standard deviation of the distribution.

Let $X$ denote the height the athlete can clear in various jumps. Then $X$ is $N(\mu, \sigma^2)$ where $\mu$ and nknown.

ow there is a probability of $\frac{1}{5} = 0.2$, if he can clear a height of 1.68m, *i.e.* $P(X>1.68)=0.2$ and

a probability of $\frac{9}{10} = 0.9$, if he can clear a height of 1.52m, that is $P(X \geq 1.52)=0.9$, implying that 52)=0.1.




e probability (area) between $\mu$ and $x = 1.68$ is $= 0.3$, and between $\mu$ and $x = 1.52$ is $0.5 - 0.1 = 0.4$. **Table 9.2** page (365) inversely we find that 3) = 0.84.

nce $x$ lies to the right of $\mu$, therefore $z$ is positive at this point.

d $(z/P = 0.4) = 1.28$, since $x$ lies to the left of $\mu$, so $z$ is negative at this point.

Substitution in the relation $x = \mu + z\sigma$, gives

$$\mu + 0.84\sigma = 1.68$$

$$\mu - 1.28\sigma = 1.52$$

Subtracting, we get $2.12\sigma = 0.16$ or $\sigma = 0.075$

Putting $\sigma = 0.075$ in $\mu + 0.84\sigma = 1.68$, we obtain

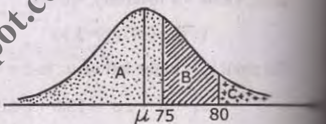
$$\mu + 0.84\,(0.075) = 1.68$$

or $\quad \mu = 1.68 - 0.063 = 1.617$

Hence the estimated value of $\mu$ is 1.617 and of $\sigma$ is 0.075.

**Example 9.15** A collection of human skulls is divided into three classes $A$, $B$ and $C$ acc[illegible] the value of a "length-breadth index" $X$. Skulls with $X<75$ are classified as $A$(long-headed), th[illegible] $75<X<80$ as $B$(medium) and those with $X>80$ as $C$ (short-headed). The percentages in the three c[illegible] this collection are 58, 38 and 4. Find approximately the mean and the standard deviation of [illegible] assumption that $X$ is normally distributed.

Let $\mu$ and $\sigma$ be the mean and the standard deviation of the normal distribution respectively. Then the area of skull $A$, whose length-breadth index is under 75 is 0.58, the area of skull $B$ whose index lies between 75 and 80 is 0.38, and the area under skull $C$, whose index is over 80 is 0.04. The accompanying sketch of the normal curve shows the given information.






The area between $\mu$ and $X=75$ is $0.58-0.50=0.08$, while the area between $\mu$ and $X$ [illegible] $0.08+0.38=0.46$. In Table 9.2, we find that these probabilities do not appear there, so we take [illegible] probabilities to 0.08 and 0.46 which are 0.0793 and 0.4599. Therefore

$$(z/P = 0.0793) = 0.20 \text{ and } (z/P = 0.4599) = 1.75$$

These $z$ values are positive as the $x$ values lie to the right of $\mu$.

Since $x = \mu + z\sigma$, therefore we get

$$\mu + 0.20\sigma = 75, \text{ and}$$

$$\mu + 1.75\sigma = 80.$$

Solving these two equations simultaneously, we obtain

$$\mu = 74.4 \text{ and } \sigma = 3.23.$$

**Example 9.16** A lawyer commutes daily from his suburban home to his midtown offi[illegible] average, the trip one way takes 24 minutes, with a standard deviation of 3.8 minutes. As[illegible] distribution of trip times to be normally distributed.

a) What is the probability that a trip will take at least $\frac{1}{2}$ hour?

b) If the office opens at 9:00 AM and he leaves his house at 8:45 AM daily, what percentage of the time is he late for work?

c) If he leaves the house at 8:35 AM and coffee is served at the office from 8:50 AM until 9:00AM, what is the probability that he misses coffee?

d) Find the length of time above which we find the slowest 15% of the trips.

e) Find the probability that 2 of the next 3 trips will take at least $\frac{1}{2}$ hour.

(P.U., M.Sc. 1989; I.U., M.Sc. 1986, 93)

et the *r.v.* $X$ denote the trip time in minutes. Then $X$ is $N[24, (3.8)^2]$.

a) We need to calculate the probability that a trip will take *at least* ½ hour, *i.e.* $P(X \geq 30)$.

$$\therefore \quad P(X \geq 30) = P\left(\frac{X-24}{3.8} \geq \frac{30-24}{3.8}\right) = P(Z \geq 1.58) = 0.0571$$

b) He leaves home at 8:45 AM and the office opens at 9:00 AM implies that he has 15 minutes to reach the office. He will be late for work if he takes more than 15 minutes. Thus we need $P(X>15)$.

$$\therefore \quad P(X \geq 15) = P\left(\frac{X-24}{3.8} > \frac{15-24}{3.8}\right) = P(Z > -2.37)$$

$$= 0.5 + P(0 \leq Z \leq 2.37) = 0.5 + 0.4911 = 0.9911$$

he percentage of the time he will be late for work is 99.11%.

c) He leaves home at 8:35AM and coffee is served from 8:50 AM until 9:00 AM. He will miss coffee if he reaches office after 9:00 AM, i.e. if he takes 25 minutes or more time. Thus we need $P(X \geq 25)$.

$$\therefore \quad P(X \geq 25) = P\left(\frac{X-24}{3.8} > \frac{25-24}{3.8}\right) = P(Z \geq 0.26)$$

$$= 0.5 - P(0 \leq Z \leq 0.26) = 0.5 - 0.1026 = 0.3974$$

d) To calculate the length of time ($x$) above which we find the slowest 15% of the trips, we need to calculate the value of $x$ such that $P(X \geq x) = 0.15$.

Thus $\quad P(X \geq x) = 0.15 \text{ or } P\left(\frac{X-24}{3.8} \geq \frac{x-24}{3.8}\right) = 0.15$

or $\quad P\left(Z \geq \frac{x-24}{3.8}\right) = 0.15$

find from the Tables of Standard Normal distribution that the value of $z$ corresponding to a tail is 1.04.

$$\therefore \quad x = \mu + z\sigma = 24 + (1.04)(3.8) = 27.952 \text{ minutes}$$

e) Using the binomial distribution with $n$=3 and $p$=0.0571 (where $p$ is the probability ... will take at least ½ hour), we find that the probability that 2 out of the next 3 trips ... least ½ hour is $\binom{3}{2}(0.0571)^2(0.9429) = 0.0092$.

**9.5.6 Normal Approximation to the Binomial Distribution.** The binomial distribution ... can be closely approximated by the normal distribution when $n$ is sufficiently large and neither ... close to zero. As a rule of thumb, the normal distribution provides a reasonable approx... binomial distribution if both $np$ and $nq$ are equal to or greater than 5.

The probability for a binomial random variable $X$ to take the value $x$ is

$$f(x) = \binom{n}{x} p^x q^{n-x}, \text{ for } 0 \le x \le n \text{ and } q + p = 1$$

The variable $X$ has a mean of $\mu = E(X) = np$ and $\mathrm{Var}(X) = npq$.

To derive the limiting form, we define a new random variable $Z$ by the relation

$$Z = \frac{X - np}{\sqrt{npq}}$$

Now, obviously $E(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

Thus we have to show that $Z$ is normally distributed in the limit as $n \to \infty$ and $p$ is fi... actually the de Moivre-Laplace theorem). We shall use the moment generating function (mgf ... the theorem.



The *mgf* of $Z$ is

$$\begin{aligned}
M(t) &= E(e^{tZ}) = E\left[e^{t(X-np)/\sqrt{npq}}\right] \\
&= e^{-npt/\sqrt{npq}} E\left[e^{xt/\sqrt{npq}}\right] \\
&= e^{-npt/\sqrt{npq}} \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} e^{xt/\sqrt{npq}} \\
&= e^{-npt/\sqrt{npq}} \sum_{x=0}^{n} \binom{n}{x} \left(pe^{t/\sqrt{npq}}\right)^x q^{n-x} \\
&= e^{-npt/\sqrt{npq}} \left(q + pe^{t/\sqrt{npq}}\right)^n \\
&= \left(qe^{-pt/\sqrt{npq}} + pe^{qt/\sqrt{npq}}\right)^n \\
&= \left[q\sum_{r=0}^{\infty}\left(\frac{-pt}{\sqrt{npq}}\right)^r \Big/ r! + p\sum_{r=0}^{\infty}\left(\frac{qt}{\sqrt{npq}}\right)^r \Big/ r!\right]^n
\end{aligned}$$

$$= \left[ q\left\{1-\frac{pt}{\sqrt{npq}}+\frac{1}{2!}\left(\frac{pt}{\sqrt{npq}}\right)^2-\frac{1}{3!}\left(\frac{pt}{\sqrt{npq}}\right)^3+\frac{1}{4!}\left(\frac{pt}{\sqrt{npq}}\right)^4-\ldots\right\}\right.$$

$$\left. +\, p\left\{1+\frac{qt}{\sqrt{npq}}+\frac{1}{2!}\left(\frac{qt}{\sqrt{npq}}\right)^2+\frac{1}{3!}\left(\frac{qt}{\sqrt{npq}}\right)^3+\frac{1}{4!}\left(\frac{qt}{\sqrt{npq}}\right)^4+\ldots\right\}\right]$$

$$=\left[1+\frac{t^2}{2!}.\frac{1}{n}+\frac{t^3}{3!}.\frac{q-p}{n\sqrt{npq}}+\frac{t^4}{4!}.\frac{q^2-qp+p^2}{n^2qp}+\ldots\right]^n$$

Taking logs to the base $e$, we have

$$\log_e M(t) = n\log_e\left[1+\frac{t^2}{2!}.\frac{1}{n}+\frac{t^3}{3!}.\frac{q-p}{n\sqrt{npq}}+\frac{t^4}{4!}.\frac{1-3pq}{n^2qp}+\ldots\right]$$

$$=\frac{t^2}{2!}+\frac{t^3}{3!}.\frac{q-p}{\sqrt{npq}}+\frac{t^4}{4!}.\frac{1-6pq}{npq}+\ldots$$

Thus $$\lim_{n\to\infty}\log_e M(t)=\frac{t^2}{2!} \text{ or } \lim_{n\to\infty}M(t)=e^{t^2/2}$$

We know that $e^{t^2/2}$ is the *m.g.f.* of a standard normal variable.

Hence we find that in the limit $Z$ has a standard normal distribution.

It is important to note that a binomial variable is *discrete* whereas the normal curve probability is a probability for an *interval*. Therefore in using normal curve areas to approximate binomial probabilities, a discrete value of the binomial variable is to be replaced by an interval before the $z$ values are computed. Accordingly, a discrete value $x$ becomes the interval from x–0.5 to x+0.5; and this sort of adjustment is called the *continuity correction*. Thus, the discrete value 5, adjusted means 4.5 to 5.5.

**Example 9.17** A fair coin is tossed 20 times. Find the probability that the number of heads occurring is between 10 and 14 inclusive by using (a) the binomial distribution, (b) the normal approximation to the binomial distribution.

(a) Let $X$ denote the number of heads occurring. Then the *p.d.* of $X$ is

$$P(X=x)=f(x)=\binom{20}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{20-x}$$

and the desired probability is $P(10 \le X \le 14)$.

Now $$P(10\le X\le 14)=\sum_{x=10}^{14}\binom{20}{x}\left(\frac{1}{2}\right)^{20}$$

$$= \binom{20}{10}\left(\frac{1}{2}\right)^{20} + \binom{20}{11}\left(\frac{1}{2}\right)^{20} + \binom{20}{12}\left(\frac{1}{2}\right)^{20} + \binom{20}{13}\left(\frac{1}{2}\right)^{20} + \binom{20}{14}\left(\frac{1}{2}\right)^{20}$$

$$= 0.1762 + 0.1602 + 0.1201 + 0.0739 + 0.0370$$

$$= 0.5674$$

**Alternatively**

$$P(10 \le X \le 14) = \sum_{x=10}^{14} b(x; 20, 0.5)$$

$$= \sum_{x=0}^{14} b(x;\ 20,\ 0.5) - \sum_{x=0}^{9} b(x;\ 20,\ 0.5)$$

$$= 0.9793 - 0.4119 \qquad \text{(from binomial tables)}$$

$$= 0.5674$$

b) Since $np = 20(0.5) = 10 > 5$ and $nq = 20(0.5) = 10 \ge 5$, so we will use the normal distribution to approximate the binomial distribution.

Now $\mu = np = 20\ (0.5) = 10$, and

$$\sigma = \sqrt{npq} = \sqrt{20(0.5)\ (0.5)} = 2.24.$$

For the normal approximation, the interval of discrete value $10 \le X \le 14$ is replaced by the interval $9.5 \le X \le 14.5$. We compute the $z$-values as below:

At $x = 9.5$, we find $z = \dfrac{9.5 - 10}{2.24} = -0.22$, and

at $x = 14.5$, we get $z = \dfrac{14.5 - 10}{2.24} = +2.01$

Hence, using Table 9.2, we find

$$P(10 \le X \le 14) = P(-0.22 \le Z \le 2.01)$$

$$= P(-0.22 \le Z \le 0) + P(0 \le Z \le 2.01)$$

$$= 0.0871 + 0.4778 = 0.5649$$

Hence the probability of obtaining heads between 10 and 14 is 0.5649.



**Example 9.18** A pair of dice is rolled 180 times. Use the normal approximation method to find the probability that a total of 7 occurs. (i) at least 25 times, (ii) between 33 and 41 times inclusive and (iii) exactly 30 times.

Let $X$ denote the number of times a total of 7 occurs when a pair of dice is rolled. Then $X$ is a binomial variable with $n$=180 and $p = \dfrac{1}{6}$ (probability of getting a total of 7 with 2 dice).

Since $n$ is large and $p$ is not too small, so we use the normal approximation method with $np = 180\times\frac{1}{6} = 30$, and

$$\sigma = \sqrt{npq} = \sqrt{180\times\frac{1}{6}\times\frac{5}{6}} = 5;$$

and the desired probabilities.

i) The interval *at least* 25 includes 25; therefore it starts at 24.5 and extends to $\infty$, i.e. *at least* 25 becomes the interval 24.5 to $\infty$.

The corresponding $z$ value is $z = \frac{24.5-30}{5} = -1.1$.

Hence, using Table 9.2, we find

$$P(\text{at least } 25) = \sum_{x=25}^{180} b\left(x; 180, \frac{1}{6}\right) = p(-1.1 \le Z \le \infty)$$
$$= P(-1.1 \le Z \le 0) + P(0 \le Z \le \infty)$$
$$= 0.3643 + 0.5 = 0.8643$$

Thus the probability of obtaining at least 25 times seven is 0.8643.

ii) The interval of discrete values $33 \le X \le 41$ is replaced by the interval $32.5 \le X \le 41.5$, and the corresponding $z$ values are:

$$\text{at } x = 32.5,\ z = \frac{32.5-30}{5} = 0.5, \text{ and}$$
$$\text{at } x = 41.5,\ z = \frac{41.5-30}{5} = 2.3.$$

Hence using Table 9.2, we find

$$P(33 \le X \le 41) = P(0.5 \le Z \le 2.3)$$
$$= P(0 \le Z \le 2.3) - P(0 \le Z \le 0.5)$$
$$= 0.4893 - 0.1915 = 0.2978.$$

iii) The discrete value 30, adjusted for continuity, becomes the interval 29.5 to 30.5, and the corresponding $z$ values are:

$$\text{at } x = 29.5,\ z = \frac{29.5-30}{5} = -0.1, \text{ and}$$
$$\text{at } x = 30.5,\ z = \frac{30.5-30}{5} = +0.1.$$

Hence using Table 9.2, we obtain

$$P(X=30) = P(-0.1 \le Z \le 0.1)$$
$$= 2(0.0398) = 0.0796.$$

**9.5.7 Normal Approximation to the Poisson Distribution.** The Poisson distribution $p(x;\ \mu)$ ca also be approximated by the normal distribution when $\mu \to \infty$. The probability for a Poison r.v. $X$ to ta the value $x$ is

$$P(X = x) = \frac{e^{-\mu}\mu^x}{x!}, \qquad \text{for } x = 0, 1, 2, \ldots, \infty$$

and $\quad E(X) = \text{Var}(X) = \mu$.

Let $\quad Z = \dfrac{X - \mu}{\sqrt{\mu}}$ so that $E(Z) = 0$ and $\text{Var}(Z) = 1$.

In order to show that as $\mu \to \infty$, the limiting distribution of $Z$ is standard normal distributio we use the moment generating function.

Now, the $m.g.f.$ of $Z$ is

$$\begin{aligned}
M(t) &= E(e^{tZ}) = E\left[e^{t(X-\mu)/\sqrt{\mu}}\right] \\
&= e^{-t\sqrt{\mu}}\ E\left[e^{tx/\sqrt{\mu}}\right] \\
&= e^{-t\sqrt{\mu}}\ \sum_{x=0}^{\infty} e^{tx/\sqrt{\mu}} . \frac{e^{-\mu}\ \mu^x}{x!} \\
&= e^{-t\sqrt{\mu}}\ e^{-\mu} \sum_{x=0}^{\infty} \frac{\left(\mu e^{-t\sqrt{\mu}}\right)^x}{x!} \\
&= e^{-t\sqrt{\mu}}\ .\ e^{-\mu}\left[1 + \mu e^{t/\sqrt{\mu}} + \frac{\left(\mu e^{-t/\sqrt{\mu}}\right)^2}{2!} + \ldots\right] \\
&= e^{-t\sqrt{\mu}}\ e^{-\mu}\ .\ e^{\mu e^t/\sqrt{\mu}} \\
&= e^{-t\sqrt{\mu}}\ .\ e^{\mu(e^t/\sqrt{\mu}-1)}
\end{aligned}$$

Taking logs to the base $e$, we get

$$\begin{aligned}
\log_e M(t) &= -t\sqrt{\mu} + \mu(e^{-t/\sqrt{\mu}-1}) \\
&= -t\sqrt{\mu} + \mu\left[1 + \frac{t}{\sqrt{\mu}} + \frac{t^2}{2!\ \mu} + \frac{t^3}{3!\ \mu\sqrt{\mu}} + \ldots - 1\right]
\end{aligned}$$

$$= -t\sqrt{\mu} + t\sqrt{\mu} + \frac{t^2}{2!} + \frac{t^3}{3!\sqrt{\mu}} + \frac{t^4}{4!\,\mu} + \ldots$$

$$= \frac{t^2}{2!} + \frac{t^3}{3!\sqrt{\mu}} + \frac{t^4}{4!\,\mu} + \ldots$$

Therefore $\lim\limits_{\mu\to\infty} \log_e M(t) = \dfrac{t^2}{2!}$ or $\lim\limits_{\mu\to\infty} M(t) = e^{t^2/2}$

which is the *m.g.f.* of the standard normal distribution.

Hence we see the Poisson distribution approaches the normal distribution as $\mu \to \infty$.

**Example 9.19** The number of calls received by an office switch board per hour follows a Poisson distribution with parameter 25. Find the probabilities that in one hour (a) there are between 23 and 26 calls (inclusive), (b) more than 30 calls, using the normal approximation to the Poisson distribution.

Let $X$ be the *r.v.* the number of calls received in one hour. Then $X$ is $p(x;\, 25)$. We require (a) $P(23 \le X \le 26)$ and (b) $P(X>30)$.

Using the normal approximation, $X$ is $N(25, 25)$.

(a) $P(23 \le X \le 26)$ becomes on continuous scale $P(22.5 \le X \le 26.5)$. The $z$-values are:

At $x = 22.5$, $z = \dfrac{22.5-25}{5} = -0.5$, and

at $x = 26.5$, $z = \dfrac{26.5-25}{5} = 0.3$.

$$\therefore \quad P(22.5 \le X \le 26.5) = P(-0.5 \le Z \le 0.3)$$

$$= P(0<Z<0.5) + P(0<Z<0.3)$$

$$= 0.1915 + 0.1179 = 0.3094$$

(b) $P(X>30)$ becomes on continuous scale $P(X>30.5)$

$$\therefore \quad P(X>30.5) = P\left(\frac{X-25}{5} > \frac{30.5-25}{5}\right) = P(Z>1.1)$$

$$= 0.5 - P(0<Z<1.1)$$

$$= 0.5 - 0.3643 = 0.1357$$

**9.5.8 Fitting a Normal Distribution.** There are two possible situations to deal with.

(a) *Given an observed frequency distribution* with $k$ classes, we need to calculate frequencies which we would expect for a normal distribution with the same mean and standard deviation as the given data.

b) Given only values of mean $\mu$ and variance $\sigma^2$, we need to determine the number of (and hence their width) for the presentation of the distribution.

**Case (a). To fit a normal distribution** to an observed frequency distribution when neither the nor the variance $\sigma^2$ is known, we proceed as below:

i) We estimate the two parameters $\mu$ and $\sigma^2$ by calculating $\overline{x}$ and $s^2$ from the frequency distribution.

ii) We calculate the standard normal $z$-values corresponding to the upper class-bound subtracting the estimated mean from each upper class-boundary and dividing by normal distribution is defined from $-\infty$ to $+\infty$, we therefore extend the lowest class to $-\infty$ and the last (highest) class forward to $+\infty$.

iii) We find the cumulative probability $P(Z<z) = \Phi(z)$ associated with each $z$-value, Tables of standard normal distribution.

iv) We then obtain the probability, $\hat{p}$ (or area) for each class by successive subtraction of

v) We finally get the expected (theoretical) frequencies by multiplying each of probabilities by the total frequency of the given distribution.

This procedure is also known as fitting a normal distribution to a given frequency distri *area method.*



**Case (b). To fit a normal distribution** when we are given only the values of the two parame $\sigma^2$, we proceed as follows:

i) We determine the practical range of the distribution as $\mu-3\sigma$ to $\mu+3\sigma$, since mo normal distributions are practically covered by a range of 6 standard deviations.

ii) We choose all the classes (between 6 and 15 classes) of a convenient width within this

iii) We proceed as in Case (a) above to calculate the probabilities of the theoretical p distribution. The expected frequencies can be obtained when the total number of obse (total frequency) is available.

The following examples illustrate the procedure for fitting a normal distribution:

**Example 9.20** Fit a normal distribution to the following frequency distribution of weigh

| Weight (kg) | 28–31, | 32–35, | 36–39, | 40–43, | 44–47, | 48–51, | 52–55, | 56–59, | 60–63, |
|---|---|---|---|---|---|---|---|---|---|
| $f_i$ | 1 | 14 | 56 | 172 | 245 | 263 | 154 | 67 | 23 |

We first calculate the mean and standard deviation of this distribution to estimate $\mu$ and two parameters of the normal distribution.

Computation of mean and standard deviation.

| Weight (kg) | $x_i$ | $f_i$ | $u_i = \left(\dfrac{x_i - 49.5}{4}\right)$ | $f_i u_i$ | $f_i u_i^2$ |
|---|---|---|---|---|---|
| 28–31 | 29.5 | 1 | –5 | –5 | 25 |
| 32–35 | 33.5 | 14 | –4 | –56 | 224 |
| 36–39 | 37.5 | 56 | –3 | –168 | 504 |
| 40–43 | 41.5 | 172 | –2 | –344 | 688 |
| 44–47 | 45.5 | 245 | –1 | –245 | 245 |
| 48–51 | 49.5 | 263 | 0 | 0 | 0 |
| 52–55 | 53.5 | 156 | 1 | 156 | 156 |
| 56–59 | 57.5 | 67 | 2 | 134 | 268 |
| 60–63 | 61.5 | 23 | 3 | 69 | 207 |
| 64–67 | 65.5 | 3 | 4 | 12 | 48 |
| $\Sigma$ | --- | 1000 | --- | –447 | 2365 |

Now, $\bar{x} = a + \dfrac{\sum f_i u_i}{n} \times h$

$$= 49.5 + \frac{(-477)}{1000} \times 4 = 49.5 - 1.79 = 47.71 \text{ kg, and}$$

$$s = h \times \sqrt{\frac{\sum fu^2}{n} - \left(\frac{\sum fu}{n}\right)^2}$$

$$= 4 \times \sqrt{\frac{2365}{1000} - \left(\frac{-477}{1000}\right)^2} = 4\sqrt{2.365 - 0.1998}$$

$$= 4\sqrt{2.1652} = 4 \times 1.47 = 5.88 \text{ kg.}$$

The necessary calculations for the expected frequencies of the fitted distribution are shown below:

| Weight (kg) | upper class boundary | z $\left(= \dfrac{u.c.b. - \bar{x}}{s}\right)$ | P(Z<z) $\Phi$ (z) | Probability ($\hat{p}$) | Expected frequency ($\hat{p} \times \sum f$) |
|---|---|---|---|---|---|
| $-\infty$–27 | 27.5 | –3.44 | 0.0003 | 0.0003 | |
| 28–31 | 31.5 | –2.76 | 0.0029 | 0.0026 | 3 |
| 32–35 | 35.5 | –2.08 | 0.0188 | 0.0159 | 16 |
| 36–39 | 39.5 | –1.40 | 0.0808 | 0.0620 | 62 |
| 40–43 | 43.5 | –0.71 | 0.2389 | 0.1581 | 158 |
| 44–47 | 47.5 | –0.03 | 0.4880 | 0.2491 | 249 |
| 48–51 | 51.5 | +0.64 | 0.7389 | 0.2509 | 251 |
| 52–55 | 55.5 | 1.32 | 0.9066 | 0.1677 | 168 |
| 56–59 | 59.5 | 2.01 | 0.9778 | 0.0712 | 71 |
| 60–63 | 63.5 | 2.68 | 0.9963 | 0.0185 | 18 |
| 64–67 | 67.5 | 3.37 | 0.9996 | 0.0033 | 4 |
| 68–$\infty$ | $\infty$ | $\infty$ | 1.0000 | 0.0004 | |
| $\Sigma$ | --- | --- | --- | --- | 1,000 |

**Example 9.21** Fit a normal distribution given mean $\mu = 60$ and standard deviation $\sigma = 2.5$.

Since the bulk of the normal distribution lies between $\mu - 3\sigma$ and $\mu + 3\sigma$, therefore the rang classes would be $60 \pm 3(2.5)$ i.e. 52.5 to 67.5.

As range = 15, we use 8 classes with a common width $h$=2. We then construct the class 52.5–54.5, 54.5–56.5, …, 66.5–68.5. Following are the necessary computations:

| Classes | upper class boundary | z $\left(= \frac{u.c.b. - \mu}{s}\right)$ | P(Z<z) $\Phi$ (z) | Probability $(\hat{p})$ |
|---|---|---|---|---|
| Upto 52.5 | 52.5 | –3.0 | 0.0013 | 0.0013 |
| 52.5–54.5 | 54.5 | –2.2 | 0.0139 | 0.0126 |
| 54.5–56.5 | 56.5 | –1.4 | 0.0808 | 0.0669 |
| 56.5–58.5 | 58.5 | –0.6 | 0.2743 | 0.1935 |
| 58.5–60.5 | 60.5 | 0.2 | 0.5793 | 0.3050 |
| 60.5–62.5 | 62.5 | 1.0 | 0.8413 | 0.2620 |
| 62.5–64.5 | 64.5 | 1.8 | 0.9641 | 0.1228 |
| 64.5–66.5 | 66.5 | 2.6 | 0.9953 | 0.0312 |
| Over 66.5 | ∞ | ∞ | 1.0000 | 0.0047 |

If the total frequency were available, we could have obtained the expected frequencie multiplication of the probabilities by the total frequency.

**9.5.9 Ordinates of Normal Distribution.** The *ordinate* (height) is the value of $f(x)$ correspo to a specified value of $X$. For convenience, the ordinates of the standard normal curve at various dis from the mean have been tabulated. **Table 9.3** on page (383) gives the ordinates obtained fro function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad \text{where} \quad z = \frac{x-\mu}{\sigma}.$$

for different positive values of $z$. Because of symmetry, ordinates at positive values of $z$ equ ordinates at negative values of $z$.

We calculate the heights of the ordinates for an **observed frequency distribution** having $k$ classes common class-width $h$ by:

i) calculating $\bar{x}$ and $s$, as estimates of $\mu$ and $\sigma$, from the given distribution;

ii) converting the class-marks, $x_i$, into standard normal $z$ – values by the relation $z = \frac{x_i - \bar{x}}{s}$

iii) finding the ordinates $\phi(z)$ corresponding to each $z$ – value from the *Table of Ordinates*

iv) multiplying $\phi(z)$ by the factor $\frac{nh}{s}$, where $n$ is the total frequency.

We need the heights of a number of ordinates when we wish to draw the graph of the fitted cur

**9.3 Ordinates of the Normal Curve $\phi(z)$**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .3989 | .3989 | .3989 | .3988 | .3986 | .3984 | .3982 | .3980 | .3977 | .3973 |
| 0.1 | .3970 | .3965 | .3961 | .3956 | .3951 | .3945 | .3939 | .3932 | .3025 | .3918 |
| 0.2 | .3910 | .3902 | .3894 | .3885 | .3876 | .3867 | .3857 | .3847 | .3836 | .3825 |
| 0.3 | .3814 | .3802 | .3790 | .3778 | .3765 | .3752 | .3789 | .3725 | .3712 | .3697 |
| 0.4 | .3683 | .3668 | .3653 | .3637 | .3621 | .3605 | .3589 | .3572 | .3555 | .3538 |
| 0.5 | .3521 | .3603 | .3485 | .3467 | .3448 | .3429 | .3410 | .3391 | .3372 | .3352 |
| 0.6 | .3332 | .3312 | .3292 | .3271 | .3251 | .3230 | .3209 | .3187 | .3166 | .3144 |
| 0.7 | .3123 | .3101 | .3079 | .3056 | .3034 | .3011 | .2969 | .2966 | .2943 | .2920 |
| 0.8 | .2897 | .2874 | .2850 | .2827 | .2803 | .2780 | .2756 | .2732 | .2709 | .2685 |
| 0.9 | .2661 | .2637 | .2613 | .2589 | .2585 | .2541 | .2516 | .2492 | .2468 | .2444 |
| 1.0 | .2420 | .2396 | .2371 | .2347 | .2323 | .2299 | .2275 | .2251 | .2227 | .2203 |
| 1.1 | .2179 | .2155 | .2131 | .2107 | .2083 | .2059 | .2036 | .2012 | .1989 | .1965 |
| 1.2 | .1942 | .1919 | .1895 | .1872 | .1849 | .1825 | .1804 | .1781 | .1758 | .1736 |
| 1.3 | .1714 | .1691 | .1660 | .1647 | .1626 | .1604 | .1582 | .1561 | .1539 | .1518 |
| 1.4 | .1497 | .1476 | .1456 | .1435 | .1415 | .1394 | .1374 | .1354 | .1334 | .1315 |
| 1.5 | .1295 | .1276 | .1257 | .1238 | .1219 | .1200 | .1182 | .1163 | .1145 | .1127 |
| 1.6 | .1109 | .1092 | .1074 | .1057 | .1040 | .1023 | .1006 | .0989 | .0973 | .0957 |
| 1.7 | .0940 | .0925 | .0909 | .0893 | .0878 | .0863 | .0848 | .0833 | .0818 | .0804 |
| 1.8 | .0790 | .0775 | .0761 | .0748 | .0734 | .0721 | .0707 | .0694 | .0681 | .0669 |
| 1.9 | .0656 | .0644 | .0632 | .0620 | .0608 | .0596 | .0584 | .0573 | .0562 | .0551 |
| 2.0 | .0540 | .0529 | .0519 | .0508 | .0498 | .0488 | .0478 | .0468 | .0459 | .0449 |
| 2.1 | .0440 | .0431 | .0422 | .0413 | .0404 | .0395 | .0387 | .0379 | .0371 | .0363 |
| 2.2 | .0355 | .0347 | .0339 | .0332 | .0325 | .0317 | .0310 | .0303 | .0297 | .0290 |
| 2.3 | .0283 | .0277 | .0270 | .0264 | .0258 | .0252 | .0246 | .0241 | .0235 | .0229 |
| 2.4 | .0224 | .0219 | .0213 | .0208 | .0203 | .0198 | .0194 | .0189 | .0184 | .0180 |
| 2.5 | .0175 | .0171 | .0167 | .0163 | .0158 | .0154 | .0161 | .0147 | .0143 | .0139 |
| 2.6 | .0136 | .0132 | .0129 | .0126 | .0122 | .0119 | .0116 | .0113 | .0110 | .0107 |
| 2.7 | .0104 | .0101 | .0099 | .0096 | .0093 | .0091 | .0088 | .0086 | .0084 | .0081 |
| 2.8 | .0079 | .0077 | .0075 | .0073 | .0071 | .0069 | .0067 | .0065 | .0063 | .0061 |
| 2.9 | .0060 | .0058 | .0056 | .0055 | .0053 | .0051 | .0050 | .0048 | .0047 | .0046 |
| 3.0 | .0044 | .0043 | .0042 | .0040 | .0039 | .0038 | .0037 | .0036 | .0035 | .0034 |
| 3.1 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 | .0025 | .0025 |
| 3.2 | .0024 | .0023 | .0022 | .0022 | .0021 | .0020 | .0020 | .0019 | .0018 | .0018 |
| 3.3 | .0017 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 | .0013 | .0013 |
| 3.4 | .0012 | .0012 | .0012 | .0011 | .0011 | .0010 | .0010 | .0010 | .0009 | .0009 |
| 3.5 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 | .0007 | .0007 | .0006 |
| 3.6 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 | .0005 | .0005 | .0005 | .0004 |
| 3.7 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 | .0003 | .0003 | .0003 |
| 3.8 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 | .0002 | .0002 | .0002 | .0002 |
| 3.9 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0001 | .0001 |

eproduced from Table II of Fisher and Yates: "*Statistical Tables for Biological, gricultural and Medical Research*, published by Oliver and Boyd, Ltd., Edinburgh, y permission of the author and publishers".

**Example 9.22** Find the ordinates of the standard normal curve at (i) $z = 0.64$, (ii) $z =$ (iii) $z = -0.08$.

i) To find the ordinate at $z = 0.64$, in **Table 9.3** on page 383, we move downward the column marked $Z$ to reach the entry 0.6, and then move across that row to the column 0.04 to find entry 0.3251, which is the desired ordinate.

ii) Similarly, ordinate at $z = 2.18$ is 0.0371.

iii) By symmetry, (ordinate at $z = -0.08$) = (ordinate at $z = 0.08$)

$$= 0.3977$$

**Example 9.23** Find the ordinates of the frequency distribution of weighs given in Example

We have calculated $\bar{x} = 47.71$ kg and $s = 5.88$ kg for the distribution of weights in Example

The procedure for calculating the heights of the ordinates for the given frequency distribution shown in the table below:

| $x_i$ | $z_i = \dfrac{x_i - \bar{x}}{s}$ | $\phi(z_i)$ | Ordinates $\dfrac{nh}{s}\phi(z)$ |
|---|---|---|---|
| 29.5 | −3.10 | 0.0033 | 2.24 |
| 33.5 | −2.42 | 0.0213 | 14.49 |
| 37.5 | −1.74 | 0.0878 | 59.73 |
| 41.5 | −1.06 | 0.2275 | 154.76 |
| 45.5 | −0.38 | 0.3712 | 252.52 |
| 49.5 | +0.30 | 0.3814 | 258.45 |
| 53.5 | 0.98 | 0.2468 | 167.89 |
| 57.5 | 1.66 | 0.1006 | 68.44 |
| 61.5 | 2.35 | 0.0252 | 17.14 |
| 65.5 | 3.03 | 0.0041 | 2.79 |



The entries in the last column of the table have been obtained by multiplying each value of

by $\dfrac{nh}{s}$, *i.e.* $\dfrac{1000 \times 4}{5.88} = 680.27$.

## EXERCISES

### OBJECTIVE

a) Answer 'True' or 'False'. If the statement is not true then replace the underlined words that make the statement true:

i) The mean and standard deviation of the exponential distribution are <u>equal</u>.

ii) <u>All</u> continuous random variables follow a Normal Probability Distribution.

Not all normal distribution can be transformed to a Standard Normal Distribution.

For any continuous random variable X; $P(X > 1.0) = P(X \geq 1.0)$.

The area under the normal curve left to its mean is $-0.5$.

The normal distribution is symmetrical about zero.

The mean, median and mode of normal probability distribution are not equal.

The mean of the standard Z score is one and its standard deviation equal to zero.

The total area under the curve of any normal distribution is two.

If the computed value of Z is zero, then the value of normal random variable is greater than the mean of this variable.

The total area under any normal curve is always 0.5.

Z scores for a standard normal r.v. is set of all whole numbers.

The second quartile of the standard normal variable is larger than its median.

Transforming a normal distribution to a standard normal distribution will not change the mean of the distribution.

The standard score unit is the same as the data unit.



## ULTIPLE CHOICE QUESTIONS

The area under the standard normal curve between $-3.0$ and $-2.0$ is

a) 0.0228
b) 0.4472
c) .02165
d) 0.3413

Which is the characteristic of the normal distribution?

a) It is bell shaped and symmetric curve.
b) For any normal r.v. X. $P(X \leq \mu) = P(X \geq \mu)$.
c) The total area under the curve is unity.
d) All of above.

All normal distributions are:

a) Symmetrical.
b) Having two parameters $\mu$ and $\sigma$.
c) Bell shaped.
d) All of above.

iv) For a standard normal probability distribution the mean and standard deviation are:

a) $\mu = 1$ and $\sigma = 1$

b) $\mu = 0$ and $\sigma = 1$

c) $\mu = 50$ and $\sigma = 10$

d) All of above.

v) The middle area under the normal curve with $\mu \pm 2\sigma$ is

a) 0.6827

b) 1.0000

c) 0.9545

d) 0.9973

vi) For a normal distribution with $\mu = 50$ and $\sigma = 10$, how much area will be scanned
X = 50?

a) 0.35

b) 0.95

c) 0

d) 0.5

vii) In a normal distribution, mean deviation is equal to

a) $1\sigma$

b) $0.8\sigma$

c) $0.6745\sigma$

d) $2.0\sigma$



viii) The normal distribution will be less spread out when

a) The mean is small

b) The median is small

c) The mode is small

d) The standard deviation is small

xi) The lifetime of general tires is normally distributed with an average of 40,000
a standard deviation of 5000 kilometers. The probability that a randomly se
more than 50,000 kilometers is

a) 0.6789

b) 0.9772

c) 0.0228

d) 0.1600

x) Which of the following statements is correct for standard normal distribution?

a) $P(Z > -2.0) = P(Z > 2.0)$

b) $P(Z > -2.0) = P(Z < 2.0)$

c) $P(Z > -2.0) = P(Z < -2.0)$

d) All of above.

## BJECTIVE

a) Find the mean, variance and *m.g.f.* of the uniform distribution.

b) Find the moment generating function and the first four moments of the rectangular distribution on (–1/2, 1/2).

a) Describe an exponential distribution and derive its mean and standard deviation.

b) Suppose the average length of life of a colour television tube is 12 months. What is the probability that the length of life is equal to or greater than 18 months? Assume an exponential distribution.

a) If $X$ has an exponential distribution given by

$$f(x) = \frac{1}{2}e^{-x/2}, \qquad 0 \le x < \infty .$$

what are the mean, variance and *m.g.f.* of $X$? Also calculate $P(X>3)$ and $P(X>5 \mid X>2)$.

b) The distance, $x_i$, in kilometers travelled by customers to the "Cheap Supermarket" are distributed with the density function

$$f(x) = \frac{1}{5}e^{-x/5}, \qquad 0 \le x < \infty .$$

$= 0$, elsewhere.

Find the proportion of customers travelling less than 1 kilometer and the proportion travelling more than 15 kilometers to the Super market.

Show that the standard deviation of the density function $f(x) = \alpha e^{-\alpha x}$, where $x$ takes all values from 0 to $\infty$, is $\frac{1}{\alpha}$.

Write down the *m.g.f.* for the distribution given by

$$f(x) = \frac{1}{a}e^{-x/a}, \qquad 0 \le x < \infty .$$

Derive the first four moments about the mean.

$$f(x) = xe^{-x}, \qquad 0 \le x \le \infty .$$

$$= 0, \qquad \text{otherwise.}$$

the first four moment using moment generating function. (P.U., B.A. (Hons.) Part-III, 1963)

9.6 Show that for the exponential distribution.

$$dy = y_0 e^{-x/\sigma} dx, \qquad 0 \le x \le \infty,\ \sigma > 0,$$

$y_0$ being constant, the mean and the standard deviation are each equal to $\sigma$ and the interquartile range is $\sigma \log_e 3$. Also find $\mu'_r$ and show that $\beta_1 = 4$ and $\beta_2 = 9$.

(P.U., B.A.

9.7 a) Let $X$ have probability density

$$f(x) = \frac{1}{2} e^{-|x|}, \qquad -\infty < x < \infty.$$

Find the expectation and variance of $X$.

b) The density function of a random variable $X$ is given by

$$f(x) = \frac{a}{e^{-x} + e^{x}}; \qquad -\infty < x < \infty.$$

Find (i) the constant $a$, (ii) the probability that in two independent observations $X$ will take on values less than 1.

(P.U., B.A. (Hons.) Part-

9.8 a) The Gamma distribution is given by

$$f(x) = \frac{1}{\Gamma(m)} e^{-x} x^{m-1}, \qquad 0 < x < \infty,$$

where $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx$.

Show that the mean and the variance are each equal to $m$ and the third moment about mean is $2m$.

b) Find the mode of the Gamma distribution.

9.9 a) Find the $rth$ moment of the Gamma distribution (i) without using the $m.g.f.$, (ii) using $m.g.f.$

b) Show that the sum of two independent Gamma variables with parameters $m$ and $n$ is a Gamma variable with parameter $m + n$.

9.10 a) The Beta distribution of the fist kind is given by

$$f(x) = \frac{1}{\beta(l, m)} x^{l-1} (1-x)^{m-1}, \qquad 0 < x < 1,$$

where $\beta(l, m) = \int_0^1 e^{l-1} (1-x)^{m-1} dx$.

Show that the mean is $\dfrac{l}{l+m}$ and the variance is $\dfrac{lm}{(l+m)^2 (l+m+1)}$.

b) Find the harmonic mean of the Beta distribution.

(P.U., B.A. (Hons.) Part–III, 1967)

a) Defining a $\gamma(m)$ variate as one with a probability density function of the form

$$f(x) = \frac{x^{m-1}e^{-m}}{\Gamma(m)}, \text{ for } x \geq 0, \text{ m being a +ve constant.}$$

Obtain the distribution of $\frac{X}{Y}$ where $X$ and $Y$ are independent $\gamma(m)$ and $\gamma(n)$ variates respectively.

b) Show that the mean values of the positive square root of a $\gamma(l)$ variate is $\frac{\Gamma(l+1/2)}{\Gamma(l)}$. Hence prove that the mean deviation of normal variate from its mean is $\sigma\sqrt{2/\pi}$.

(P.U., B.A. (Hons.) Part–III, 1966)

a) Define the Normal Distribution and obtain its mean and variance.

b) Show that for the normal distribution, the mean, mode and median are the same.

(B.Z.U., B.A./B.Sc. 1976)

State the mathematical equation of the Normal distribution and prove that

i) the area under the normal curve is unity;

ii) the normal curve has points of inflection which are equidistant from the mean;

(P.U., B.A./B.Sc. 1986, 88, 93)

iii) all odd order moments about mean vanish. (P.U., B.A./B.Sc. 1973)

The continuous random variable $X$ has p.d.f. $f(x)$, where

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$$

a) Derive the mean and the variance of the distribution.

b) Show that the maximum value of $f(x)$ occurs at $x = \mu$.

c) Show that there are points of inflection at $x = \mu + \sigma$ and $x = \mu - \sigma$.

a) If $f(x) = ke^{-(x^2-6x+9)/24}$ is the equation of a normal curve, find the value of $k$, the mean and standard deviation. (P.U., B.A./B.Sc. 1984)

b) Show that for the normal distribution, the mean deviation from the mean is approximately $\frac{4}{5}$ of its standard deviation. (P.U., B.A./B.Sc. 1983, 87)

Prove that, for the normal distribution, the quartile deviation, the mean deviation and the standard deviation are approximately in the ratio 10:12:15. (B.Z.U., B.A./B.Sc. 1990)

9.17 a) Obtain the moment generating function of the standardized normal distribution.

b) Show that for the normal distribution, moments of odd order about mean are all ... moments of even order are given by

$$\mu_{2n}=\left(\frac{\sigma^2}{2}\right)^n\frac{(2n)!}{(n)!}$$

9.18 a) Show that the *m.g.f.* of the normal distribution with mean $\mu$ are variance $\sigma^2$, is

$$M_x(t)=e^{\mu t+t^2\sigma^2/2}$$

b) What percentage of the normal distribution with mean $\mu$ and variance $\sigma^2$ is ... between the points (i) $\mu$ and ($\mu+1.54\sigma$), (ii) ($\mu-1.73\sigma$) and ($\mu+0.56\sigma$)?

(P.U., B.A./B.Sc

9.19 a) If $X$ is normally distributed with zero mean and $\sigma=0.6$, find $P(X>0)$ and $P(0.2<X<1...$

b) For a normal distribution with mean 1 and standard deviation 3, find the probabilities

i) $P(3.43\le X\le 6.19)$, ii) $P(-1.43\le X\le 6.19)$.

9.20 A normal distribution has mean = 12 and $\sigma=2$, find the area under the curve

a) from $X$=10 to $X$=13.5, b) from $X$=11.4 to $X$=14.2, c) from $X$=6 to $X$=18.

9.21 Let $X$ be $N(100, 225)$. Find the following probabilities:

a) $P(X\le 92.5)$, b) $P(X\le 107.5)$, c) $P(X\ge 124)$,
d) $P(112\le X\le 128.5)$, e) $P(91\le X\le 127)$, f) $P(X\ge 76)$.

9.22 a) If $X$ is $N(\mu, \sigma^2)$ and if $Y=aX+b$, then show that $Y$ is $N(a\mu+b, a^2\sigma^2)$.

b) Let $Y=5X+10$ and $X$ be normally distributed with a mean 10 and variance 25. F... following:

i) $P(Y\le 54)$, (ii) $P(Y\ge 68)$, (iii) $P(52\le Y\le 67)$. (B.Z.U., B.A./B.Sc

9.23 a) Scores on a certain nation-wide college entrance examination follow a normal dis... with a mean of 500 and a standard deviation of 100. Find the probability that a stud... score (i) over 650, (ii) less than 250, and (iii) between 325 and 675.

b) Given that the height of college boys is normally distributed with mean $5'-2''$ and s... deviation $4''$, and that the minimum height required for joining the *N.C.C.* is $5'-$... the percentage of boys who would be rejected on account of their height.

9.24 a) If the heights ($X$) of college students are normally distributed with mean 69 and va... find the probability that (i) $X<65$ and (ii) $65\le X\le 70$.

b) If the *m.g.f.* of $X$ is $M(t)=e^{-6t+32t^2}$, find

$P(-4\le X<16)$ and $P(-10<X\le 0)$.

9.25 Suppose that weights of 2000 male students are normally distributed with mean 155 pounds and standard deviation 20 pounds. Find the number of students with weights (i) less than or equal to 100 pounds, (ii) between 120 and 130 pounds, (iii) between 150 and 175 pounds, (iv) greater than or equal to 200 pounds.

9.26 a) The mean life of stockings used by an army was 40 days, with a standard deviation of 8 days. Assume the life of the stockings follows a normal distribution. If 100,000 pairs are issued, how many would need replacement before 35 days? After 46 days?

(B.Z.U., B.A./B.Sc. 1990)

b) The time taken by a milkman to deliver milk to the GOR Estate is normally distributed with mean 12 minutes and standard deviation 2 minutes. He delivers milk everyday. Estimate the number of days during the year when he takes (i) longer than 17 minutes, (ii) less than 10 minutes, (iii) between 9 and 13 minutes.

$N=365$

9.27 The scores made by candidates in a certain test are normally distributed with mean 500 and standard deviation 100;

a) What percent of the candidates received scores (i) greater than 700, (ii) less than 400, (iii) between 400 and 600, (iv) which differ from mean by more than 150?

(v) which differ from mean by no more than 150

b) If a candidate gets a score of 680, what percent of the candidates have higher scores than he?

(P.U., B.A./B.Sc. 1980)

9.28 A man goes by car to his office, and the route through the city centre takes him, on the average, 27 minutes with a standard deviation of 5 minutes. With the opening of a new ring road, the man can bypass the congestion of the city centre, but the journey now takes, on the average, 29 minutes with a standard deviation of 2 minutes. Assuming that both journey times are normally distributed, determine which route is the better one if the man has (i) 28 minutes, and (ii) 32 minutes to reach his office for an appointment. (I.U., M.Sc. 1990)

**Hint.** The better route is one that gives the smaller probability of the man's being late for appointment.

9.29 a) A random variable $X$ is $N(100, 16)$. If $P(X > a) = 0.5636$, find the value of $a$.

b) A possible measure of kurtosis (i.e. flatness) is given by $k = \dfrac{Q.D.}{P_{90} - P_{10}}$, where $Q.D.$ is the Semi-Interquartile range, and $P$'s are the percentiles. Use the standard normal table to estimate the value of $k$ for a normal distribution.

9.30 In a normal distribution with $\mu = 47.6$ and $\sigma = 16.2$, find (i) the probability that a single observation will be larger than 50, (ii) two points such that a single observation has a 97% probability of falling between them, (iii) $P_{10}$, $P_{30}$ and $P_{99}$. (P.U., B.A./B.Sc. 1976)

9.31 a) A random sample of 1,000 iron rods are tested for their length and it is found that the mean and the standard deviation are 14.40 metres and 2.50 metres respectively. If the lengths of the rods are normally distributed, then find (i) how many rods will be between 12 and 16 metres? (ii) what are the chances that any rod selected at random will be 15 metres length or above?

b) The mean score of 1000 students appearing for an examination is 34.4 and the stand deviation is 16.6. How many candidates may be expected to obtain marks between 30 and 60 assuming the normality of the distribution? Under the same assumptions, determine also the limits of marks of the central 70% of the candidates.

9.32 A soft drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the amount of drink is normally distributed with a standard deviation equal to 15 millimeters,

a) what fraction of the cups will contain more than 240 milliliters?

b) what is the probability that a cup contains between 191 and 209 milliliters?

c) how many cups will likely overflow if 230 milliliters cups are used for the next 1000 drin

d) below what value do we get the smallest 25% of the drinks?

(P.U., B.A./B.Sc.

9.33 a) The heights of applicants to the police force are normally distributed with mean 170 cm and standard deviation 3.8cm. If 30% of applicants are rejected because they are too small, what is the minimum acceptable height for the police force?

b) The average life of a certain type of small motors is 10 years with standard deviation of years. The manufacturer replaces free all motors that fail while under guarantee. If he is willing to replace only 3% of the motors that fail, how long a guarantee should he offer? Assume that the lives of the motors follow normal distribution.

9.34 An architect is designing the interior doors in a men's gymnasium. He wants to make them enough so that 95 percent of the men using the doors will have at least a one-foot clearance. Assuming that the heights will be normally distributed, with a mean of 70 inches and a standard deviation of 3 inches, how high must the architect make the doors? (P.U., B.A./B.Sc.

9.35 a) In a normal distribution, the lower and upper quartiles are respectively 8 and 17. Find the mean and standard deviation of the normal distribution. (B.I.S.E., Lahore

b) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.

c) Let $X$ be normally distributed with mean $\mu$ and variance $\sigma^2$, so that $P(X<89) =$ [illegible] and $P(X>94) = 0.05$. Find $\mu$ and $\sigma^2$. (P.U., B.A./B.Sc.

9.36 Assuming that the number of marks scored by a candidate is normally distributed, find the mean and the standard deviation, if the number of first class students (60% or more marks) is [illegible], number of failed students (less than 30% marks) is 90 and the total number of candidates appearing for the examination is 450.

9.37 A boy is trying to climb a slippery pole and finds that he can climb to a height of at least [illegible] once in five attempts, and to a height of at least 1.70m nine times out of ten attempts. Assuming that the heights he can reach in various attempts form a normal distribution, calculate the mean and standard deviation of the distribution. Calculate also the heights that the boy can expect to [illegible] once in one thousand attempts. (I.U., M.Sc.

38 a) Prove that the Binomial distribution $(q+p)^n$ tends to become a normal distribution for large values of $n$.

b) Supposing that the death rate from Malaria is 20%, find the probability that the number of deaths in a particular village is between 70 and 80 out of 500.
(P.U., B.Sc. Hons. Part-II, 1972)

39 a) Explain the reason why the correction for continuity is usually made when we apply the normal approximation to the binomial distribution.

b) Find the probability that 200 tosses of a fair coin will result in (i) between 80 and 120 heads inclusive, (ii) less than 90 heads, and (iii) exactly 100 heads. (P.U., B.A./B.Sc. 1977)

40 a) A coin is tossed 200 times. Find the probability of getting (i) between 105 and 110 heads inclusive, (ii) less than 95 heads. (P.U., B.A./B.Sc. 1983)

b) Find the probability of obtaining between six and nine heads inclusive in 15 tosses of an ideal coin by applying (i) the binomial distribution, (ii) the normal approximation to the binomial, with correction for continuity. (P.U., B.A./B.Sc. 1978)

a) A telephone exchange receives, on average, 5 calls per minute. Find the probability that in a 20-minute period no more than 102 calls are received.

b) If $X$ is $b(x; 20, 0.4)$ find $P(6 \le X \le 10)$. Then find the approximations to this probability using (i) the Poisson distribution, (ii) the normal distribution.

a) Find the ordinates of the normal curve at (i) $z$=0.064, (ii) $z$=1.27, (iii) $z$=0.84 and (iv) $z$=–2.08.

b) Fit a normal distribution given mean $\mu$=27.0, standard deviation $\sigma$=2.2 and total frequency=209.

The following table gives the distribution of statures among the first year students of a university:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Stature (in.): | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 |
| Frequency: | 2 | 10 | 11 | 38 | 57 | 93 | 106 | 126 | 109 | 87 | 75 | 23 | 9 | 4 |

a) Test the normality of the distribution by comparing the proportion of the cases lying between $\bar{x} \pm s$, $\bar{x} \pm 2s$, $\bar{x} \pm 3s$ for the distribution and for the normal distribution.

b) Fit a normal distribution to the data, using area Tables.

Fit a normal distribution by area method to the following data. Also calculate ordinates.

| Classes | 40–, | 45–, | 50–, | 55–, | 60–, | 65–, | 70–, | 75–, | 80–, | 85– |
|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 7 | 8 | 25 | 38 | 51 | 60 | 45 | 32 | 10 | 4 |

Compare actual and expected frequencies on a graph.

9.45 Calculate the frequencies of the normal distribution which has the same total frequency, mean and standard deviation as the following distribution

| Group | 10–, | 12–, | 14–, | 16–, | 18–, | 20–, | 22–, | 24–, | 26– |
|---|---|---|---|---|---|---|---|---|---|
| $f$ | 4 | 30 | 106 | 206 | 272 | 219 | 120 | 37 | 6 |

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Contents

# CHAPTER 10

# SIMPLE REGRESSION AND CORRELATION

# SIMPLE REGRESSION AND CORRELATION

## 0.1 INTRODUCTION

The term *regression* was introduced by the English biometrician, Sir Francis Galton (1822–1911) describe a phenomenon which he observed in analyzing the heights of children and their parents. He und that, though tall parents have tall children and short parents have short children, the average height children tends to *step back* or to *regress* toward the average height of all men. This tendency toward e average height of all men was called a *regression* by Galton.

Today, the word *regression* is used in a quite different sense. It investigates the *dependence* of one riable, conventionally called the *dependent variable*, on one or more other variables, called *ependent variables,* and provides an equation to be used for estimating or predicting the average value the dependent variable from the known values of the independent variable. The dependent variable is umed to be a random variable whereas the independent variables are assumed to have *fixed* values, *i.e.* y are chosen non-randomly. The relation between the expected value of the dependent variable and the ependent variable is called a *regression relation.* When we study the dependence of a variable on a gle independent variable, it is called a *simple* or *two-variable regression*. When the dependence of a able on two or more than two independent variables is studied, it is called *multiple regression.* thermore, when the dependence is represented by a straight line equation, the regression is said to be *ar,* otherwise it is said to be *curvilinear*.

It is relevant to note that in regression study, a variable whose variation we try to explain is a *endent variable* while an *independent variable* is a variable that is used to explain the variation in the endent variable.

*Some more terminology:* The dependent variable is also called the *regressand*, the *predictand,* the *nse* or the *explained variable* whereas the independent or the non-random variable is also referred to e *regressor*, the *predictor*, the *regression variable* or the *explanatory variable.*

## 2 DETERMINISTIC AND PROBABILISTIC RELATIONS OR MODELS

The relationship among variables may or may not be governed by an exact physical law. For enience, let us consider a set of $n$ pairs of observation $(X_i, Y_i)$. If the relation between the variables is *linear*, then the mathematical equation describing the linear relation is generally written as

$$Y_i = a + bX_i,$$

e $a$ is the value of $Y$ when $X$ equals zero and is called the *Y-intercept,* and $b$ indicates the change in $Y$ one-unit change in $X$ and is called the *slope* of the line. Substituting a value for $X$ in the equation, we completely determine a *unique* value of $Y$. The linear relation in such a case is said to be a *inistic model.* An important example of the deterministic model is the relationship between Celsius ahrenheit scales in the form of $F = 32 + \frac{9}{5}C$. Another example is the area of a circle expressed by lation, area $= \pi r^2$. Such relations cannot be studied by regression.

In contrast to the above, the linear relationship in some situations is *not exact.* For example, we precisely determine a person's weight from his height as the relationship between them is not ed to follow an exact linear form. The weights for given values of age are reasonably assumed to e measurement of random errors. The deterministic relation in such cases is then modified to allow

for the inexact relationship between the variables and we get what is called a *non-determin* ... *probabilistic model* as

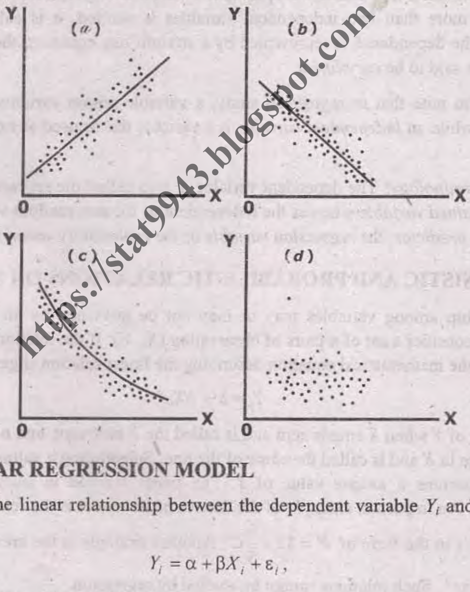
$$Y_i = a + bX_i + e_i, \qquad (i = 1, 2, \ldots, n)$$

where $e_i$'s are the unknown random errors.

## 10.3 SCATTER DIAGRAM

A first step in finding whether or not a relationship between two variables exists, is to plo... pair of independent-dependent observations $\{(X_i, Y_i), i = 1, 2, \ldots, n\}$ as a point on graph paper, usi... X-axis for the regression variable and the Y-axis for the dependent variable. Such a diagram is cal... *scatter diagram* or a *scatter plot*. If a relationship between the variables exists, then the points ... scatter diagram will show a tendency to cluster around a straight line or some curve. Such a line or ... around which the points cluster, is called the *regression line* or *regression curve* which can be ... estimate the expected value of the random variable $Y$ from the values of the nonrandom variable $X$.

The scatter diagrams shown below reveal that the relationship between two variable in ... positive and linear, in (b) is negative and linear, in (c) is curvilinear and in (d) there is no relationship.






## 10.4 SIMPLE LINEAR REGRESSION MODEL

We assume that the linear relationship between the dependent variable $Y_i$ and the value $X_i$ ... regressor $X$ is

$$Y_i = \alpha + \beta X_i + \varepsilon_i,$$

where the $X_i$'s are fixed or predetermined values,

the $Y_i$'s are observations randomly drawn from a population,

the $\varepsilon_i$'s are error components or random deviations,

$\alpha$ and $\beta$ are population parameters, $\alpha$ is the intercept and the slope $\beta$ is called *regression coefficient,* which may be positive or negative depending upon the direction of the relationship between $X$ and $Y$.

Furthermore, we assume that

i) $E(\varepsilon_i) = 0$, *i.e.* the expected value of error term is zero, it implies that the expected value of $Y$ is related to $X$ in the population by a straight line;

ii) $\text{Var}(\varepsilon_i) = E(\varepsilon_i^2) = \sigma^2$ for all *i, i.e.* the variance of error term is constant. It means that the distribution of error has the same variance for all values of $X$. (*Homoscedasticity assumption*);

iii) $E(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$, *i.e.* error terms are independent of each other (*assumption of no serial or auto correlation between* $\varepsilon$*'s*);

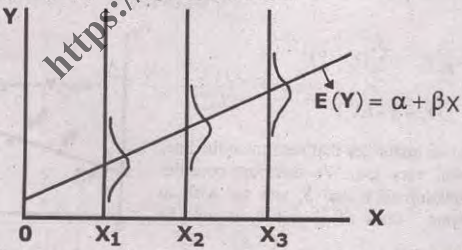
iv) $E(X, \varepsilon_i) = 0$, *i.e.* $X$ and $\varepsilon$ are also independent of each other;

v) $\varepsilon_i$*'s* are normally distributed with a mean of zero and a constant variance $\sigma^2$. This implies that $Y$ values are also normally distributed. The distributions of $Y$ and $\varepsilon$ *are identical except that they have different means. This assumption is required for estimation and testing of hypothesis on linear regression.*

According to this population regression model, each $Y_i$ is an observation from a normal distribution mean $= \alpha + \beta X$ and variance $= \sigma^2$. Thus the relation may be expressed alternatively as

$$E(Y) = \alpha + \beta X,$$

implies that the expected value of $Y$ is linearly related to $X$ and the observed value of $Y$ deviates the line $E(Y) = \alpha + \beta X$ by a random component $\varepsilon$, *i.e.* $\varepsilon_i = Y_i - (\alpha + \beta X_i)$. The following graph the assumed line, giving $E(Y)$ for the given values of $X$.




in practice, we have a sample from some population, therefore we desire to estimate the regression line from the sample data. Then the basic relation in terms of sample data may be

$$Y_i = a + bX_i + e_i,$$

b and $e_i$ are the estimates of $\alpha$, $\beta$ and $\varepsilon_i$. The estimated regression is generally written as

$+bX_i$.

Many possible regression lines could be fitted to the sample data, but we choose that parti which *best* fits that data. The *best* regression line is obtained by estimating the regression para the most commonly used *method of least squares* which we describe in the following subsection.

**10.4.1 An Aside—The Principle of Least Squares.** *The principle of least squares (LS) of determining the values of the unknown parameters that will minimize the sum of squares of residuals) where errors are defined as the differences between observed values and the corre values predicted or estimated by the fitted Model equation.*

The parameter values thus determined, will give the *least* sum of the squares of errors known as *least squares estimates*. The method of least squares that gets its name from the of a sum of squared deviations is attributed to Karl F. Gauss (1777–1855). Some people belie method was discovered at the same time by Adrien M. Legendre (1752–1833), Piere S. (1749–1827) and others. Markov's name is also mentioned in connection with its further develo recent years, efforts have been made to find better methods of fitting but the least squares metho dominant and is used as one of the important methods of estimating the population parameters.

**10.4.2 Least-Squares Estimates in Simple Linear Regression.** Let there be a set of obser $\{(X_i, Y_i), i = 1, 2, \ldots, n\}$,

$$Yi = \alpha + \beta Xi + \varepsilon_i$$

or in terms of sample data as

$$Yi = a + bXi + e_i,$$

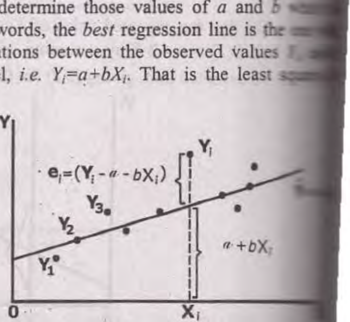
where $a$ and $b$ are the *least-squares estimates* of $\alpha$ and $\beta$, $e_i$ commonly called *residual*, is the of the observed $Y_i$ from its estimate provided by $\hat{Y}_i = a + bX_i$.



According to the principle of least-squares, we determine those values of $a$ and $b$ minimize the sum of squares of the residuals. In other words, the *best* regression line is the minimizes the sum of the squares of the vertical deviations between the observed values $Y_i$ corresponding values predicted by the regression model, *i.e.* $\hat{Y}_i = a + bX_i$. That is the least minimizes

$$S(a, b) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y})^2$$

$$= \sum (Y_i - a - bX_i)^2$$




As $a$ and $b$, the two quantities that determine the line, vary, $S(a, b)$ will vary too. We therefore consider $S(a, b)$ as a function of $a$ and $b$, and we wish to determine at what values of $a$ and $b$, it will be minimum.

Minimizing $S(a, b)$, we need to set its partial derivatives *w.r.t* $a$ and $b$ equal to zero. The

$$\frac{\partial S(a, b)}{\partial a} = 2\sum (Y_i - a - bX_i)(-1) = 0, \text{ and}$$

$$\frac{\partial S(a, b)}{\partial b} = 2\sum (Y_i - a - bX_i)(-X_i) = 0$$

Simplifying, we obtain the following two equations, called the *normal equations* (the word *normal* used here in the sense of regular or standard).

$$\sum Y_i = na + b\sum X_i \text{ and } \sum X_i Y_i = a\sum X_i + b\sum X_i^2$$

These two normal equations are solved simultaneously for values of $a$ and $b$ either by direct elimination or by using determinants.

i) **Direct Elimination:** Multiplying the first equation by $\sum X_i$ and the second equation by $n$, we get $\sum X \sum Y = na\sum X + b(\sum X)^2$ and $n\sum XY = na\sum X + nb\sum X^2$

Subtracting, we get

$$n\sum XY - \sum X \sum Y = b[n\sum X^2 - (\sum X)^2]$$

Therefore
$$b = \frac{n\sum XY - \sum X\sum Y}{n\sum X^2 - (\sum X)^2} = \frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sum(X-\overline{X})^2}$$

which is the least-squares estimate of the regression co-efficient $\beta$.

Similarly, we get

$$a = \frac{\sum X^2 \sum Y - \sum X \sum XY}{n\sum X^2 - (\sum X)^2}$$

the least squares estimate of $\alpha$.

Alternatively, we divide the first normal equation by n, and get the least-squares estimate of $\alpha$ as

$$a = \overline{Y} - b\overline{X}.$$

This also shows that the estimated regression line passes through ($\overline{X}$, $\overline{Y}$), the means of the data.

ii) **By means of determinants,** the solution is

$$b = \frac{\begin{vmatrix} \sum XY & \sum X \\ \sum \mathbf{Y} & n \end{vmatrix}}{\begin{vmatrix} \sum X^2 & \sum X \\ \sum X & n \end{vmatrix}} = \frac{n\sum XY - (\sum X)(\sum Y)}{n\sum X^2 - (\sum X)^2}, \text{ and}$$

$$a = \frac{\begin{vmatrix} \sum X^2 & \sum XY \\ \sum X & \sum Y \end{vmatrix}}{\begin{vmatrix} \sum X^2 & \sum X \\ \sum X & n \end{vmatrix}} = \frac{(\sum X^2)(\sum Y) - (\sum X)(\sum XY)}{n\sum X^2 - (\sum X)^2},$$

These estimates give us the regression equation

$$\hat{Y}_i = a + bX_i$$
$$= \overline{Y} + b(X - \overline{X}).$$

where $b = \dfrac{n\sum XY - (\sum X)(\sum Y)}{n\sum X^2 - (\sum X)^2}$,

Since $Y$ is a random variable, therefore the deviations in the $Y$ direction are taken into acc determining the *best-fitting* line.

*It is very important to note that, when both X and Y are observed at random, i.e. the sample are from a bivariate population, there are two regression equations, each obtained by choos variable as dependent whose average value is to be estimated and treating the other varia independent. In case of a single random variable, the single regression equation is used to estim values of either the dependent or the independent variable.* In case of two regression lines, it is cust to denote the regression coefficients of $Y$ on $X$ and of $X$ on $Y$ by $b_{yx}$ and $b_{xy}$ respectively.

**Example 10.1** Compute the least squares regression equation of $Y$ on $X$ for the follow What is the regression coefficient and what does it mean?

| $X$ | 5 | 6 | 8 | 10 | 12 | 13 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 16 | 19 | 23 | 28 | 36 | 41 | 44 | 45 | 50 |

The estimated regression line of $Y$ on $X$ is

$$\hat{Y} = a + bX,$$

and the two normal equations are

$$\sum Y = na + b\sum X$$
$$\sum XY = a\sum X + b\sum X^2$$

To compute the necessary summations, we arrange the computations in the table below:



| | $X$ | $Y$ | $XY$ | $X^2$ |
|---|---|---|---|---|
| | 5 | 16 | 80 | 25 |
| | 6 | 19 | 114 | 36 |
| | 8 | 23 | 184 | 64 |
| | 10 | 28 | 280 | 100 |
| | 12 | 36 | 432 | 144 |
| | 13 | 41 | 533 | 169 |
| | 15 | 44 | 660 | 225 |
| | 16 | 45 | 720 | 256 |
| | 17 | 50 | 850 | 289 |
| Total | 102 | 302 | 3853 | 1308 |

Now $\bar{X} = \dfrac{\sum X}{n} = \dfrac{102}{9} = 11.33$, $\bar{Y} = \dfrac{\sum Y}{n} = \dfrac{302}{9} = 33.56$,

$$b = \frac{n\sum XY - (\sum X)(\sum Y)}{n\sum X^2 - (\sum X)^2} = \frac{9(3853) - (102)(302)}{9(1308) - (102)^2}$$

$$= \frac{34677 - 30804}{11772 - 10404} = \frac{3873}{1368} = 2.831, \text{ and}$$

$$a = \overline{Y} - b\,\overline{X} = 33.56 - (2.831)\,(11.33) = 1.47.$$

ce the desired estimated regression line of $Y$ on $X$ is

$$\hat{Y} = 1.47 + 2.831X.$$

The estimated regression co-efficient, $b = 2.831$, which indicates that the values of $Y$ increase by units for a unit increase in $X$.

**Example 10.2** In an experiment to measure the stiffness of a spring, the length of the spring under nt loads was measured as follows:

| $X$=Loads (1b) | 3 | 5 | 6 | 9 | 10 | 12 | 15 | 20 | 22 | 28 |
|---|---|---|---|---|---|---|---|---|---|---|
| $Y$=length (in) | 10 | 12 | 15 | 18 | 20 | 22 | 27 | 30 | 32 | 34 |

Find the regression equations appropriate for predicting

i) the length, given the weight on the spring;

ii) the weight, given the length of the spring. (W.P.C.S., 1964)

The data come from a bivariate population, *i.e.* both $X$ and $Y$ are random, therefore there are two on lines. To find the regression equation for predicting length ($Y$), we take $Y$ as dependent e and treat $X$ as independent variable (*i.e.* non-random). For the second regression, the choice of ables is reversed.

The computations needed for the regression lines are given in the following table:

| | $X$ | $Y$ | $X^2$ | $Y^2$ | $XY$ |
|---|---|---|---|---|---|
| | 3 | 10 | 9 | 100 | 30 |
| | 5 | 12 | 25 | 144 | 60 |
| | 6 | 15 | 36 | 225 | 90 |
| | 9 | 18 | 81 | 324 | 162 |
| | 10 | 20 | 100 | 400 | 200 |
| | 12 | 22 | 144 | 484 | 264 |
| | 15 | 27 | 225 | 729 | 405 |
| | 20 | 30 | 400 | 900 | 600 |
| | 22 | 32 | 484 | 1024 | 704 |
| | 28 | 34 | 784 | 1156 | 932 |
| Total | 130 | 220 | 2288 | 5486 | 3467 |

https://stat9943.blogspot.com

i) The estimated regression equation appropriate for predicting the length, $Y$, given the weight $X$, is

$$\hat{Y} = a_0 + b_{yx}\,X,$$

where $b_{YX} = \dfrac{n\sum XY - (\sum X)\,(\sum Y)}{n\sum X^2 - (\sum X)^2} = \dfrac{(10)\,(3467) - (130)\,(220)}{(10)\,(2288) - (130)^2}$

$$= \frac{6070}{5980} = 1.02, \text{ and}$$

$$a_0 = \overline{Y} - b_{yx}\,\overline{X} = 22 - (1.02)\,(13) = 8.74$$

Hence the desired estimated regression equation is

$$\hat{Y} = 8.74 + 1.02\,X$$

ii) The estimated regression equation appropriate for predicting the weight, $X$, given the

$$\hat{X} = a_1 + b_{xy}\,Y,$$

$$\text{where } b_{XY} = \frac{n\sum XY - (\sum X)\,(\sum Y)}{n\sum Y^2 - (\sum Y)^2} = \frac{(10)\,(3467) - (130)\,(220)}{(10)\,(5486) - (220)^2}$$

$$= \frac{6070}{6460} = 0.94, \text{ and}$$

$$a_1 = \overline{X} - b_{xy}\overline{Y} = 13 - (0.94)\,(22) = -7.68$$

Hence $\hat{X} = 0.94Y - 7.68$ is the estimated regression equation appropriate for predicting the weig given the length ($Y$).

**10.4.3 Properties of the Least-Squares Regression Line.** The least-squares linear regre has the following properties:

$b_{yx} \neq b_{xy}$

i) The least squares regression line always goes through the point ( $\overline{X}$, $\overline{Y}$ ), the means of

ii) The sum of the deviations of the observed values of $Y_i$ from the least squares regressi always equal to zero, *i.e.* $\sum(Y_i - \hat{Y}) = 0$.

iii) The sum of the squares of the deviations of the observed values from the leas regression line is a minimum, *i.e.* $\sum(Y_i - \hat{Y}_i)^2 = \text{minimum}$.

iv) The least-squares regression line obtained from a random sample is the line of *best fi* *a* and *b* are the unbiased estimates of the parameters $\alpha$ and $\beta$.



**10.4.4 Standard Deviation of Regression or Standard Error of Estimate.** The observed of ($X$, $Y$) do not all fall on the regression line but they scatter away from it. The degree of (or dispersion) of the observed values about the regression line is measured by what is called the *deviation of regression* or the *standard error of estimate* of $Y$ on $X$. For the population data, the deviation that measures the variation of observations about the true regression line $E(Y) = \alpha +$ denoted by $\sigma_{Y.X}$ and is defined by

$$\sigma_{Y.X} = \sqrt{\frac{\Sigma[Y - (\alpha + \beta X)]^2}{N}}$$

where $N$ is the population size.

For sample data, we estimate $\sigma_{Y.X}$ by $s_{y.x}$ which is defined as

$$s_{y.x} = \sqrt{\frac{\sum(Y - \hat{Y})^2}{n-2}},$$

$\hat{Y} = a + bX$, the estimated regression line. This is actually an unbiased estimate of $\sigma_{Y.X}$, the on standard deviation about the regression line. The standard error of estimate, $s_{y.x}$ will be zero all the observed values fall on the regression line. It is interesting to note that the ranges $\hat{Y} \pm 2s_{y.x}$ and $\hat{Y} \pm 3s_{y.x}$ contain about 68%, 95.4% and 99.7% observations respectively.

To find $\sum(Y - \hat{Y})^2$, we have to calculate $\hat{Y}$ from the estimated regression line for the observed of $X$, which is not an easy task. We therefore use an alternative form obtained as below:

$$-\hat{Y})^2 = \sum(Y_i - a - bX_i)^2$$
$$= \sum Y_i(Y_i - a - bX_i) - a\sum(Y_i - a - bX_i) - b\sum X_i(Y_i - a - bX_i)$$
$$= \sum Y_i^2 - a\sum Y_i - b\sum X_iY_i - a[\sum Y_i - na - b\sum X_i] - b[\sum X_iY_i - a\sum X_i - b\sum X_i^2]$$

But $\sum Y_i - na - b\sum X_i = 0$ and $\sum X_iY_i - a\sum X_i - b\sum X_i^2 = 0$ as they are the normal equations.

$$\sum(Y_i - \hat{Y})^2 = \sum Y_i^2 - a\sum Y_i - b\sum X_iY_i$$

Hence $s_{y.x} = \sqrt{\dfrac{\sum Y_i^2 - a\sum Y_i - b\sum X_iY_i}{n-2}}$,

is the number of pairs.

**Example 10.3** Using the data in Example 10.1,

find the values of $\hat{Y}$ and show that $\sum(Y - \hat{Y}) = 0$, and

compute the standard error of estimate $s_{y.x}$. = S.D

The calculations needed to find the values of $\hat{Y}$ and the standard error of estimate $s_{y.x}$ are given in below:

| $X$ | $Y$ | $\hat{Y}$ (=1.47+2.831X) | $Y - \hat{Y}$ | $(Y - \hat{Y})^2$ | $Y^2$ |
|---|---|---|---|---|---|
| 5 | 16 | 15.625 | 0.375 | 0.140625 | 256 |
| 6 | 19 | 18.456 | 0.544 | 0.295936 | 361 |
| 8 | 23 | 24.118 | −1.118 | 1.249924 | 529 |
| 10 | 28 | 29.780 | −1.780 | 3.168400 | 784 |
| 12 | 36 | 35.442 | 0.558 | 0.311364 | 1296 |
| 13 | 41 | 38.273 | 2.727 | 7.436529 | 1681 |
| 15 | 44 | 43.935 | 0.065 | 0.004225 | 1936 |
| 16 | 45 | 46.766 | −1.766 | 3.118756 | 2025 |
| 17 | 50 | 49.597 | 0.403 | 0.162409 | 2500 |
| 102 | 302 | 301.992 | 0.008 | 15.888168 | 110368 |

i) The estimated values $\hat{Y}$ appear in the third column of the table on page 403, and
turns out to be 0.008. This small difference is due to rounding off.

ii) The standard error of estimate of $Y$ on $X$ is

$$s_{y.x} = \sqrt{\frac{\sum(Y-\hat{Y})^2}{n-2}} = \sqrt{\frac{15.888168}{7}} = \sqrt{2.269738} = 1.51$$

Using the alternative form for the calculation of $s_{y.x}$, we get

$$s_{y.x} = \sqrt{\frac{\sum Y^2 - a\sum Y - b\sum XY}{n-2}}$$

$$= \sqrt{\frac{11368-(1.47)(302)-(2.831)(3853)}{9-2}}$$

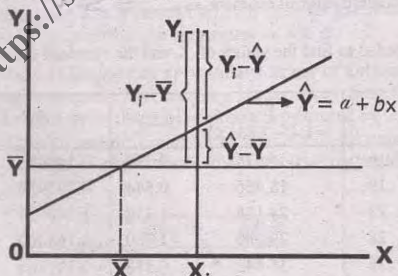
$$= \sqrt{\frac{16.217}{7}} = \sqrt{2.316714} = 1.52.$$

**10.4.5 Co-efficient of Determination.** The variability among the values of the dependent
$Y$, called the *total variation*, is given by $\sum(Y-\overline{Y})^2$. This is composed of two parts (i) that
explained by (associated with) the regression line, *i.e.* $\sum(\hat{Y}-\overline{Y})^2$, (ii) that which the regression
to explain, *i.e.* $\sum(Y-\hat{Y})^2$ (see figure). In symbols

$$\sum(Y-\overline{Y})^2 = \sum(Y-\hat{Y})^2 + \sum(\hat{Y}-\overline{Y})^2$$

Total variation = Unexplained variation + Explained variation






The *co-efficient of determination* which measures the proportion of variability of the values
dependent variable ($Y$) explained by its linear relation with the independent variable ($X$), is defined
ratio of the explained variation to the total variation. We use the symbol $\rho^2$ for the population

and symbol $r^2$ for the estimate obtained from sample. Thus the sample co-efficient of determination is given by

$$r^2 = \frac{\text{Explained variation}}{\text{Total variatioin}} = \frac{\sum(\hat{Y}-\overline{Y})^2}{\sum(Y-\overline{Y})^2}$$

$$= 1 - \frac{\sum(Y-\hat{Y})^2}{\sum(Y-\overline{Y})^2}.$$

An alternative form for calculating the coefficient of determination is

$$r^2 = \frac{a\sum Y + b\sum XY - (\sum Y)^2/n}{\sum Y^2 - (\sum Y)^2/n}.$$

When all the observed values fall on the regression line, then $Y = \hat{Y}$ and $\sum(Y-\overline{Y})^2 = \sum(\hat{Y}-\overline{Y})^2$, and hence $r^2 = 1$. When the observed values are such that $\hat{Y} = \overline{Y}$, then $\sum(\hat{Y}-\overline{Y})^2 = 0$, and hence $r^2 = 0$. This shows that $0 \le r^2 \le 1$. A value of $r^2 = 1$, signifies that 100% of the variability in the dependent variable is associated with the regression equation. When $r^2 = 0$, it means that none of the variability in the dependent variable is explained by $X$-variable. A value of $r^2 = 0.93$, indicates that 93% of the variability in $Y$ is *explained* by its linear relationship with the independent variable $X$ and 7% of the variation is due to chance or other factors.

**Example 10.4** Taking length ($Y$) as dependent variable for the data in Example 10.2, calculate (i) the total variation, (ii) the unexplained variation, (iii) the explained variation, and (iv) the co-efficient of determination and interpret the coefficient.

In Example 10.2, we found that

$$\sum Y = 220,\ \sum Y^2 = 5486,\ \sum XY = 3467,\ b = 1.02,\ a = 8.74 \text{ and } n = 10.$$

We now find

Total variation $= \sum(Y-\overline{Y})^2 = \sum Y^2 - (\sum Y)^2/n$

$= 5486 - (220)^2/10 = 646$

Unexplained variation $= \sum(Y-\hat{Y})^2 = \sum Y^2 - a\sum Y - b\sum XY$

$= 5486 - (8.74)(220) - (1.02)(3467)$

$= 5486 - 5459.14 = 26.86$

Explained variation $=$ Total variation – unexplained variation

$= 646 - 26.86 = 619.14$

The coefficient of determination, $r^2$, is given by

$$r^2 = \frac{\text{Explained variation}}{\text{Total variation}} = \frac{\sum(\hat{Y}-\overline{Y})^2}{\sum(Y-\overline{Y})^2}$$

$$= \frac{619.14}{646} = 0.958$$

A value of $r^2 = 0.958$ indicates that 95.8% of the variability in $Y$, the length of the spring, is demonstrated by its linear relationship with $X$, the weight on the spring.

## 10.5 CORRELATION

*Correlation*, like covariance, is a measure of the degree to which any two variables *vary* together. In other words, two variables are said to be *correlated* if they tend to simultaneously vary in some direction. If both the variables tend to increase (or decrease) together, the correlation is said to be direct or *positive*, *e.g.* the length of an iron bar will increase as the temperature increases. If one variable tends to increase as the other variable decreases, the correlation is said to be *negative* or *inverse*, *e.g.* the volume of gas will decrease as the pressure increases. It is worth remarking that in correlation, we assess the strength of the relationship (or interdependence) between two variables; both the variables are random variables, and they are treated symmetrically, *i.e.* there is no distinction between dependent and independent variable. In regression, by contrast, we are interested in determining the dependence of one variable that is random, upon the other variable that is non-random or fixed, and in predicting the average value of the dependent variable by using the known values of the other variable.

**10.5.1 Pearson Product Moment Correlation Co-efficient.** A numerical measure of strength of the *linear* relationship between any two variables is called the *Pearson's product moment correlation co-efficient* or sometimes, the *coefficient of simple correlation or total correlation.* The sample linear correlation coefficient for $n$ pairs of observations $(X_i, Y_i)$ usually denoted by the letter $r$, is defined by

$$r = \frac{\Sigma(X-\bar{X})(Y-\bar{Y})}{\sqrt{\Sigma(X-\bar{X})^2\,\Sigma(Y-\bar{Y})^2}}$$

The population correlation co-efficient for a bivariate distribution, denoted by $\rho$, has already been defined as

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$



For computational purposes, we have an alternative form of $r$ as

$$r = \frac{\sum XY - (\sum X)(\sum Y)/n}{\sqrt{[\sum X^2 - (\sum X)^2/n][(\sum Y^2 - (\sum Y)^2/n)]}}$$

$$= \frac{n\sum XY - \sum X \sum Y}{\sqrt{[n\sum X^2 - (\sum X)^2][n\sum Y^2 - (\sum Y)^2]}}$$

This is a more convenient and useful form, especially when $\bar{X}$ and $\bar{Y}$ are not integers. The coefficient of correlation $r$ is a pure number (*i.e.* independent of the units in which the variables are measured) and it assumes values that can range from +1 for perfect positive linear relationship, to −1 for perfect negative linear relationship with the intermediate value of zero indicating no linear relationship between $X$ and $Y$. The sign of $r$ indicates the direction of the relationship or correlation.

It is important to note that $r = 0$ does not mean that there is no relationship at all. For example, if all the observed values lie exactly on a circle, there is a perfect *non-linear* relationship between the variables but $r$ will have a value of zero as $r$ only measures the linear correlation.

The linear correlation co-efficient, is also the square root of the linear co-efficient of determination,

We have $\quad \hat{Y} = \overline{Y} + b(X - \overline{X})$

or $\quad \hat{Y} - \overline{Y} = b(X - \overline{X})$

Squaring both sides, we get

$$(\hat{Y} - \overline{Y})^2 = b^2(X - \overline{X})^2$$

Substituting in the ratio, we find

$$\frac{\Sigma(\hat{Y} - \overline{Y})^2}{\Sigma(Y - \overline{Y})^2} = \frac{b^2\,\Sigma(X - \overline{X})^2}{\Sigma(Y - \overline{Y})^2}$$

$$= \frac{\Sigma(X - \overline{X})^2}{\Sigma(Y - \overline{Y})^2}\left[\frac{\Sigma(X - \overline{X})\,(Y - \overline{Y})}{\Sigma(X - \overline{X})^2}\right]^2$$

$$= \left[\frac{\Sigma(X - \overline{X})\,(Y - \overline{Y})}{\sqrt{\Sigma(Y - \overline{Y})^2\;\Sigma(X - \overline{X})^2}}\right]^2 = r^2$$



**Example 10.5** Calculate the product moment co-efficient of correlation between $X$ and $Y$ from the following data:

| | | | | | |
|---|---|---|---|---|---|
| $X$ | 1 | 2 | 3 | 4 | 5 |
| $Y$ | 2 | 5 | 3 | 8 | 7 |

(P.U., B.A./B.Sc. 1973)

The calculations needed to compute $r$ are given below:

| $X$ | $Y$ | $(X-\overline{X})$ | $(X-\overline{X})^2$ | $(Y-\overline{Y})$ | $(Y-\overline{Y})^2$ | $(X-\overline{X})(Y-\overline{Y})$ |
|---|---|---|---|---|---|---|
| 1 | 2 | −2 | 4 | −3 | 9 | 6 |
| 2 | 5 | −1 | 1 | 0 | 0 | 0 |
| 3 | 3 | 0 | 0 | −2 | 4 | 0 |
| 4 | 8 | 1 | 1 | 3 | 9 | 3 |
| 5 | 7 | 2 | 4 | 2 | 4 | 4 |
| 15 | 25 | 0 | 10 | 0 | 26 | 13 |

Here $\overline{X} = \frac{\sum X}{n} = \frac{15}{5} = 3$, and $\overline{Y} = \frac{\sum Y}{n} = \frac{25}{5} = 5$

$$\therefore \quad r = \frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sqrt{\sum(X-\overline{X})^2 \sum(Y-\overline{Y})^2}} = \frac{13}{\sqrt{10\times 26}} = \frac{13}{16.1} = 0.8$$

**Alternatively**, the following table is set up for calculation of $r$.

| $X$ | $Y$ | $X^2$ | $Y^2$ | $XY$ |
|---|---|---|---|---|
| 1 | 2 | 1 | 4 | 2 |
| 2 | 5 | 4 | 25 | 10 |
| 3 | 3 | 9 | 9 | 9 |
| 4 | 8 | 16 | 64 | 32 |
| 5 | 7 | 25 | 49 | 35 |
| 15 | 25 | 55 | 151 | 88 |



$$\therefore \quad r = \frac{\sum XY - (\sum Y)(\sum Y)/n}{\sqrt{[\sum X^2 - (\sum X)^2/n][\sum Y^2 - (\sum Y)^2/n]}}$$

$$= \frac{88-(15)\,25/5}{\sqrt{[55-(15)^2/5][151-(25)^2/5]}} = \frac{13}{\sqrt{10\times 26}} = 0.8$$

**10.5.2 Correlation and Causation.** The fact that correlation exists between two variab[les does] not imply any *cause-and-effect* relationship. Two unrelated variables such as the sale of bananas an[d the] death rate from cancer in a city, may produce a high positive correlation which may be due to a th[ird] unknown variable (namely, the city population). The larger the city, the more consumption of ban[anas] and the higher will be the death rate from cancer. Clearly, this is a *false* or a merely *incidental* co[rrelation] which is the result of a third variable, the city size. Such a false correlation between two unco[nnected] variables is called *nonsense* or *spurious* correlation. We therefore should be very careful in inte[rpreting] the correlation coefficient as a measure of relationship or interdependence between two variables.

**10.5.3 Properties of r.** The sample correlation co-efficient $r$ has the following properties:

i) The correlation co-efficient $r$ is symmetrical with respect to the variables $X$ and $Y$, *i.e.* $r_{XY} = \ldots$

ii) The correlation co-efficient lies between $-1$ and $+1$, *i.e.* $-1 \le r \le +1$.

iii) The correlation co-efficient is independent of the origin and scale.

**Proof:** Let $u$ and $v$ be the two new variables defined by $u = \frac{X-a}{h}$ and $v = \frac{Y-b}{k}$ so that $X = a + \ldots$ $Y = b + kv$, where $a$ and $b$ are the new origins and $h$ and $k$ are the units of measurement.

Let $r_{XY}$ denote the correlation co-efficient between $X$ and $Y$ and $r_{uv}$, the correlation co-e[fficient] between $u$ and $v$.

Substituting these values in $r_{XY}$, *viz.*

$$r_{XY} = \frac{\Sigma(X-\overline{X})\,(Y-\overline{Y})}{\sqrt{\Sigma(X-\overline{X})^2\ \Sigma(Y-\overline{Y})^2}}, \text{ we get}$$

$$r_{XY} = \frac{\Sigma[(a+hu)-(a+h\overline{u})]\,[(b+kv)-(b+k\overline{v})]}{\sqrt{\Sigma[(a+hu)-(a+h\overline{u})]^2 . \Sigma[(b+kv)-(b+k\overline{v})]^2}},$$

here $\overline{X} = a + h\overline{u}$ and $\overline{Y} = b + k\overline{v}$. Therefore

$$r_{XY} = \frac{hk\,\Sigma(u-\overline{u})\,(v-\overline{v})}{hk\sqrt{\Sigma(u-\overline{u})^2 . \Sigma(v-\overline{v})^2}} = r_{uv}$$

This property is very useful in numerical evaluation of $r$, since due to this property, we can choose venient origin and scale.

) In case of a bivariate population where both $X$ and $Y$ are random variables, $r$ is the geometric mean between the two regression co-efficient.

That is, if $b_{yx}$ is the regression coefficient of the regression line of $Y$ on $X$ and $b_{xy}$ is the regression ent of the regression line of $X$ on $Y$, and $r$ is the coefficient of correlation, then $r^2 = b_{yx}.b_{xy}$ implies $r = \pm\sqrt{b_{yx}.b_{xy}}$.

Since the signs of the regression coefficients depend on the same expression $\Sigma(Y-\overline{X})\,(Y-\overline{Y})$ so $b_{yx}$ and $b_{xy}$ are both positive or $b_{yx}$ and $b_{xy}$ are both negative. Therefore

$$r = +\sqrt{b_{yx}.b_{xy}}, \text{ if } b_{yx} \text{ and } b_{xy} \text{ are positive,}$$

$$r = -\sqrt{b_{yx}.b_{xy}}, \text{ if } b_{yx} \text{ and } b_{xy} \text{ are negative.}$$

the value of $r$ always takes the same sign as the regression coefficients.

The regression co-efficients and the regression lines for a bivariate population, by using the n of the correlation co-efficient, may be expressed as

$$b_{yx} = r\frac{S_y}{S_x};\ b_{xy} = r\frac{S_x}{S_y}$$

$$Y-\overline{Y} = r\frac{S_y}{S_x}(X-\overline{X});\ and\ \ X-\overline{X} = r\frac{S_x}{S_y}(Y-\overline{Y}),$$

the letters have their usual meaning.

**Example 10.6** Calculate the co-efficient of correlation between the values of $X$ and $Y$ given below:

| $X$ | 78 | 89 | 97 | 69 | 59 | 79 | 68 | 61 |
|---|---|---|---|---|---|---|---|---|
| $Y$ | 125 | 137 | 156 | 112 | 107 | 136 | 123 | 108 |

Let $u = X - 69$ and $v = Y - 112$. Then $r_{xy} = r_{uv}$. The calculations needed to find $r$ are given in the the next page:

| $X$ | $Y$ | $u$ | $v$ | $u^2$ | $v^2$ | $uv$ |
|---|---|---|---|---|---|---|
| 78 | 125 | 9 | 13 | 81 | 169 | 117 |
| 89 | 137 | 20 | 25 | 400 | 625 | 500 |
| 97 | 156 | 28 | 44 | 784 | 1936 | 1232 |
| 69 | 112 | 0 | 0 | 0 | 0 | 0 |
| 59 | 107 | −10 | −5 | 100 | 25 | 50 |
| 79 | 136 | 10 | 24 | 100 | 576 | 540 |
| 68 | 123 | −1 | 11 | 1 | 121 | −11 |
| 61 | 108 | −8 | −4 | 64 | 16 | 32 |
| 600 | 1004 | 48 | 108 | 1530 | 3468 | 2160 |

Now $$r=\frac{\sum uv-(\sum u)(\sum v)/n}{\sqrt{\left[\sum u^2-\frac{(\sum u)^2}{n}\right]\left[\sum v^2-\frac{(\sum v)^2}{n}\right]}}$$

$$=\frac{2160-\frac{48\times 108}{8}}{\sqrt{\left[1530-\frac{(48)^2}{8}\right]\left[3468-\frac{(108)^2}{8}\right]}}$$

$$=\frac{2160-648}{\sqrt{(1530-288)\times(3468-1458)}}=\frac{1512}{1578}=0.96.$$

Hence the correlation coefficient between $X$ and $Y$ is 0.96.



**Example 10.7** If $b_{ij}$ is the regression coefficient of $X_i$ on $X_j$, then calculate the product moment coefficient of correlation in each case, given

i) $b_{12}=-0.1,\ b_{21}=-0.4$ ii) $b_{13}=0.27,\ b_{31}=0.6$

iii) $b_{23}=0.67,\ b_{32}=0.38$.

The product moment coefficient of correlation between $X_i$ and $X_j$ is given by

$$r_{ij}=\sqrt{b_{ij}\times b}$$

i) Here $b_{12}=-0.1$, and $b_{21}=-0.4$

$\therefore$ $$r_{12}=-\sqrt{(-0.1)(-0.4)}=-0.20.$$

$r$ is negative since both regression coefficients are negative.

ii) Here both regression coefficients are positive, so $r$ is positive. Thus

$$r_{13}=+\sqrt{b_{13}\times b_{31}}=+\sqrt{(0.27)(0.6)}=+0.40.$$

iii) Here we have

$$r_{23} = \sqrt{(0.67)(0.38)} = 0.50 \quad (\because\ b_{23} \text{ and } b_{32} \text{ are positive})$$

**10.5.4 Correlation Co-efficient for Grouped Data.** In a simple frequency table, the data are arranged with respect to one variable only. If the arrangement is made according to two variables simultaneously in say, $m$ columns and $k$ rows, the frequency table thus obtained is called a *correlation table* or a *bivariate frequency table*. The number of observations falling in the $(i, j)th$ cell, is called the $(i, j)th$ cell frequency and is denoted by $f_{ij}$. The correlation co-efficient, if it exists, can be calculated from such a two-way frequency table by using the class midpoints as the value of the observations. The formula for $r$ then becomes

$$r = \frac{\sum f_{ij} X_j Y_i - \frac{1}{n}(\sum f_{.j} X_j)(\sum f_{i.} Y_i)}{\sqrt{\left[\sum f_{.j} X_j^2 - \frac{1}{n}(\sum f_{.j} X_j)^2\right]\left[\sum f_{i.} Y_i^2 - \frac{1}{n}(\sum f_{i.} Y_i)^2\right]}}$$

where $f_{i.} = \sum_{j=1}^{m} f_{ij}$, the frequency of $Y$ values, $f_{.j} = \sum_{j=1}^{k} f_{ij}$, the frequency of $X$ values and $n$ is the total frequency.



**Example 10.8** Calculate the co-efficient of linear correlation from the table given below:

| Grades in Statistics ($Y$) | Grades in Mathematics ($X$) | | | | | | Total |
|---|---|---|---|---|---|---|---|
| | 40–49 | 50–59 | 60–69 | 70–79 | 80–89 | 90–99 | |
| 90–99 | -- | -- | -- | 2 | 4 | 4 | 10 |
| 80–89 | -- | -- | 1 | 4 | 6 | 5 | 16 |
| 70–79 | -- | -- | 5 | 10 | 8 | 1 | 24 |
| 60–69 | 1 | 4 | 9 | 5 | 2 | -- | 21 |
| 50–59 | 3 | 6 | 6 | 2 | -- | -- | 17 |
| 40–49 | 3 | 5 | 4 | -- | -- | -- | 12 |
| Total | 7 | 15 | 25 | 23 | 20 | 10 | 100 |

(P.U., B.A./B.Sc. 1968)

Let us introduce two new variables $u$ and $v$ given by the relations $u = \frac{X - 64.5}{10}$ and $v = \frac{Y - 74.5}{10}$. Then the calculations needed for finding $r$ are arranged in the table on page (412).

| $Y_i$ | $X_j$ | 44.5 | 54.5 | 64.5 | 74.5 | 84.5 | 94.5 | $f_{i\cdot}$ | $f_{i\cdot}v_i$ | $f_{i\cdot}v_i^2$ | $f_{ij}u_jv_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $v_i$ \ $u_j$ | −2 | −1 | 0 | 1 | 2 | 3 | | | | |
| 94.5 | 2 | -- | -- | -- | 2 [4] | 4 [16] | 4 [24] | 10 | 20 | 40 | 44 |
| 84.5 | 1 | -- | -- | 1 [0] | 4 [4] | 6 [12] | 5 [15] | 16 | 16 | 16 | 31 |
| 74.5 | 0 | -- | -- | 5 [0] | 10 [0] | 8 [0] | 1 [0] | 24 | 0 | 0 | 0 |
| 64.5 | −1 | 1 [2] | 4 [4] | 9 [0] | 5 [−5] | 2 [−4] | --- | 21 | −21 | 21 | −3 |
| 54.5 | −2 | 3 [12] | 6 [12] | 6 [0] | 2 [−4] | --- | --- | 17 | −34 | 68 | 20 |
| 44.5 | −3 | 3 [18] | 5 [15] | 4 [0] | --- | --- | --- | 12 | −36 | 108 | 33 |
| $f_{\cdot j}$ | | 7 | 15 | 25 | 23 | 20 | 10 | 100 | −55 | 253 | 125 |
| $f_{\cdot j}u_j$ | | −14 | −15 | 0 | 23 | 40 | 30 | 64 | | | ↑ |
| $f_{\cdot j}u_j^2$ | | 28 | 15 | 0 | 23 | 80 | 90 | 236 | | | |
| $f_{ij}u_jv_i$ | | 32 | 31 | 0 | −1 | 24 | 39 | 125 | ← | Check | |

The number in the corner of each cell represents the product $f_{ij}u_jv_i$, where $f_{ij}$ is the cell frequ
Thus $f_{1,4}\, u_4v_1 = 2(1)\,(2)$=4 and $f_{1,5}u_5v_1$=4(2) (2) = 16 and so on. The totals in the last column and the
row are equal and represent $\sum f_{ij}u_jv_i$.

Now $$r_{XY} = r_{uv} = \frac{n\sum fuv - (\sum fu)(\sum fv)}{\sqrt{[n\sum fu^2 - (\sum fu)^2]\,[n\sum fv^2 - (\sum fv)^2]}}$$

(subscripts dropped for convenience in printing)

$$= \frac{(100)(125) - (64)(-55)}{\sqrt{[(100)(236) - (64)^2][(100)(253) - (-55)^2]}}$$

$$= \frac{16020}{\sqrt{(19504)(22275)}} = 0.77$$

**Example 10.9** (a) Correlation between $X$ and $Y$ is $r$, show that correlation between $aX$ and $bY$ is $r$ or $-r$ according as $a$ and $b$ have the same or different signs.

b) Find correlation between $X$ and $Y$ connected by

$$aX + bY + c = 0.$$

(P.U., B.A. (Hons) Part-II, 196

a) Let $u = aX$, so that $\overline{u} = a\overline{X}$,
and $v = bY$, so that $\overline{v} = b\overline{Y}$

Then $(u-\overline{u}) = a(X-\overline{X})$ and $(v-\overline{v}) = b(Y-\overline{Y})$

By definition, we have

$$r_{uv} = \frac{\sum(u-\overline{u})(v-\overline{v})}{\sqrt{\sum(u-\overline{u})^2 \sum(v-\overline{v})^2}}$$

$$= \frac{ab\sum(X-\overline{X})(Y-\overline{Y})}{\sqrt{a^2\sum(X-\overline{X})^2 b^2 \sum(Y-\overline{Y})^2}}$$

$$= \frac{ab}{\sqrt{a^2b^2}} r_{XY}$$

$= +r$, if $a$ and $b$ are of the same signs.

$= -r$, if $a$ and $b$ are of the different signs.

b) We are given $aX + bY + c = 0$

Thus $a\sum X + b\sum Y + nc = 0$, where $n$ is the number of pairs of values $(X_i, Y_i)$

Dividing by $n$, we get

$a\overline{X} + b\overline{Y} + c = 0$, $\overline{X}$ and $\overline{Y}$ being the means of $X$ and $Y$ sets of observations. Subtracting, we have

$$a(X-\overline{X}) + b(Y-\overline{Y}) = 0$$

or $$(Y-\overline{Y}) = -\frac{a}{b}(X-\overline{X})$$

Now $$r_{XY} = \frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sqrt{\sum(X-\overline{X})^2 \sum(Y-\overline{Y})^2}}$$

$$= \frac{-\frac{a}{b}\sum(X-\overline{X})^2}{\sqrt{\left[\sum(X-\overline{X})^2\right]\left[\frac{a^2}{b^2}\sum(X-\overline{X})^2\right]}} = \frac{-a/b}{\sqrt{\frac{a^2}{b^2}}}$$

$= -1$, if $a$ and $b$ are of the same signs.

$= +1$, if $a$ and $b$ are of the opposite signs.

## RANK CORRELATION

Sometimes, the actual measurements or counts of individuals or objects are either not available or assessment is not possible. They are then arranged *in order* according to some characteristic of interest. Such an ordered arrangement is called a *ranking* and the *order given to an individual or object* is called its *rank*. The correlation between two such sets of rankings is known as *Rank Correlation*.

**10.6.1 Derivation of Rank Correlation.** Let a set of $n$ objects be ranked with respect to cha… $A$ as $x_1, x_2, \ldots, x_i, \ldots, x_n$, and according to character $B$ as $y_1, y_2, \ldots, y_i, \ldots, y_n$. We assume that no tw… more objects are given the same ranks (*i.e.* are tied). Then obviously $x_i$ and $y_i$ are some two numbers fr… 1 to $n$.

Since both $x_i$ and $y_i$ are the first $n$ natural numbers, therefore, we have

$$\sum_{i=1}^{n} x = \sum_{i=1}^{n} y = \sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{n(n+1)}{2},$$

$$\sum_{i=1}^{n} x^2 = \sum_{i=1}^{n} y^2 = \sum_{i=1}^{n} (i)^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)\,(2n+1)}{6},$$

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}(y_i-\bar{y})^2 = \sum_{i=1}^{n} y_i^{\,2} - \frac{(\sum y_i)^2}{n}$$

$$= \frac{n(n+1)\,(2n+1)}{6} - \frac{n(n+1)^2}{4} = \frac{n(n^2-1)}{12}$$

Let $d_i$ denote the difference in ranks assigned to the $ith$ individual or object, *i.e.* $d_i = x_i - y_i$.

Then $$\sum_{i=1}^{n} d_i^{\,2} = \sum_{i=1}^{n}(x_i - y_i)^2$$

$$= \sum(x_i^{\,2} + y_i^{\,2} - 2x_iy_i) = \sum x_i^{\,2} + \sum y_i^{\,2} - 2\sum x_iy_i$$

Substituting for $\sum x_i^{\,2}$ and $\sum y_i^{\,2}$, we get

$$\sum_{i=1}^{n} d_i^{\,2} = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)(2n+1)}{6} - 2\sum x_iy_i$$

or $$\sum x_iy_i = \frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\sum d_i^{\,2}$$

The product moment co-efficient of correlation between the two sets of rankings is

$$r = \frac{\sum xy - \dfrac{(\sum x)(\sum y)}{n}}{\sqrt{\left[\sum x^2 - \dfrac{(\sum x)^2}{n}\right]\left[\sum y^2 - \dfrac{(\sum y)^2}{n}\right]}}$$

Substitution gives

$$r_s = \frac{\left[\dfrac{n(n+1)(2n+1)}{6} - \dfrac{1}{2}\sum d_i^{\,2}\right] - \dfrac{n(n+1)^2}{4}}{\dfrac{n(n^2-1)}{12}}$$

$$= \frac{\left[\dfrac{n(n+1)(2n+1)}{6} - \dfrac{n(n+1)^2}{4}\right] - \dfrac{1}{2}\sum d_i^{\,2}}{\dfrac{n(n^2-1)}{12}}$$

$$= \frac{\frac{n(n^2-1)}{12} - \frac{1}{2}\sum d_i^2}{\frac{n(n^2-1)}{12}} = 1 - \frac{6\sum d_i^2}{n(n^2-1)}.$$

This formula is usually denoted by $r_s$ in order to have a distinction. It is often called Spearman's ficient of rank correlation, in honour of the psychometrician Charles Edward Spearman –1945), who first developed the procedure in 1904.

t is to be noted that $\sum d_i^2$ has the least value and is zero when the numbers are in complete ement. When they are in complete disagreement, $\sum d_i^2$ attains the maximum value and is equal to $\frac{n(n^2-1)}{3}$.

Substituting these values in the formula, we see that

$r_s = 1$ for $\sum d_i^2 = 0$, and

$r_s = -1$ for $\sum d_i^2 = \frac{n(n^2-1)}{3}$.

Thus $r_s$ also lies between −1 and +1.

**Example 10.10** Find the co-efficient of rank correlation from the following rankings of 10 in Statistics and Mathematics.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Statistics ($x$): | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Mathematics ($y$): | 2 | 4 | 3 | 1 | 7 | 5 | 8 | 10 | 6 | 9 |

(P.U., B.A. (Hons.) Part-I, 1964)

We calculate the co-efficient of rank correlation as follows:

| $x_i$ | $y_i$ | $d_i(=x_i-y_i)$ | $d_i^2$ |
|---|---|---|---|
| 1 | 2 | −1 | 1 |
| 2 | 4 | −2 | 4 |
| 3 | 3 | 0 | 0 |
| 4 | 1 | 3 | 9 |
| 5 | 7 | −2 | 4 |
| 6 | 5 | 1 | 1 |
| 7 | 8 | −1 | 1 |
| 8 | 10 | −2 | 4 |
| 9 | 6 | 3 | 9 |
| 10 | 9 | 1 | 1 |
| --- | --- | 0 | 34 |

Hence, using Spearman's co-efficient of rank correlation, we get

$$r_s = 1 - \frac{6\sum d_i^2}{n(n^2-1)} = 1 - \frac{6\times 34}{10\times 99} = 1 - 0.2 = +0.8.$$

This indicates a high correlation between Statistics and Mathematics.

**10.6.2 Rank Correlation for Tied Ranks.** The Spearman's co-efficient of rank correla applies only when no ties are present. In case there are ties in ranks, the ranks are adjusted by assig the mean of the ranks which the tied *objects* or observations would have if they were ordered. F example, if two *objects* or observations are tied for fourth and fifth, they are both given the mean ran 4 and 5, *i.e.* 4.5. The sum of adjusted ranks remains $\frac{n(n+1)}{2}$ but $\sum(x_i-\bar{x})^2 \neq \sum(y_i-\bar{y})^2 \neq \frac{n(n^2-}{12}$ has been shown that each set of ties involving $t$ observations reduces the values of $d^2$ by a quantity to $\frac{1}{12}(t^3-t)$. In such a situation, one of the following two methods is to be used:

First, for each tie, add a quantity $\frac{1}{12}(t^3-t)$ to $\sum d^2$ before substituting the values Spearman's co-efficient of rank correlation in order to adjust the formula for the tied observations.

*Second*, use the product moment co-efficient of correlation to find the correlation between the sets of adjusted ranks.

**Example 10.11** Two members of a selection committee rank eight persons according to suitability for promotion as follows:

| Persons | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Member 1 | 1 | 2.5 | 2.5 | 4 | 5 | 6 | 7 | 8 |
| Member 2 | 2 | 4 | 1 | 3 | 6 | 6 | 6 | 8 |

Calculate the co-efficient of rank correlation.

We observe that both the sets of rankings contain ties. The coefficient of rank correla therefore calculated as below:

| Person | Member 1 | Member 2 | $d$ | $d^2$ |
|---|---|---|---|---|
| A | 1 | 2 | −1 | 1 |
| B | 2.5 | 4 | −1.5 | 2.25 |
| C | 2.5 | 1 | 1.5 | 2.25 |
| D | 4 | 3 | 1 | 1 |
| E | 5 | 6 | −1 | 1 |
| F | 6 | 6 | 0 | 0 |
| G | 7 | 6 | 1 | 1 |
| H | 8 | 8 | 0 | 0 |
| $\Sigma$ | 36 | 36 | 0 | 8.5 |

For tie between $B$ and $C$, (first rankings) $t$=2 and for $E$, $F$ and $G$ (second rankings) $t$=3, the quantity to be added to $\sum d^2$ is

$$\frac{1}{12}(2^3-2)+\frac{1}{12}(3^3-3)=2.5.$$

Hence $r_s = 1-\dfrac{6[8.5+2.5]}{8(64-1)} = 1-\dfrac{66}{504} = 1-0.131 = 0.869.$

**rnative Method:**

We see that the first member has tied $B$ and $C$, while the second member has tied $E$, $F$ and $G$. Let note the ranks given by the first member by $x_i$ and those of second member by $y_i$. Then we proceed low:

| $x_i$ | $y_i$ | $x_i^2$ | $y_i^2$ | $x_iy_i$ |
|---|---|---|---|---|
| 1 | 2 | 1 | 4 | 2 |
| 2.5 | 4 | 6.25 | 16 | 10 |
| 2.5 | 1 | 6.25 | 1 | 2.5 |
| 4 | 3 | 16 | 9 | 12 |
| 5 | 6 | 25 | 36 | 30 |
| 6 | 6 | 36 | 36 | 36 |
| 7 | 6 | 49 | 36 | 42 |
| 8 | 8 | 64 | 64 | 64 |
| 36 | 36 | 203.5 | 202 | 198.5 |

Hence the co-efficient of rank correlation is

$$r=\frac{\sum x_iy_i-(\sum x_i)(\sum y_i)/n}{\sqrt{[\sum x_i^2-(\sum x_i)^2/n][\sum y_i^2-(\sum y_i)^2/n]}}$$

$$=\frac{198.5-(36)(36/8)}{\sqrt{[203.5-(36)^2/8][202-(36)^2/8]}}$$

$$=\frac{198.5-162}{\sqrt{(203.5-162)(202-162)}}=\frac{36.5}{\sqrt{(41.5)(40)}}$$

$$=\frac{36.5}{40.47}=0.896,$$

indicates a high degree of agreement between the two members.

**.6.3 Co-efficient of Concordance.** The Spearman's co-efficient of rank correlation measures ement between two sets of rankings only, but in practice; the individuals or objects are sometimes by more than two people. We then need a co-efficient to measure agreement among more than of rankings. Such a co-efficient is obtained as below:

Let there be $m$ rankings of $n$ individuals or objects instead of two. Obviously in case of complete ent, the rank totals will form the series $m$, $2m$, $3m$,…, $nm$.

The mean of these totals is

$$\overline{X} = (m + 2m + 3\,m + \ldots + nm) \div n.$$

$$= \frac{m(1+2+3+\ldots+n)}{n} = \frac{m(n+1)}{2},$$

and the variance of these sums, which is the maximum possible, is

$$\text{Var(Total)} = \frac{1}{n}[m^2 + (2m)^2 + (3m)^2 + \ldots + (nm)^2] - \left[\frac{m(n+1)}{2}\right]^2$$

$$= \frac{m^2[1^2 + 2^2 + 3^2 + \ldots + n^2]}{n} - \left[\frac{m(n+1)}{2}\right]^2$$

$$= \frac{m^2(n+1)(2n+1)}{6} - \frac{m^2(n+1)^2}{4} = \frac{m^2(n^2-1)}{12}.$$

But the totals of observed ranks will not necessarily be the same. Let $S$ denote the sum squares of deviations of the totals of the observed ranks from their common mean, .*i.e.* $\frac{m(n+1)}{2}$, *Co-efficient of Concordance*, $W$, is defined as the ratio of the variance of the totals of the observed to the variance in case of complete agreement. Thus, we have

$$W = \frac{S}{n} \div \frac{m^2(n^2-1)}{12} = \frac{12S}{m^2(n^3-n)}.$$

This co-efficient is due to Maurice *G.* Kendall (1907–1983) and varies from 0 to 1. When represents complete agreement.

Imp **Example 10.12** The following data give rankings of six persons for their ability by three $Q$ and $R$. Calculate the co-efficient of concordance.

| Persons | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Judge $P$ | 3 | 1 | 6 | 2 | 5 | 4 |
| Judge $Q$ | 4 | 3 | 2 | 5 | 1 | 6 |
| Judge $R$ | 2 | 1 | 6 | 5 | 4 | 3 |

(P.U., B.A. (Hons.), Part-

Here the totals of the observed ranks are 9, 5, 14, 12, 10 and 13; $m$=3 and $n$=6 so mean $= \frac{m(n+1)}{2} = \frac{3(6+1)}{2} = 10.5$.

Thus $S = (9-10.5)^2+(5-10.5)^2+(14-10.5)^2+(12-10.5)^2+(10-10.5)^2+(13-10.5)^2$

$= (-1.5)^2+(-5.5)^2+(3.5)^2+(1.5)^2+(-0.5)^2+(2.5)^2 = 53.50$

Hence $W = \frac{12S}{m^2(n^3-n)} = \frac{12 \times 53.5}{9(216-6)} = \frac{642}{1890} = +0.34$.

## EXERCISES

### …JECTIVE

Answer 'True' or 'False'. If the statement is not true then replace the underlined words with words that make the statement true:

i) A high value of correlation between Y and X <u>indicates</u> a high likelihood of a cause and effect relationship between Y and X.

ii) <u>Correlation analysis</u> finds the equation of the line for two variables.

iii) The co-efficient of correlation lies between <u>0 and +1</u>.

iv) The correlation co-efficient <u>is not</u> independent of the origin and scale.

v) <u>Regression analysis</u> measures the strength of the linear relationship between two variables.

vi) In regression analysis X and Y <u>must both</u> be normally distributed.

vii) The <u>method of least squares</u> gives the line of best fit.

viii) If the co-efficient of determination $r^2$ is equal to ½, then it indicates that 50% of the variation is due to <u>chance</u> or <u>other factors</u>.

ix) If the slope of the regression line has a negative sign, then the coefficient of determination <u>also is negative</u>.

x) If all the points in a scatter diagram fall on the regression line, then the standard error of estimate equals <u>positive value</u>.



### MULTIPLE CHOICE QUESTIONS

When the slope of regression line is negative, the following statistic is also negative

a) r

b) $r^2$

c) Standard error of estimate

d) Standard error of slope co-efficient

If there is no linear relationship between the two variables then which one of the following does not hold?

a) $a = 0$

b) $b = 0$

c) $r^2 = 0$

d) The regression line is either vertical; or horizontal.

iii) If the correlation co-efficient r = 0.7, then the proportion of variation for Y explained by X is

a) 0.49

b) 0.50

c) 0.70

d) $\sqrt{0.70}$

iv) The dependent variable is also known as

a) Explained variable

b) Response variable

c) Predicted variable

d) All of above

v) In the regression equation $Y = \alpha + \beta x + \varepsilon$, both X and Y variables are

a) Random

b) Fixed

c) X is fixed and Y is random

d) Y is fixed and X is random

vi) The variation of the Y values around the regression line is measured by

a) $\sum(Y-\bar{Y})^2$

b) $\sum(Y-\hat{Y})^2$

c) $\sum(\hat{Y}-\bar{Y})^2$

d) None of above

vii) If both the dependent and independent variables increase simultaneously, the correlation coefficient will be in the range of

a) 0 to +1

b) 0 to –1

c) 1 to 2

d) –1 to +1

viii) Which of the following statements is incorrect about correlation coefficient?

a) It passes through the means of the data

b) It is symmetrical with respect to X and Y

c) It is independent of origin and scale

d) It is the geometric mean between the two regression coefficients

ix) If the unexplained variation between variables X and Y is 0.40 then $r^2$ is

a) 0.75
b) 0.60
c) 0.40
d) None of the above

x) The strength of a linear relationship between two variables Y and X is measured by

a) $r^2$
b) $b_{yx}$
c) r
d) None of above

## BJECTIVE

a) Explain what is meant by (i) *regression*, (ii) *regressand*, (iii) *regressor*, and (iv) regression co-efficient.

b) Differentiate between a deterministic and a probabilistic relationship, giving examples.

a) What is a scatter diagram? Describe its role in the theory of regression.

b) What is a linear regression model? Explain the assumptions underlying the linear regression model.

a) Explain the *principle of least-squares*.

b) Explain briefly how the principle of least squares is used to find a regression line based on a sample of size $n$. Illustrate on a rough sketch the distances whose squares are minimized, taking care to distinguish the dependent and independent variables.

a) Find least-squares estimates of parameters in a simple linear regression model $Y_i = \alpha + \beta X_i + e_i$, where $e_i$'s are distributed independently with mean zero and constant variance.

b) What are the properties of the least-squares regression line?

(P.U., B.A./B.Sc. 1992)

c) Show that the regression line passes through the means of observations.

(P.U., D.St. 1962)

a) Describe briefly how you would obtain the line of regression of one variable ($Y$) on another variable ($X$), using the method of least-squares.

(P.U., B.A./B.Sc. 1975)

b) What is meant by the *standard error of estimate*? If the regression line of $Y$ on $X$ is given by $\hat{Y} = a + bX$, prove that the standard error of estimate $s_{y.x}$ is given by

$$s_{y.x} = \sqrt{\frac{\sum Y^2 - a\sum Y - b\sum XY}{n-2}}.$$

10.6 Given the following set of values:

| $X$ | 20 | 11 | 15 | 10 | 17 | 19 |
|---|---|---|---|---|---|---|
| $Y$ | 5 | 15 | 14 | 17 | 8 | 9 |

a) Determine the equation of the least squares regression line.

b) Find the predicted values of $Y$ for $X$=10, 11, 15, 17, 19, 20.

c) Use the predicted values found in (b) to find the standard error of estimate.

10.7 Given these ten pairs of $(X, Y)$ values:

| $X$ | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 2.1 | 2.5 | 3.1 | 3.0 | 3.8 | 3.2 | 4.3 | 3.9 | 4.4 | 4.8 |

a) Plot a scatter diagram for the above data.

b) Carry out the necessary computations to obtain the least-squares estimates of the parameters in the simple linear regression $Y_i = \alpha + \beta X_i + e_i$.

c) Compute the residuals and verify that they add to zero.

d) Use the regression equation to predict the values of $Y$ when $X$=10.

10.8 For each of the following data, determine the estimated regression equation $\hat{Y} = a + bX$.

a) $\overline{X} = 10; \overline{Y} = 20; \sum XY = 1{,}000; \sum X^2 = 2{,}000; n = 10.$

b) $\sum X = 528; \sum Y = 11{,}720; \sum XY = 193{,}640; \sum X^2 = 11{,}440; n = 32.$

c) $\sum X = 1{,}239; \sum Y = 79; \sum XY = 1{,}613; \sum X^2 = 17{,}322; \sum Y^2 = 293; n = 100.$

d) $n = 10, \sum X = 1710, \sum Y = 760, \sum X^2 = 293{,}162, \sum Y^2 = 59{,}390, \sum XY = 130{,}628.$

e) $\overline{X} = 52, \overline{Y} = 237, \sum(X - \overline{X})^2 = 2800, \sum(X - \overline{X})(Y - \overline{Y}) = 9871.$

10.9 The owner of a retailing organization is interested in the relationship between price at which a commodity is offered for sale and the quantity sold. The following sample data have been collected.

| Price | 25 | 45 | 30 | 50 | 35 | 40 | 65 | 75 | 70 | 60 |
|---|---|---|---|---|---|---|---|---|---|---|
| Quantity sold | 118 | 105 | 112 | 100 | 111 | 108 | 95 | 88 | 91 | 96 |

a) Plot a scatter diagram for the above data.

b) Using the method of least squares, determine the equation for the estimated regression line. Plot this line on the scatter diagram.

c) Calculate the standard deviation of regression, $s_{y.x}$. (B.Z.U., M.A. Econ.,

10.10 Given the following sets of values:

| $Y$ | 6.5 | 5.3 | 8.6 | 1.2 | 4.2 | 2.9 | 1.1 | 3.0 |
|---|---|---|---|---|---|---|---|---|
| $X$ | 3.2 | 2.7 | 4.5 | 1.0 | 2.0 | 1.7 | 0.6 | 1.9 |

a) Compute the least-squares regression equation for $Y$ values on $X$ values, that is the equation $\hat{Y} = a + bX$.

b) Compute the standard error of estimate, $s_{y.x}$.

c) Compute the least-squares regression equation for $X$ values on $Y$ values, that is the equation $\hat{X} = a_0 + b_0 Y$.

d) Compute the standard error of estimate, $s_{x.y}$.

11 a) Explain what is meant by the co-efficient of determination.

b) Compute the co-efficient of determination for the following data and interpret the co-efficient.

| Income ($X$) (000) | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|
| Expenditure ($Y$) (000) | 7 | 21 | 23 | 34 | 6 | 53 |

2 a) What is the total variation, the explained variation and the unexplained variation?

b) Compute (i) the total variation, (ii) the explained variation and (iii) the unexplained variation for the data in 10.11(b). How much of the variability in $Y$ is explained by the linear regression model?

a) Differentiate between regression and correlation, giving examples.
(P.U., B.A./B.Sc. 1979)

b) Describe the properties of the correlation coefficient.

c) What values may $r$ assume? Interpret the meaning when $r = -1, 0, +1$.
(P.U., B.A./B.Sc. 1980)

a) Define the terms correlation and product moment co-efficient of correlation. Prove that the correlation co-efficient is independent of the origin and scale. (P.U., B.A./B.Sc. 1981)

b) Compute the correlation co-efficient between the variables $X$ and $Y$ represented in the following table:

| $X$ | 2 | 4 | 5 | 6 | 8 | 11 |
|---|---|---|---|---|---|---|
| $Y$ | 18 | 12 | 0 | 8 | 7 | 5 |

c) Multiply each $X$ value by 2 and add 6. Multiply each $Y$ value by 3 and subtract 15. Find the correlation co-efficient between the two new sets of values, explaining why you do or do not obtain the same result as in part (b).

a) Show that, if $r_{XY}$ is the correlation co-efficient calculated from a set of paired data $(X_1, Y_1)$, $(X_2, Y_2)$, ..., $(X_n, Y_n)$, then $r_{uv}$, the correlation co-efficient for $u_i = aX_i + b$ and $v_i = cY_i + d$ (with $a \neq 0$ and $c \neq 0$), is given by $r_{uv} = r_{XY}$.

b) Calculate the correlation co-efficient by first multiplying each $X$ and $Y$ by 10 and then subtracting 70 from each $X$ and 60 from each $Y$.

| $X$ | 8.2 | 9.6 | 7.0 | 9.4 | 10.9 | 7.1 | 9.0 | 6.6 | 8.4 | 10.5 |
|---|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 8.7 | 9.6 | 6.9 | 8.5 | 11.3 | 7.6 | 9.2 | 6.3 | 8.4 | 12.4 |

10.16 a) Explain the term correlation. It is known that $r_{XY} = 0.7$. Find (i) $r_{YX}$, (ii) $r_{uv}$, where $u = -$ and $v = 3Y$.

b) Calculate the coefficient of correlation for a sample of 20 pairs of observations, given tha

$$\overline{X} = 2, \overline{Y} = 8, \sum X^2 = 180, \sum Y^2 = 1424 \text{ and } \sum XY = 404. \qquad \text{(P.U., B.Sc. Hons. 19}$$

10.17 The following data were computed from personnel records of a manufacturing firm:

$X$ = number of years of service, $Y$ = weekly wage rate

$n = 23$; $\sum X = 2{,}433$; $\sum Y = 4{,}245$; $\sum X^2 = 281{,}019$; $\sum Y^2 = 841{,}786$ and $\sum XY = 482{,}788$.

i) Compute the correlation co-efficient.

ii) If the correlation co-efficient indicates that there does exist a relationship between $X$ an compute the least-squares line of regression. What do the values of $a$ and $b$ signify?

(P.U., B.A./B.Sc. 19

10.18 Find the product moment co-efficient of correlation between traffic density and accident rate fro the following information available. Find also the coefficient of determination and interpret it.

| Traffic Density | 30 | 35 | 40 | 45 | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|---|---|---|---|
| Accident Rate | 2 | 4 | 5 | 5 | 8 | 15 | 24 | 30 | 32 |

10.19 Given marks as

| Student | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Economics paper | 36 | 56 | 41 | 46 | 59 | 46 | 65 | 31 | 68 | 41 | 70 | 36 |
| Physics Paper | 62 | 42 | 60 | 53 | 36 | 50 | 42 | 66 | 44 | 58 | 65 | 71 |

Find the co-efficient of correlation and interpret it.



10.20 Calculate the co-efficient of correlation and obtain the lines of regression of the following data

| Price ($X$) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| Demand ($Y$) | 25 | 24 | 20 | 20 | 19 | 17 | 16 | 13 | 10 | 6 |

(P.U., M.A. Econ. 19

10.21 a) Find the correlation co-efficient between $X$ and $Y$, given

| $X$ | 5 | 12 | 4 | 16 | 18 | 21 | 22 | 23 | 25 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 11 | 16 | 15 | 20 | 17 | 19 | 25 | 24 | 21 |

(P.U., M.A. Econ. 2

b) Find the co-efficient of correlation between persons employed and cloth manufactured in a textile mill. Interpret the result

| Persons employed | 137 | 209 | 113 | 189 | 176 | 200 | 219 |
|---|---|---|---|---|---|---|---|
| Cloth manufactured ('000 yds) | 23 | 47 | 22 | 40 | 39 | 51 | 49 |

(P.U., B.A./B.Sc. 1960)

0.22 The following table gives the distribution of the total population and those who are wholly or partially blind among them. Find out if there is any relation between age and blindness.

| Age | No. of Persons in thousand | Blind |
|---|---|---|
| 0 – 9 | 100 | 55 |
| 10 – 19 | 60 | 40 |
| 20 – 29 | 40 | 40 |
| 30 – 39 | 36 | 40 |
| 40 – 49 | 24 | 36 |
| 50 – 59 | 11 | 22 |
| 60 – 69 | 6 | 18 |
| 70 – 79 | 3 | 15 |

(P.U., B.A./B.Sc. 1983)

**Hint.** First calcuiate the numbers of blink per lakh and then correlate with the midpoints of age groups.



23 A computer while calculating the correlation co-efficient between two variables $X$ and $Y$ from 25 pairs of observations obtained the following sums:

$$\sum X = 125,\ \sum X^2 = 650,\ \sum Y = 100,\ \sum Y^2 = 460,\ \sum XY = 508$$

It was, however, later discovered at the time of checking that he had copied down two pairs as $\begin{array}{c|c} X & Y \\ \hline 6 & 14 \\ \hline 8 & 6 \end{array}$ while the correct values were $\begin{array}{c|c} X & Y \\ \hline 8 & 12 \\ 6 & 8 \end{array}$. Obtain the correct value of the co-efficient of correlation. (P.C.S. 1972; P.U., B.A./B.Sc. 1974)

If the equations of the least squares regression lines are:

a) $Y$=20.8–0.219$X$ ($Y$ on $X$), and $X$ = 16.2–0.785$Y$ ($X$ or $Y$);

b) $Y$=2.64+0.648$X$ ($Y$ on $X$), and $X$ = –1.91+0.917$Y$ ($X$ or $Y$);

c) $Y$=1.94$X$+10.83 ($Y$ on $X$), and $X$ = 0.15$Y$+6.18 ($X$ or $Y$);

d) $Y$=15–1.96$X$ ($Y$ on $X$), and $Y$ = 15.91–2.22$X$ ($X$ or $Y$);

Find the product moment coefficient of correlation in each case.

10.25 Find the co-efficient of correlation for the frequency distribution of two variables given by the following table.

Practical

| Y \ X | 5 – 14 | 15 – 24 | 25 – 34 | 35 – 44 | 45 – 54 |
|---|---|---|---|---|---|
| 0 – 9 | 3 | 1 | --- | --- | --- |
| 10 – 19 | 12 | 8 | 14 | 1 | --- |
| 20 – 29 | 2 | 13 | 40 | 12 | 3 |
| 30 – 39 | --- | 3 | 40 | 27 | 7 |
| 40 – 49 | --- | --- | 6 | 4 | 4 |

Also find the regression equation of $Y$ and $X$.

Practical

10.26 Compute correlation co-efficient from the following correlation table for weights and heights of women students.

| Height in inches | Weight in pounds | | | | | |
|---|---|---|---|---|---|---|
| | 90 | 110 | 130 | 150 | 170 | 190 |
| 57 | --- | --- | --- | --- | 1 | --- |
| 60 | 8 | 21 | 8 | 1 | --- | --- |
| 63 | 3 | 50 | 57 | 12 | --- | 2 |
| 66 | 1 | 24 | 54 | 19 | 3 | --- |
| 69 | --- | 1 | 8 | 5 | 3 | --- |
| 72 | --- | --- | 2 | --- | --- | 1 |

(P.U., B.A./B.Sc. 19..)



10.27 a) Describe in simple words the following concepts:

(i) co-efficient of correlation, (ii) scatter diagram, (iii) least squares principle and (iv) estimate of regression co-efficient.

b) The following results were obtained for a bivariate frequency distribution after making the transformation $u = \dfrac{x-1250}{500}$ and $v = \dfrac{y-500}{200}$ : $n = 66, \sum fu = -4, \sum fu^2 = 109, \sum fv = -$..

$\sum fv^2 = 115$, $\sum fuv = 91$. Calculate the coefficient of correlation and obtain the equations of the lines of regression in the simplest form. (P.U., B.A./B.Sc. 19..)

10.28 If $X_1$, $X_2$ and $X_3$ are uncorrelated variables, each having the same standard deviation, obtain the co-efficient of correlation between $(X_1+X_2)$ and $(X_2+X_3)$. (P.U., B.A. Hons. Part-II, 19..)

10.29 a) What is rank correlation? Derive Spearman's co-efficient of rank correlation.

(P.U., B.A./B.Sc. 1960, 71, 82, 84 ..)

b) The ranks of the same 16 students in Mathematics and Physics were as follows:

(1, 1); (2, 10); (3, 3); (4, 4); (5, 5); (6, 7); (7, 2); (8, 6), (9, 8); (10, 11); (11, 15); (12, 9); (13, 14); (14, 12); (15, 16); (16, 13); the two numbers within brackets denoting the ranks of the same student in Maths, and Physics respectively. Calculate the rank correlation co-efficient for proficiencies of this group in two subjects. (P.U., B.A./B.Sc. 19..)

30 a) If $n$ pairs of values of two variables $a$ and $b$ are given, where each variable is ranked in order (1 to $n$), show that the co-efficient of correlation between ranks is given by

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

where $d$ is the difference between the ranks of $a$ and $b$. (P.U., B.A./B.Sc. 1989)

b) Obtain the product moment coefficient of correlation between the following values:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$ | 7.4 | 9.0 | 11.0 | 2.5 | 4.6 | 6.5 |
| $b$ | 8.5 | 6.1 | 2.4 | 6.7 | 12.6 | 3.3 |

Rank the values and hence find a rank correlation coefficient between the two sets.

31 a) Describe circumstances in which you would use: (i) rank correlation co-efficient; (ii) product moment correlation coefficient.

b) The following table shows how 10 students, arranged in alphabetical order, were ranked according to their achievements in both laboratory and lecture portions of a statistics course. Find the co-efficient of rank correlation.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Laboratory | 8 | 3 | 9 | 2 | 7 | 10 | 4 | 6 | 1 | 5 |
| Lecture | 9 | 5 | 10 | 1 | 8 | 7 | 3 | 4 | 2 | 6 |

(P.U., B.A./B.Sc. 1969)

32 Ten competitors in a beauty contest are ranked by three judges in the following order.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| First Judge | 1 | 6 | 5 | 10 | 3 | 2 | 4 | 9 | 7 | 8 |
| Second Judge | 3 | 5 | 8 | 4 | 7 | 10 | 2 | 1 | 6 | 9 |
| Third Judge | 6 | 4 | 9 | 8 | 1 | 2 | 3 | 10 | 5 | 7 |

Use the rank correlation co-efficient to discuss which pair of judges have the nearest approach to common tastes in beauty. (P.U., B.A./B.Sc., 1960, B.Sc. (Hons.) Part-I, 1971)

33 In a painting competition, various entries are ranked by three judges. Use Spearman's rank correlation co-efficient to discuss which pair of judges has the nearest approach to common tastes.

| Entry | A | B | C | D | E | F | G | H | K | L |
|---|---|---|---|---|---|---|---|---|---|---|
| Judge $X$ | 5 | 2 | 6 | 8 | 1 | 7 | 4 | 9 | 3 | 10 |
| Judge $Y$ | 1 | 7 | 6 | 10 | 4 | 5 | 3 | 8 | 2 | 9 |
| Judge $Z$ | 6 | 4 | 9 | 8 | 1 | 2 | 3 | 10 | 5 | 7 |

(P.U., D.St., 1964)

34 a) What are tied ranks? Explain how you would find the co-efficient of rank correlation for tied ranks.

b) Compute the co-efficient of rank correlation for the following ranks;

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $X$ | 8 | 3 | 6.5 | 3 | 6.5 | 9 | 3 | 1 | 5 |
| $Y$ | 8 | 9 | 6.5 | 2.5 | 4 | 5 | 6.5 | 1 | 2.5 |

**10.35** Establish the formula for the 'co-efficient of concordance'. Find the same for the following data:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $Y$ | 7 | 10 | 4 | 1 | 6 | 8 | 9 | 5 | 2 | 3 |
| $Z$ | 9 | 6 | 10 | 3 | 5 | 4 | 7 | 8 | 2 | 1 |

♦♦♦♦♦♦♦♦♦♦♦

Contents

# CHAPTER 11

# MULTIPLE REGRESSION AND CORRELATION

# MULTIPLE REGRESSION AND CORRELATION

## INTRODUCTION

The technique of simple regression which involves one dependent variable and one independent ble is often inadequate in most real-world situations where a variable depends upon two or more endent variables or regressors. For example, the yield of a crop depends upon the fertility of the fertilizer applied, rainfall, quality of seed, etc. Likewise, the systolic blood pressure of a person ds upon one's weight, age, etc. In such cases, the technique of simple regression may be expanded ude several independent variables. A regression which involves two or more independent variables ed a *multiple regression*. Thus, in case of multiple linear

regression where $k$ independent variables influence the dependent variable $Y$, the general format of del is

$$Y_i = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_k X_{ki} + \varepsilon_i, \ (i = 1, 2, ..., n)$$

$\varepsilon_i$ 's are the random errors,

$\alpha$ and $\beta_i$ 's are the unknown population parameters, $\alpha$ is the intercept and $\beta_1$ , $\beta_2$ , ..., $\beta_k$ are the regression coefficients for variables $X_1$, $X_2$, ..., $X_k$ respectively,

$X_{1i}$, $X_{2i}$, ..., $X_{ki}$ are the fixed values of $k$ independent variables, the first of the two subscripts attached to each regressor denotes the variable and the second refers to the observation number,

We assume that

i) $E(\varepsilon_i) = 0$ for all $i$. This implies that for given values of $X_i$'s,

$$E(Y_i) = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_k X_{ki}.$$

ii) $\text{Var}(\varepsilon_i) = E(\varepsilon_i^2) = \sigma^2$ for all $i$, i.e. the variance of error terms is constant.

iii) $E(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$, *i.e.* error terms are independent of each other.

iv) $E(X, \varepsilon_i) = 0$ for all regressors, i.e. $\varepsilon$ and each $X$ variable are independent.

v) $\varepsilon_i$'s are normally distributed with a mean of zero and a constant variance $\sigma^2$.

vi) We assume further in a multiple regression model that there exists no exact linear relationship between any two of the regressors.

The corresponding regression equation estimated from sample data then takes the following form

$$\hat{Y}_i = a + b_1 X_{1i} + b_2 X_{2i} + ... + b_k X_{ki},$$

$a$ and $b_i$'s are the least-squares estimates of the population parameters $\alpha$ and $\beta_i$ 's. The parameters or their estimates $b_i$'s are called the *partial regression co-efficients* as $\beta_i$ or its estimate 2,..., $k$) measures the change in the mean value of $Y$ for a unit change in $X_i$, while all other ables remain unchanged.

## MULTIPLE LINEAR REGRESSION WITH TWO REGRESSORS

For two independent variables $X_1$ and $X_2$, the predicting equation for an individual $Y$ value is

$$Y_i = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i,$$

and the *estimated* multiple linear regression based on sample data is

$$\hat{Y} = a + b_1 X_{1i} + b_2 X_{2i}$$

for a set of $n$ observations, each of which is a number triple ($X_{1i}$, $X_{2i}$, $Y_i$). The error or residual in ea
is given as

$$e_i = Y_i - \hat{Y}_i = Y_i - (a + b_1 X_{1i} + b_2 X_{2i})$$

Using the least-squares criterion, we determine those values of $a$, $b_1$ and $b_2$ which will minimize th
of squared residual, $\sum e_i^2$. To minimize $\sum e_i^2$, we find $\frac{\partial \sum e_i^2}{\partial a}$, $\frac{\partial \sum e_i^2}{\partial b_1}$ and $\frac{\partial \sum e_i^2}{\partial b_2}$ and set equal to

Thus

$$\frac{\partial \sum e_i^2}{\partial a} = -2\sum[Y_i - (a + b_1 X_{1i} + b_2 X_{2i})] = 0,$$

$$\frac{\partial \sum e_i^2}{\partial b_1} = -2\sum X_{1i}[Y_i - (a + b_1 X_{1i} + b_2 X_{2i})] = 0,$$

$$\frac{\partial \sum e_i^2}{\partial b_2} = -2\sum X_{2i}[Y_i - (a + b_1 X_{1i} + b_2 X_{2i})] = 0.$$

Simplifying, we get the following three normal equations

$$\sum Y = na + b_1 \sum X_1 + b_2 \sum X_2,$$
$$\sum X_1 Y = a\sum X_1 + b_1 \sum X_1^2 + b_2 \sum X_1 X_2,$$
$$\sum X_2 Y = a\sum X_2 + b_1 \sum X_1 X_2 + b_2 \sum X_2^2$$

The values of $a$, $b_1$ and $b_2$ are determined by solving these three normal equations sim
and are substituted into

$$\hat{Y} = a + b_1 X_1 + b_2 X_2$$

to obtain the estimated multiple linear egression equation.

**Example 11.1** A statistician wants to predict the incomes of restaurants, using two indep
variables: the number of restaurant employees and restaurant floor area. He collected the follow

| Income (000) $Y$ | Floor area (000 sq. ft) $X_1$ | Number of employees $X_2$ |
|---|---|---|
| 30 | 10 | 15 |
| 22 | 5 | 8 |
| 16 | 10 | 12 |
| 7 | 3 | 7 |
| 14 | 2 | 10 |

Calculate the estimated multiple linear regression equation (*i.e.* $\hat{Y} = a + b_1X_1 + b_2X_2$) for the above

The estimated multiple linear regression equation is

$$\hat{Y} = a + b_1X_1 + b_2X_2$$

$a$, $b_1$ and $b_2$ are the least squares estimates of $\alpha$, $\beta_1$ and $\beta_2$. The three normal equations are:

$$\sum Y = na + b_1 \sum X_1 + b_2 \sum X_2$$
$$\sum X_1Y = a\sum X_1 + b_1 \sum X_1^2 + b_2 \sum X_1X_2,$$
$$\sum X_2Y = a\sum X_2 + b_1 \sum X_1X_2 + b_2 \sum X_2^2.$$

The calculations needed to find $a$, $b_1$ and $b_2$ are showing in the following table:

| $Y$ | $X_1$ | $X_2$ | $X_1^2$ | $X_2^2$ | $X_1X_2$ | $X_1Y$ | $X_2Y$ |
|---|---|---|---|---|---|---|---|
| 30 | 10 | 15 | 100 | 225 | 150 | 300 | 450 |
| 22 | 5 | 8 | 25 | 64 | 40 | 110 | 176 |
| 16 | 10 | 12 | 100 | 144 | 120 | 160 | 192 |
| 7 | 3 | 7 | 9 | 49 | 21 | 21 | 49 |
| 14 | 2 | 10 | 4 | 100 | 20 | 28 | 140 |
| 89 | 30 | 52 | 238 | 582 | 351 | 619 | 1007 |



Substituting the sums in the normal equations, we get

$$5a + 30b_1 + 52b_2 = 89$$

$$30a + 238b_1 + 351b_2 = 619$$

$$52a + 351b_1 + 582b_2 = 1007$$

Solving them simultaneously, we obtain

$$a = -1.33,\ b_1 = 0.38 \text{ and } b_2 = 1.62.$$

Hence the desired estimated multiple linear regression is

$$\hat{Y} = -1.33 + 0.38X_1 + 1.62X_2.$$

**11.2.1 Expression of Multiple Linear Regression in Deviation Form.** The computational cedure is considerably simplified by working with the deviations from the respective means of the ables. With two independent variables, the estimated multiple regression equation is

$$\hat{Y}_i = a + b_1X_{1i} + b_2X_{2i}, \qquad (i = 1, 2, \ldots, n)$$

As the regression equation goes through the point of means, we have

$$\hat{Y} = a + b_1\overline{X}_1 + b_2\overline{X}_2.$$

Subtracting, we get

$$\hat{y}_i = b_1 x_{1i} + b_2 x_{2i},$$

where $\hat{y}_i = \hat{Y}_i - \overline{Y}$, $x_{1i} = X_{1i} - \overline{X}_1$ and $x_{2i} = X_{2i} - \overline{X}_2$.

Then $e_i = y_i - \hat{y}_i = y_i - b_1 x_{1i} - b_2 x_{2i}$, and

$$\sum e_i^2 = \sum (y_i - b_1 x_{1i} - b_2 x_{2i})^2.$$

Differentiating $\sum e_i^2$ partially w.r.t $b_1$ and $b_2$, and equating to zero, we get

$$\frac{\partial \sum e_i^2}{\partial b_1} = -2\sum x_{1i}(y_i - b_1 x_{1i} - b_2 x_{2i}) = 0$$

$$\frac{\partial \sum e_i^2}{\partial b_2} = -2\sum x_{2i}(y_i - b_1 x_{1i} - b_2 x_{2i}) = 0$$

which yield, on simplification, the following two normal equations:

$$\sum x_1 y = b_1 \sum x_1^2 + b_2 \sum x_1 x_2,$$
$$\sum x_2 y = b_1 \sum x_1 x_2 + b_2 \sum x_2^2,$$

where the subscript $i$ is dropped for convenience in printing.

Solving these two equations simultaneously, we get

$$b_1 = \frac{(\sum x_1 y)(\sum x_2^2) - (\sum x_2 y)(\sum x_1 x_2)}{(\sum x_1^2)(\sum x_2^2) - (\sum x_1 x_2)^2}, \text{ and}$$

$$b_2 = \frac{(\sum x_2 y)(\sum x_1^2) - (\sum x_1 y)(\sum x_1 x_2)}{(\sum x_1^2)(\sum x_2^2) - (\sum x_1 x_2)^2}.$$

Then $a$, the constant, is determined by

$$a = \overline{Y} - b_1 \overline{X}_1 + b_2 \overline{X}_2.$$

This is an alternative approach to solving the normal equations directly.

**Example 11.2** Compute the estimated multiple linear regression $\hat{Y} = a + b_1 X_1 + b_2 X_2$ for the data in Example 11.1, using the multiple regression in the deviation form.

In Example 11.1, we found that

$$\sum Y = 89, \sum X_1 = 30, \sum X_2 = 52, \sum X_1^2 = 238, \sum X_2^2 = 582,$$
$$\sum X_1 X_2 = 351, \sum X_1 Y = 619, \sum X_2 Y = 1007 \textit{ and } n = 5.$$

Now we first calculate

$$\sum x_1^2 = \sum X_1^2 - \frac{(\sum X_1)^2}{n} = 238 - \frac{(30)^2}{5} = 58,$$

$$\sum x_2^2 = \sum X_2^2 - \frac{(\sum X_2)^2}{n} = 582 - \frac{(52)^2}{5} = 41.2,$$

$$\sum x_1 x_2 = \sum X_1 X_2 - \frac{(\sum X_1)(\sum X_2)}{n} = 351 - \frac{(30)(52)}{5} = 39,$$

$$\sum x_1 y = \sum X_1 Y - \frac{(\sum X_1)(\sum Y)}{n} = 619 - \frac{(30)(89)}{5} = 85,$$

$$\sum x_2 y = \sum X_2 Y - \frac{(\sum X_2)(\sum Y)}{n} = 1007 - \frac{(52)(89)}{5} = 81.4,$$

Next, we compute the regression co-efficients and constant as follows:

$$b_1 = \frac{(\sum x_1 y)(\sum x_2^2) - (\sum x_2 y)(\sum x_1 x_2)}{(\sum x_1^2)(\sum x_2^2) - (\sum x_1 x_2)^2}$$

$$= \frac{(85)(41.2) - (81.4)(39)}{(58)(41.2) - (39)^2} = \frac{327.4}{868.6} = 0.38,$$

$$b_2 = \frac{(\sum x_2 y)(\sum x_1^2) - (\sum x_1 y)(\sum x_1 x_2)}{(\sum x_1^2)(\sum x_2^2) - (\sum x_1 x_2)^2}$$

$$= \frac{(81.4)(58) - (85)(39)}{(58)(41.2) - (39)^2} = \frac{1406.2}{868.6} = 1.62,$$

and $\quad a = \overline{Y} - b_1 \overline{X}_1 + b_2 \overline{X}_2$

$$= 17.8 - (0.38)(6) - (1.62)(10.4) = -1.33$$

Hence the desired multiple linear regression equation is

$$\hat{Y} = -1.33 + 0.38X_1 + 1.62X_2.$$

It is to be noted that we have exactly the same results as previously.

**11.2.2 Standard Error of Estimate.** The *standard error of estimate* is the standard deviation of ...ple regression. It measures the dispersion of $Y$ values about the population multiple regression ...tion. For a multiple regression with two independent variables $X_1$ and $X_2$, it is denoted symbolically ...$\sigma_{Y.12}$ where the subscripts indicate that $Y$ is regressed against two independent variables $X_1$ and $X_2$. ...lly, the value of $\sigma_{Y.12}$ is not known, it is therefore estimated from sample data.

...*mple standard error of estimate* (unbiased estimate), denoted by $s_{Y.12}$ is given by

$$s_{Y.12} = \sqrt{\frac{\sum(Y - \hat{Y})^2}{n-3}},$$

but it can be computed more readily by using the following relations:

$$\Sigma(Y-\hat{Y})^2 = \Sigma[Y-(a+b_1X_1+b_2X_2)]^2$$
$$= \Sigma Y^2 - a\Sigma Y - b_1\Sigma X_1Y - b_2\Sigma X_2Y.$$

A larger value of $s_{Y.12}$ means that the multiple regression equation is of little use in estimatio prediction.

**11.2.3 Co-efficient of Multiple Determination and Multiple Correlation.** The *co-efficie multiple determination*, which measures as in the case of simple regression, the proportion of variabi the values of the dependent variable $Y$ explained by its linear relation with the independent variables defined by the ratio of the variation in $Y$ explained by the regression equation to the total variation. F multiple regression with two regressors $X_1$ and $X_2$, the co-efficient of multiple determination is de symbolically by $R^2_{Y.12}$ and is computed by

$$R^2_{Y.12} = \frac{\Sigma(\hat{Y}-\overline{Y})^2}{\Sigma(Y-\overline{Y})^2},$$

where $\hat{Y} = a + b_1X_1 + b_2X_2$, but it can be readily computed by using the relation

$$\Sigma(\hat{Y}-\overline{Y})^2 = a\Sigma Y + b_1\Sigma X_1Y + b_2\Sigma X_2Y - (\Sigma Y)^2/n.$$

The co-efficient of multiple determination lies between 0 and 1, and has same meaning as in s linear regression.

The positive square root of the co-efficient of multiple determination, *i.e.* $\sqrt{R^2_{Y.12}}$ is calle *co-efficient* of multiple *correlation*. $R_{Y.12}$ measures the degree of association between $Y$ and bo regressors $X_1$ and $X_2$ combined, and is always taken to be positive.

**Example 11.3** Compute the standard error of estimate, co-efficient of multiple determinatio coefficient of multiple correlation for the data in Example 11.1.

For the data in Example 11.1, we found from the regression calculation, that

$$\Sigma Y = 89, \Sigma Y^2 = 1885, n = 5, a = -1.33.$$

$$\Sigma X_1Y = 619, \Sigma X_2Y = 1007, b_1 = 0.38, b_2 = 1.62$$

Therefore
$$s_{Y.12} = \sqrt{\frac{\Sigma Y^2 - a\Sigma Y - b_1\Sigma X_1Y - b_2\Sigma X_2Y}{n-3}}$$

$$= \sqrt{\frac{1885-(-1.33)(89)-(0.38)(619)-(1.62)(1007)}{5-3}}$$

$$= \sqrt{\frac{136.81}{2}} = \sqrt{68.405} = 8.27$$

which is the standard deviation of the multiple regression.

The coefficient of multiple determination is

$$R^2_{Y.12} = \frac{\Sigma(\hat{Y}-\bar{Y})^2}{\Sigma(Y-\bar{Y})^2}$$

$$= \frac{a\Sigma Y + b_1 \Sigma X_1 Y + b_2 \Sigma X_2 Y - (\Sigma Y)^2/n}{\Sigma Y^2 - (\Sigma Y)^2/n}$$

$$= \frac{(-1.33)(89) + (0.38)(619) + (1.62)(1007) - (89)^2/5}{1885 - (89)^2/5.}$$

$$= \frac{163.99}{300.80} = 0.55$$

This means that 55% of the variability in income is explained by its linear relationship with floor area and the number of employees.

The co-efficient of multiple correlation, $R_{Y.12}$ is

$$R_{Y.12} = \sqrt{0.55} = 0.74.$$

**11.2.4 Subscript Notation.** For the purposes of generalization and change of variables, it is convenient to adopt a notation due to *G*. Udny Yule (1871–1951). This notation involves subscripts. For example, the individual *Y* value in case of the multiple linear regression with two independent variables, is written as

$$Y_i = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$$

Using Yule's notation, this can be written as

$$X_{1i} = \beta_{1.23} + \beta_{12.3} X_{2i} + \beta_{13.2} X_{3i} + \varepsilon_i$$

where the variables are numbered 1, 2 and 3 by the use of subscripts. The subscripted number 1 denotes the dependent variable, 2 and 3 denote the independent variables $X_2$ and $X_3$ respectively, and $\beta_{1.23}$ is the value of $X_1$ when $X_2$ and $X_3$ are both equal to zero.

There are three subscripts attached to each parameter. The subscripts preceding the point are called *primary subscripts* and those following the point are known as *secondary subscripts*. The dependent variable is always indicated by the first primary subscripts, while the second primary subscript indicates the variable to which the β co-efficient is attached. The secondary subscript(s) indicates which other variable(s) has been included in the regression equation. The secondary subscript, if more than one, may be written in any order.

The advantage of this notation is that it indicates the number of variables involved in the regression equation and also shows which is the dependent variable and which are the independent variables.

The estimated multiple regression equation of $X_1$ on $X_2$ and $X_3$ is

$$\hat{X}_1 = b_{1.23} + b_{12.3} X_2 + b_{13.2} X_3.$$

It should be noted that in general $b_{12.3}$ is different from $b_{13.2}$.

Allowing a change of variables, the estimated regression equation of $X_2$ on $X_1$ and $X_3$ is given by

$$\hat{X}_2 = b_{2.13} + b_{23.1}X_3 + b_{21.3}X_1.$$

Similarly, the estimated regression equation of $X_3$ on $X_1$ and $X_2$ is

$$\hat{X}_3 = b_{3.12} + b_{31.2}X_1 + b_{32.1}X_2.$$

If the two variables, measured from their means, be $x_1$ and $x_2$ then the two simple regression equations of $x_1$ on $x_2$ and of $x_2$ and $x_1$ are

$$x_1 = b_{12}x_2 \text{ and } x_2 = b_{21}x_1$$

The residuals may be expressed as

$$x_{1.2} = x_1 - b_{12}x_2 \text{ and } x_{2.1} = x_2 - b_{21}x_1.$$

If $x_1$, $x_2$ and $x_3$ are three variables, measured from their respective means, then the multiple regression equation of $x_1$ on $x_2$ and $x_3$ is

$$x_1 = b_{12.3}x_2 + b_{13.2}x_3$$

and its residual is expressed by

$$x_{1.23} = x_1 - b_{12.3}x_2 - b_{13.2}x_3$$

The two normal equations may be written as

$$\sum x_2x_{1.23} = 0 \text{ and } \sum x_3x_{1.23} = 0.$$

**11.2.5 Properties of Residuals.** The residuals or errors have the following properties:

1. "The sum of the products of corresponding values of a variable and a residual is zero, provided the subscript of the variable is included among the secondary subscripts of the residual."

Let the regression equation (in deviation form) of $x_1$ on $x_2$ and $x_3$ be

$$x_1 = b_{12.3}\,x_2 + b_{13.2}\,x_3.$$

Then the two normal equations for determining the $b$'s are

$$\sum x_2x_{1.23} = 0 = \sum x_3x_{1.23},$$

where $x_{1.23} = x_1 - b_{12.3}x_2 - b_{13.2}x_3$.

Similarly, the normal equations for the regression of $x_2$ on $x_1$ and $x_3$ and of $x_3$ on $x_1$ and $x_2$, are

$$\sum x_1x_{2.13} = 0 = \sum x_3x_{2.13},$$
$$\sum x_1x_{3.12} = 0 = \sum x_2x_{3.12}.$$

2. The sum of the products (or covariance) of two residuals remains unchanged by omitting from one residual any or all of secondary subscripts which are common to both".

Let the residual defined as $x_{1.2} = x_1 - b_{12}x_2$ be considered.

Then $\sum x_{1.2}x_{1.23} = \sum x_{1.23}(x_1 - b_{12}x_2)$.

$$= \sum x_1x_{1.23} - b_{12}\sum x_2x_{1.23}$$

The second term vanishes as $\sum x_2 x_{1.23} = 0$

Then $\sum x_{1.2} x_{1.23} = \sum x_1 x_{1.23}$

Again $\sum x_{1.2} x_{1.23} = \sum x_{1.23}(x_1 - b_{12.3}x_2 - b_{13.2}x_3)$

$$= \sum x_1 x_{1.23} - b_{12.3}\sum x_2 x_{1.23} - b_{13.2}\sum x_3 x_{1.23}$$

Here again the second and third terms vanish due to their being normal equations.

Hence $\sum x_{1.2} x_{1.23} = \sum x_1 x_{1.23}$.

3. "The sum of the products (or covariance) of two residuals is zero provided all the subscripts of one residual are included among the secondary subscripts of the second."

Let us consider the residuals defined by $x_{3.2}$ and $x_{1.23}$.

Then $\sum x_{3.2} x_{1.23} = \sum x_{3.2}(x_1 - b_{12.3}x_2 - b_{13.2}x_3)$

But this vanishes because of normal equation and property 1.

Similarly, $\sum x_{2.3} x_{1.23} = 0$.

**11.2.6 Multiple Regression in terms of Linear Correlation Coefficients.** The multiple ression equation of a variable, say $X_1$, on other variables, say $X_2$ and $X_3$, can be sometimes expressed in s of $r_{12}$, $r_{13}$ and $r_{23}$, the linear correlation coefficients. The sample regression equation (in deviation ) of $x_1$ on $x_2$ and $x_3$ is given by

$$x_1 = b_{12.3}x_2 + b_{13.2}x_3$$

The two normal equations are obtained as

$$\sum x_1 x_2 = b_{12.3}\sum x_2^2 + b_{13.2}\sum x_2 x_3,$$

$$\sum x_1 x_3 = b_{12.3}\sum x_2 x_3 + b_{13.2}\sum x_3^2$$

Let $S_i^2$ be the variance of $x_i$ and let $r_{ij}$ be the linear correlation co-efficient between $x_i$ and $x_j$. Then essing the normal equations in terms of variances and linear correlation co-efficient, we get

$$nr_{12}S_1S_2 = nb_{12.3}S_2^2 + nb_{13.2}r_{23}S_2S_3$$

$$nr_{13}S_1S_3 = nb_{12.3}r_{23}S_2S_3 + nb_{13.2}S_3^2$$

Simplification gives

$$r_{12}S_1 = b_{12.3}S_2 + b_{13.2}r_{23}S_3, \text{ and}$$

$$r_{13}S_1 = b_{12.3}r_{23}S_2 + b_{13.2}S_3$$

Solving these equations simultaneously for $b$'s, we get

$$b_{12.3} = \frac{S_1}{S_2}\left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\right), \text{ and}$$

$$b_{13.2} = \frac{S_1}{S_3}\left(\frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2}\right).$$

Substituting these values in the regression equation, we obtain

$$x_1 = \left(\frac{S_1}{S_2}\right)\left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\right)x_2 + \left(\frac{S_1}{S_3}\right)\left(\frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2}\right)x_3.$$

Or dividing both sides of the equation by $S_1$, we get

$$\frac{x_1}{S_1} = \left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\right)\left(\frac{x_2}{S_2}\right) + \left(\frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2}\right)\left(\frac{x_3}{S_3}\right)$$

as the multiple regression of $x_1$ on $x_2$ and $x_3$ in terms of standard deviations and the linear correlation co-efficients of the variables involved. Similarly, the other two multiple regression equations of $x_2$ on $x_1$ and $x_3$ and of $x_3$ on $x_1$ and $x_2$ are obtained as

$$\frac{x_2}{S_2} = \left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{13}^2}\right)\left(\frac{x_1}{S_1}\right) + \left(\frac{r_{23} - r_{12}r_{13}}{1 - r_{13}^2}\right)\left(\frac{x_3}{S_3}\right), \text{ and}$$

$$\frac{x_3}{S_3} = \left(\frac{r_{13} - r_{12}r_{23}}{1 - r_{12}^2}\right)\left(\frac{x_1}{S_1}\right) + \left(\frac{r_{23} - r_{12}r_{13}}{1 - r_{12}^2}\right)\left(\frac{x_2}{S_2}\right)$$

To obtain the regression equations in terms of original values, we replace $x_1$ by $X_1 - \overline{X}_1$, $x_2$ by $X_2 - \overline{X}_2$, and $x_3$ by $X_3 - \overline{X}_3$ respectively.

## 11.3 MULTIPLE CORRELATION CO-EFFICIENT

The *co-efficient of multiple correlation* measures the degree of relationship between a variable and its estimate from the regression equation. In other words, it is a product moment correlation between a variable, say $x_1$, and its value estimated by the regression equation $x_1 = b_{12.3}x_2 + b_{13.2}x_3$. The co-efficient of multiple correlation between $x_1$ and the variables $x_2$ and $x_3$ combined, is denoted symbolically by $R_{1.23}$.



Let us denote the *estimated value* of $x_1$ by $\hat{x}_1$. Then by definition,

$$R_{1.23} = \frac{Cov(x_1, \hat{x}_1)}{\sqrt{Var(x_1)\, Var(\hat{x}_1)}} = \frac{\sum x_1 \hat{x}_1}{\sqrt{\sum x_1^2 \sum(\hat{x}_1)^2}}$$

$$\begin{aligned}
\text{Now}\quad \sum x_1 \hat{x}_1 &= \sum x_1(x_1 - x_{1.23}) && (\because \hat{x}_1 = x_1 - x_{1.23})\\
&= \sum x_1^2 - \sum x_1 x_{1.23}\\
&= \sum x_1^2 - \sum x_{1.23}x_{1.23} && (\because \sum x_1 x_{1.23} = \sum x_{1.23}x_{1.23})\\
&= n(S_1^2 - S_{1.23}^2),
\end{aligned}$$

where $S^2_{1.23}$ is the sample variance of residuals.

Also
$$\begin{aligned}\Sigma(\hat{x}_1)^2 &= \Sigma(x_1 - x_{1.23})^2\\ &= \sum x_1^2 + \sum x_{1.23}^2 - 2\sum x_1 x_{1.23}\\ &= \sum x_1^2 + \sum x_{1.23}^2 - 2\sum x_{1.23}^2 \qquad (\because \sum x_1 x_{1.23} = \sum x_{1.23}x_{1.23})\\ &= \sum x_1^2 - \sum x_{1.23}^2 = n(S_1^2 - S_{1.23}^2),\end{aligned}$$

and $\sum x_1^2 = nS_1^2$

Substituting these values in the formula, we get

$$R_{1.23} = \frac{S_1^2 - S_{1.23}^2}{S_1\sqrt{S_1^2 - S_{1.23}^2}} = \left(1 - \frac{S_{1.23}^2}{S_1^2}\right)^{1/2}$$

Squaring, we get $R_{1.23}^2 = 1 - \dfrac{S_{1.23}^2}{S_1^2}$.

The quantity $S^2_{1.23}$ can be expressed in terms of the simple correlation co-efficients between the pairs of the variables as below:

$$S_{1.23}^2 = \frac{1}{n}\sum x_{1.23}^2 = \frac{1}{n}\Sigma(x_1 - b_{12.3}x_2 - b_{13.2}x_3)^2$$

$$= \frac{1}{n}\sum x_1(x_1 - b_{12.3}x_2 - b_{13.2}x_3)$$

(second property of residuals)

$$= \frac{1}{n}\sum x_1^2 - b_{12.3}\frac{1}{n}\sum x_1x_2 - b_{13.2}\frac{1}{n}\sum x_1x_3$$
$$= S_1^2 - b_{12.3}S_1S_2r_{12} - b_{13.2}S_1S_3r_{13}$$

Substituting the values of $b_{12.3}$ and $b_{13.2}$ in terms of simple correlation co-efficient and simplifying, we get

$$S_{1.23}^2 = S_1^2\left(\frac{1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{23}r_{13}}{1 - r_{23}^2}\right)$$

Hence
$$R_{1.23}^2 = 1 - \frac{S_1^2(1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{23}r_{13})}{S_1^2(1 - r_{23}^2)}$$
$$= \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{23}r_{13}}{1 - r_{23}^2}, \text{ so that}$$

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{23}r_{13}}{1 - r_{23}^2}}$$

It should be noted that $R_{1.23}$ is necessarily positive or zero as the term $\sum x_1\hat{x}_1$ being equal to $\sum(\hat{x}$ cannot be negative. If $R_{1.23} = 1$, the $S_{1.23}^2 = 0$, *i.e.* all the residuals $x_{1.23}$ are zero; the observed and estimated values of $x_1$ coincide. The multiple correlation in this case, is called *perfect*, indicating a linear relationship between the variables.

Similarly, by the change of variables, we get

$$R_{2.31} = \sqrt{\frac{r_{23}^2 + r_{21}^2 - 2r_{12}r_{23}r_{13}}{1-r_{31}^2}}, \text{ and}$$

$$R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{12}r_{23}r_{31}}{1-r_{12}^2}},$$

**Example 11.4** An instructor of mathematics wished to determine the relationship of grades on final examination to grades on two quizzes given during the semester. Calling $X_1$, $X_2$ and $X_3$ the grades of a student on the first quiz, second quiz and final examination respectively, he made the following computations for a total of 120 students.

$$\begin{array}{lll} \overline{X}_1 = 6.8 & S_1 = 1.0 & r_{12} = 0.60 \\ \overline{X}_2 = 7.0 & S_2 = 0.8 & r_{13} = 0.70 \\ \overline{X}_3 = 74 & S_3 = 9.0 & r_{23} = 0.65 \end{array}$$

a) Find the least-squares regression equation of $X_3$ on $X_1$ and $X_2$.

b) Estimate the final grades of two students who scored respectively (1) 9 and 7, and (2) 4 and on the two quizzes.

c) Compute $R_{3.12}$. (B.Sc. Eng.



a) Since the standard deviations and linear correlation co-efficients are given, therefore the estimated regression equation of $X_3$ on $X_1$ and $X_2$ is

$$\frac{X_3-\overline{X}_3}{S_3} = \left(\frac{r_{13}-r_{12}r_{23}}{1-r_{12}^2}\right)\left(\frac{X_1-\overline{X}_1}{S_1}\right) + \left(\frac{r_{23}-r_{12}r_{13}}{1-r_{12}^2}\right)\left(\frac{X_2-\overline{X}_2}{S_2}\right)$$

Now $$\frac{r_{13}-r_{12}r_{23}}{1-r_{12}^2} = \frac{0.70-(0.60)(0.65)}{1-(0.60)^2} = \frac{0.31}{0.64}, \text{ and}$$

$$\frac{r_{23}-r_{12}r_{13}}{1-r_{12}^2} = \frac{0.65-(0.60)(0.70)}{1-(0.60)^2} = \frac{0.23}{0.64}.$$

Substituting these values, we get

$$\frac{X_3-74}{9.0} = \left(\frac{0.31}{0.64}\right)\left(\frac{X_1-6.8}{1.0}\right) + \left(\frac{0.23}{0.64}\right)\left(\frac{X_2-7.0}{0.8}\right)$$

or $$X_3 - 74 = 4.36\,(X_1 - 6.8) + 4.04\,(X_2 - 7.0)$$

$$= 4.36\,X_1 - 29.648 + 4.04\,X_2 - 28.28$$

r $\quad X_3 = 16.07 + 4.36\,X_1 + 4.04\,X_2$

s the desired least squares regression equation of $X_3$ on $X_1$ and $X_2$.

b) **Student 1:** When $X_1 = 9$ and $X_2 = 7$, we get

$$\hat{X}_3 = 16.07 + 4.36\,(9) + 4.04\,(7) = 83.59 = 84$$

**Student 2:** When $X_1 = 4$ and $X_2 = 8$, we get

$$\hat{X}_3 = 16.07 + 4.36\,(4) + 4.04\,(8) = 65.83 = 66.$$

c) The co-efficient of multiple correlation $R_{3.12}$ is

$$R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{12}r_{23}r_{13}}{1 - r_{12}^2}},$$

$$= \sqrt{\frac{(0.70)^2 + (0.65)^2 - 2(0.60)(0.65)(0.70)}{1 - (0.60)^2}}$$

$$= \sqrt{\frac{0.3665}{0.64}} = \sqrt{0.5727} = 0.757.$$

Relationship b/w any two variables by neglecting the effect of 3rd variable.

## .4 PARTIAL CORRELATION

*A partial correlation* measures the degree of linear relationship between any two variables in a ...ivariable problem under the condition that any common relationship or influence with all other ...ables (or some of them) has been removed. Stated differently, if there are three variables $X_1$, $X_2$ and ... then the correlation between $X_1$ and $X_2$ after removing the linear effect of $X_3$ from $X_1$ and from $X_2$, is ... partial correlation. The sample *co-efficient of partial correlation* measuring the strength of the ...onship (correlation) between $X_1$ and $X_2$, when the influence of $X_3$ has been removed, is demoted ...bolically by $r_{12.3}$. By removing the influence, we mean subtracting the fitted regression $\hat{X}_1$ from the ...ved values $X_1$ obtaining the residual – a part of $X_1$ not explained by $X_3$.

To derive the co-efficient of partial correlation $r_{12.3}$, we use the variables $x_1$, $x_2$ and $x_3$ which are ...ations from their means. The linear regression of $x_1$ on $x_3$ and of $x_2$ on $x_3$ are $x_1 = b_{13}x_3$ and $x_2 = b_{23}x_3$. ...oving the linear effect of $x_3$ from $x_1$ and from $x_2$ and denoting the residuals by $x_{1.3}$ and $x_{2.3}$, we get

$$x_{1.3} = x_1 - b_{13}x_3, \text{ and } x_{2.3} = x_2 - b_{23}x_3.$$

These residuals may be written as

$$x_{1.3} = x_1 - r_{13}\frac{S_1}{S_3}x_3, \text{ and } x_{2.3} = x_2 - r_{23}\frac{S_2}{S_3}x_3.$$

Now the co-efficient of partial correlation is the product moment correlation co-efficient between ...uals $x_{1.3}$ and $x_{2.3}$. Thus by definition

$$r_{12.3} = \frac{\sum x_{1.3}x_{2.3}}{\sqrt{\sum x_{1.3}^2 \sum x_{2.3}^2}}.$$

Now $\quad \sum x_{1.3}x_{2.3} = \sum\left[x_1 - r_{13}\frac{S_1}{S_3}x_3\right]\left[x_2 - r_{23}\frac{S_2}{S_3}x_3\right]$

$$= \sum\left[x_1x_2 - r_{23}\frac{S_2}{S_3}x_1x_3 - r_{13}\frac{S_1}{S_3}x_2x_3 + r_{13}r_{23}\frac{S_1S_2}{S_3^2}x_3^2\right]$$

$$= \sum x_1x_2 - r_{23}\frac{S_2}{S_3}\sum x_1x_3 - r_{13}\frac{S_1}{S_3}\sum x_2x_3 + r_{13}r_{23}\frac{S_1S_2}{S_3^2}\sum x_3^2$$

$$= n\,[r_{12}S_1S_2 - r_{23}r_{13}S_1S_2 - r_{13}r_{23}S_1S_2 + r_{13}r_{23}S_1S_2]$$

$$= n\,S_1S_2\,(r_{12} - r_{13}r_{23})$$

And $\quad \sum x_{1.3}^2 = \sum\left[x_1 - r_{13}\frac{S_1}{S_3}x_3\right]^2$

$$= \sum x_1^2 + r_{13}^2\frac{S_1^2}{S_3^2}\sum x_3^2 - 2r_{13}\frac{S_1}{S_3}\sum x_1x_3$$

$$= n[S_1^2 + r_{13}^2S_1^2 - 2r_{13}^2S_1^2]$$

$$= nS_1^2(1 - r_{13}^2).$$

Similarly, $\sum x_{2.3}^2 = nS_2^2(1 - r_{23}^2)$.

Substituting these values in the formula, we obtain

$$r_{12.3} = \frac{S_1S_2(r_{12} - r_{13}r_{23})}{S_1S_2\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

**Alternatively.** The partial correlation co-efficient between $x_1$ and $x_2$ when the influence of $x_3$ has been eliminated, is also defined as the geometric mean of the regression co-efficient $b_{12.3}$ and $b_{21.3}$ of the two partial regression lines of $x_1$ on $x_2$ and of $x_2$ on $x_1$ respectively, *i.e.*

$$r_{12.3} = \sqrt{b_{12.3} \times b_{21.3}}$$

$$= \sqrt{\frac{S_1}{S_2}\left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\right)\cdot\frac{S_2}{S_1}\left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{13}^2}\right)}$$

$$= \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}} \qquad (r_{12.3} \text{ has the same sign as } b_{12.3} \text{ and } b_{21.3})$$

In a similar way, we can prove that

$$r_{13.2}=\frac{r_{13}-r_{12}r_{23}}{\sqrt{1-r_{12}^2}\sqrt{1-r_{23}^2}},\text{ and } r_{23.1}=\frac{r_{23}-r_{12}r_{13}}{\sqrt{1-r_{12}^2}\sqrt{1-r_{13}^2}}.$$

**Example 11.5** From the following data, determine the linear regression equations of $X_1$ on $X_3$ and $X_2$ on $X_3$.

| | | | | | |
|---|---|---|---|---|---|
| $X_1$ | 7 | 12 | 14 | 17 | 20 |
| $X_2$ | 4 | 7 | 8 | 9 | 12 |
| $X_3$ | 1 | 2 | 4 | 5 | 8 |

Find the deviations of observed values of $X_1$ from the regression, viz. $X_{1.3}$. Repeat the same for $X_2$, to obtain $X_{2.3}$. Determine the simple correlation co-efficient between the two sets of deviations $X_{1.3}$ and $X_{2.3}$. (P.U., B.A./B.Sc. 1977)

The estimated simple regression equation of $X_1$ on $X_3$ is

$$\hat{X}_1=b_{1.3}+b_{13}X_3,$$

where $$b_{13}=\frac{n\sum X_1X_3-(\sum X_1)(\sum X_3)}{n\sum X_3^2-(\sum X_3)^2}\text{ and } b_{1.3}=\overline{X}_1-b_{13}\overline{X}_3.$$

The estimated simple regression equation of $X_2$ on $X_3$ is

$$\hat{X}_2=b_{2.3}+b_{23}X_3,$$

where $$b_{23}=\frac{n\sum X_2X_3-(\sum X_2)(\sum X_3)}{n\sum X_3^2-(\sum X_3)^2}\text{ and } b_{2.3}=\overline{X}_2-b_{23}\overline{X}_3.$$

The computations needed to find the $b$'s are given in the table below:

| $X_1$ | $X_2$ | $X_3$ | $X_1X_3$ | $X_2X_3$ | $X_3^2$ |
|---|---|---|---|---|---|
| 7 | 4 | 1 | 7 | 4 | 1 |
| 12 | 7 | 2 | 24 | 14 | 4 |
| 14 | 8 | 4 | 56 | 32 | 16 |
| 17 | 9 | 5 | 85 | 45 | 25 |
| 20 | 12 | 8 | 160 | 96 | 64 |
| 70 | 40 | 20 | 332 | 191 | 110 |

$$\overline{X}_1=\frac{\sum X_1}{n}=\frac{70}{5}=14,\ \overline{X}_2=\frac{\sum X_2}{n}=\frac{40}{5}=8\text{ and }\overline{X}_3=4.$$

And the regression co-efficient are obtained as

$$b_{13} = \frac{(5)(332)-(70)(20)}{(5)(110)-(20)^2} = \frac{260}{150} = 1.73,$$

$$b_{1.3} = 14-(1.73)(4) = 7.08,$$

$$b_{23} = \frac{(5)(191)-(40)(20)}{(5)(110)-(20)^2} = \frac{155}{150} = 1.03, \textit{ and}$$

$$b_{2.3} = 8-(1.03)(4) = 3.88.$$

Hence the desired regression equations are

$$\hat{X}_1 = 7.08 + 1.73X_3 \text{ and } \hat{X}_2 = 3.88 + 1.03X_3.$$

Next, we compute the residuals $X_{1.3} = X_1 - 7.08 - 1.73\ X_3$ and $X_{2.3} = X_2 - 3.88 - 1.03\ X_3$, and the s correlation between them. The necessary computations are given in the following table:

| $X_1$ | $X_2$ | $X_3$ | $X_{1.3}$ | $X_{2.3}$ | $X_{1.3}X_{2.3}$ | $X_{1.3}^2$ | $X_{2.3}^2$ |
|---|---|---|---|---|---|---|---|
| 7 | 4 | 1 | –1.81 | –0.91 | 1.6471 | 3.2761 | 0.8281 |
| 12 | 7 | 2 | 1.46 | 1.06 | 1.5476 | 2.1316 | 1.1236 |
| 14 | 8 | 4 | 0 | 0 | 0 | 0 | 0 |
| 17 | 9 | 5 | 1.27 | –0.03 | 0.0381 | 1.6129 | 0.0009 |
| 20 | 12 | 8 | –0.92 | –0.12 | 0.1104 | 0.8464 | 0.0144 |
| 70 | 40 | 20 | 0 | 0 | 3.2670 | 7.8670 | 1.9670 |

Hence the co-efficient of correlation between $X_{1.3}$ and $X_{2.3}$, which is the co-efficient of correlation between $X_1$ and $X_2$ when the influence of $X_3$ has been removed, is obtained as

$$r_{12.3} = \frac{\sum X_{1.3}X_{2.3}}{\sqrt{\sum X_{1.3}^2 \sum X_{2.3}^2}} \qquad (\because \sum X_{1.3} = \sum X_{2.3} = 0)$$

$$= \frac{3.2670}{\sqrt{(7.8670)(1.9670)}} = \frac{3.2670}{3.9340} = 0.83$$

**Example 11.6** Given $r_{12}$= 0.492, $r_{13}$= 0.927 and $r_{23}$= 0.758, find all the partial co co-efficients.

We have $$r_{12.3} = \frac{r_{12}-r_{13}r_{23}}{\sqrt{1-r_{13}^2}\sqrt{1-r_{23}^2}} = \frac{0.492-(0.927)(0.758)}{\sqrt{1-(0.927)^2}\sqrt{1-(0.758)^2}}$$

$$= \frac{-0.2107}{\sqrt{0.1407\times 0.4254}} = -0.86;$$

$$r_{23.1}=\frac{r_{23}-r_{21}r_{31}}{\sqrt{1-r_{21}^2}\sqrt{1-r_{31}^2}}=\frac{0.758-(0.492)(0.927)}{\sqrt{1-(0.492)^2}\sqrt{1-(0.927)^2}}$$

$$=\frac{0.302}{\sqrt{0.7579\times0.1407}}=0.92;\text{ and}$$

$$r_{31.2}=\frac{r_{31}-r_{32}r_{12}}{\sqrt{1-r_{32}^2}\sqrt{1-r_{12}^2}}=\frac{0.927-(0.758)(0.492)}{\sqrt{1-(0.758)^2}\sqrt{1-(0.492)^2}}$$

$$=\frac{0.5541}{\sqrt{0.4254\times0.7579}}=0.98.$$

**Example 11.7** Show that if $x_3=ax_1+bx_2$, the three partial correlations are numerically equal to $r_{\ldots 2}$ having the sign of $a$, $r_{32.1}$, the sign of $b$ and $r_{12.3}$, the opposite sign of $a/b$.

... the multiple regression equation $x_3=ax_1+bx_2$, we treat $x_3$ as dependent and $x_1$ and $x_2$ as ...ent variables. Let the three variables be measured from their respective means.

...uaring and summing over all values, we get

$$\sum x_3^2 = a^2\sum x_1^2+b^2\sum x_2^2 \qquad \text{(the product vanishes as } x_1 \text{ and } x_2 \text{ are independent)}$$

$$=n(a^2S_1^2+b^2S_2^2)$$

...ltiplying the given equation by $x_1$ and summing, we have

$$\sum x_1x_3 = a\sum x_1^2 \qquad (\textstyle\sum x_1x_2=0\text{, as } x_1 \text{ and } x_2 \text{ are independent)}$$

$$=naS_1^2$$

...ow $$r_{31}=\frac{\sum x_1x_3}{\sqrt{\sum x_1^2\sum x_3^2}}=\frac{aS_1^2}{\sqrt{S_1^2(a^2S_1^2+b^2S_2^2)}}$$

$$=\frac{aS_1}{\sqrt{a^2S_1^2+b^2S_2^2}}=\frac{aS_1}{w},\text{ where } w^2=a^2S_1^2+b^2S_2^2.$$

...milarly, $r_{23}=\dfrac{bS_2}{w}$ and $r_{12}=0$

...nce $$r_{13.2}=\frac{r_{13}-r_{12}r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{32}^2)}}=\frac{\frac{aS_1}{w}-0}{\sqrt{(1-0)\left(1-\frac{b^2S_2^2}{w^2}\right)}}$$

$$=\frac{a}{+\sqrt{a^2}}=\pm1,\text{ according as } a \text{ is +ve or –ve.}$$

In other words, $r_{13.2}$ has the sign of $a$.

Again $$r_{12.3}=\frac{r_{12}-r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}=\frac{0-\dfrac{aS_1}{w}\cdot\dfrac{bS_2}{w}}{\sqrt{\left(1-\dfrac{a^2S_1^2}{w^2}\right)\left(1-\dfrac{b^2S_2^2}{w^2}\right)}}$$

$$=\frac{-abS_1S_2}{\sqrt{a^2b^2S_1^2S_2^2}}=\frac{-ab}{+\sqrt{a^2b^2}}.$$

$\because$ $a^2b^2$ is always positive, therefore, $\sqrt{a^2b^2}$ is always positive.

Now $ab$ may be positive or negative.

Thus $r_{12.3}$ has the sign opposite to $ab$ or $\dfrac{a}{b}$.

Similarly, $$r_{32.1}=\frac{r_{23}-r_{31}r_{21}}{\sqrt{(1-r_{31}^2)(1-r_{21}^2)}}$$

$$r_{32.1}=\frac{\dfrac{bS_2}{w}-0}{\sqrt{\left(1-\dfrac{a^2S_1^2}{w^2}\right)(1-0)}}$$

$$=\frac{b}{+\sqrt{b^2}}=\pm 1,$$ according as $b$ is +ve or −ve.

Hence the result.

**11.4.1 Relationship between Multiple and Partial Correlation Co-efficients.** The correlation co-efficients can be connected with the various partial correlation co-efficients. For we have shown earlier that

$$1-R_{1.23}^2=\frac{S_{1.23}^2}{S_1^2},$$

where $$nS_{1.23}^2 =\sum x_{1.23}^2=\sum x_{1.2}x_{1.23}$$ (second property of residuals)

$$=\sum x_{1.2}(x_1-b_{12.3}x_2-b_{13.2}x_3)$$

$$=\sum x_{1.2}^2-b_{13.2}\sum x_{1.2}x_{3.2}$$ because of the properties of residuals.

$$=\sum x_{1.2}^2\left[1-b_{13.2}\frac{\sum x_{1.2}x_{3.2}}{\sum x_{1.2}^2}\right]$$

$$=nS_{1.2}^2[1-b_{13.2}b_{31.2}]=nS_{1.2}^2(1-r_{13.2}^2)$$

$$S_{1.23}^2 = S_{1.2}^2(1-r_{13.2}^2)$$
$$= S_1^2(1-r_{12}^2)(1-r_{13.2}^2)$$

$$1-R_{1.23}^2 = (1-r_{12}^2)(1-r_{13.2}^2).$$

ding in the same way, we can find

$$1-R_{1.234}^2 = (1-r_{12}^2)(1-r_{13.2}^2)(1-r_{14.23}^2).$$

## CURVILINEAR REGRESSION

Sometimes a scatter diagram indicates that the relationship between the two variables will be more ely described by a non-linear regression line. When this occurs, either we may transform one or f the variables so that the transformed data appear approximately linear or we may use a ial equation. In the former case, the estimating equation may be an exponential or a logarithmic . In the latter case, the estimating equation may be

$$\hat{Y} = a + bX + cX^2,$$

b and c are the least-squares estimates of the population parameters in

$$E(Y) = \alpha + \beta X + \gamma X^2.$$

ey are determined from the following set of normal equations:

$$\sum Y = na + b\sum X + c\sum X^2$$
$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$
$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4.$$

s quadratic equation may also be changed into a multiple linear form

$$\hat{X}_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3,$$

$\hat{X}_1 = \hat{Y}$, $a_{1.23} = a$, $b_{12.3} = b_1$, $b_{13.2} = c$, $X_2 = X$ and $X_3 = X^2$. A number of other curvilinear are available. The co-efficient of determination and standard error of estimate can be obtained same way as in the case of *linear* regressions.



## EXERCISES

### TIVE

er 'True' or 'False'. If the statement is not true then replace the underlined words with words ake the statement true:

A partial correlation coefficient measures the degree of relationship between a variable and its estimate from the regression line.

ii) A multiple correlation coefficient measures the degree of linear relationship between any two variables in a multivariable problem when the influence with all other variables has been removed.

iii) The multiple correlation coefficient is the square of the coefficient of multiple determination.

iv) The multiple correlation coefficient $R^2$ will be negative in sign when all of the two variable correlation coefficients are negative in sign.

v) The regression sum of squares in case of multiple regression is the explained variation.

vi) For a multiple regression analysis, if $\sum(Y-\bar{Y})^2 = 50$ and $\sum(Y-\hat{Y})^2 = 20$, then the multiple coefficient of determination $R^2$ is equal to 0.70.

vii) The standard error of estimate in multiple regression has $n-k$ degrees of freedom.

viii) The standard error of estimate is a measure of scatter of the observations about the regression line.

ix) The regression coefficients are the other name for multiple regression coefficients.

x) In a multiple regression the addition of new variables will always reduce the standard error of estimate.



b) **MULTIPLE CHOICE QUESTIONS**

i) The range of multiple correlation coefficient is

a) $-1$ to $+1$

b) 0 to $\infty$

c) 0 to 1

d) none of above

ii) The range of partial correlation coefficient is

a) 0 to 1

b) $-1$ to $+1$

c) 0 to $+\infty$

d) $-1$ to 0

iii) If the multiple correlation coefficient $R_{3.12} = 1$, then it implies a

a) perfect relationship

b) high relationship

c) weak linear relationship

d) perfect linear relationship

iv) In the regression analysis, the explained variation of the dependent variable Y is given by

a) $\Sigma(Y-\overline{Y})^2$

b) $\Sigma(Y-\hat{Y})^2$

c) $\Sigma(\hat{Y}-\overline{Y})^2$

d) $\Sigma(Y-\hat{Y})$

v) Which of the following is not a standard deviation?

a) Standard error of the slope coefficient

b) Mean square errors

c) Standard error of estimator

d) Standard deviation of the Y variable

vi) The coefficient of determination in multiple regression is given by

a) $R^2_{Y.13}=1-(SST/SSE)$

b) $R^2_{Y.13}=1-(SSR/SST)$

c) $R^2_{Y.13}=1-(SSE/SSR)$

d) $R^2_{Y.13}=1-(SSE/SST)$

vii) The slope $b_1$ in the multiple regression equation $\hat{Y}=a+b_1X_1+b_2X_2$ measures

a) the amount of variation in $\hat{Y}$ explained by $X_1$

b) the change in $\hat{Y}$ per unit change in $X_1$

c) the change in $\hat{Y}$ per unit change in $X_1$, holding $X_2$ constant

d) the change in $\hat{Y}$ per unit change in $X_2$, holding $X_1$ constant

viii) The predicted value of Y for $X_1 = 1$, $X_2 = 5$, and $X_3 = 10$ by using the regression line

$$\hat{Y}=30-10X_1+18X_2-7.5X_3 \text{ is}$$

a) 45

b) 15

c) 35

d) 50

ix) Which of the following statements remains always true?

a) The coefficient of multiple determination will increase when new variables are added

b) The coefficient of multiple determination will decrease when new variables are added

c) The adjusted coefficient of multiple determination will not decrease when new variables are added

d) Both a and c above

x) Which of the following relationship holds?

a) $r_{13.2} = \sqrt{b_{12.3} \times b_{21.3}}$

b) $r_{13.2} = \sqrt{b_{13.2} \times b_{31.2}}$

c) $r_{13.2} = \sqrt{b_{23.1} \times b_{32.1}}$

d) All of above

## SUBJECTIVE

11.1 a) What is a multiple regression? Explain the basic differences between simple regression and multiple regression.

b) What is meant by the co-efficient of multiple determination and multiple correlation?

c) Explain the assumptions underlying a multiple linear regression model.

11.2 Carryout the necessary computations to obtain the least-squares estimates of the parameters in the multiple regression model $Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$, given

| | | | | | |
|---|---|---|---|---|---|
| $Y$ | 12 | 10 | 9 | 13 | 20 |
| $X_1$ | 2 | 2 | 3 | 4 | 4 |
| $X_2$ | 1 | 1 | 0 | 0 | 3 |

(B.Z.U., M.A. Econ. 19

11.3 Given the data

| | | | | | |
|---|---|---|---|---|---|
| $Y$ | 2 | 5 | 7 | 8 | 5 |
| $X_1$ | 8 | 8 | 6 | 5 | 3 |
| $X_2$ | 0 | 1 | 1 | 3 | 4 |

a) Calculate the estimated regression equation, (*i.e.* $\hat{Y} = a + b_1 X_1 + b_2 X_2$) for the above data.

b) State the meaning of the partial regression co-efficients $b_1$ and $b_2$.

(B.Z.U., M.A. Econ. 19



11.4 Given the following data

| | | | | | | |
|---|---|---|---|---|---|---|
| $X_1$ | 1 | 4 | 1 | 3 | 2 | 4 |
| $X_2$ | 1 | 8 | 3 | 5 | 6 | 10 |
| $X_3$ | 2 | 8 | 1 | 7 | 4 | 6 |

a) Find the least-squares regression line where $X_1$ is the dependent variable and $X_2$ and $X_3$ are independent variables

b) Calculate the standard error of estimate, $s_{1.23}$

c) Calculate the co-efficient of multiple determination and multiple correlation and interpret the result.

The following table shows the corresponding values of three variables $X_1$, $X_2$ and $X_3$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $X_1$ | 3 | 5 | 6 | 8 | 12 | 14 |
| $X_2$ | 16 | 10 | 7 | 4 | 3 | 2 |
| $X_3$ | 90 | 72 | 54 | 42 | 30 | 12 |

a) Find the regression equation of $X_3$ on $X_1$ and $X_2$.

b) Estimate $X_3$ when $X_1 = 10$and $X_2$=6.

c) Compute $R_{3.12}$ and $s_{3.12}$. (I.U., M.Sc. 1991)

The following data were collected to determine a suitable regression equation relating the length of an infant, $Y$(cm), to age, $X_1$ (days), and weight at birth, $X_2$ (kg):

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 57.5 | 52.8 | 61.3 | 67.0 | 53.5 | 62.7 | 56.2 | 68.5 | 69.2 |
| $X_1$ | 78 | 69 | 77 | 88 | 67 | 80 | 74 | 94 | 102 |
| $X_2$ | 2.75 | 2.15 | 4.41 | 5.52 | 3.21 | 4.32 | 2.31 | 4.30 | 3.71 |

a) Fit a least-squares regression equation of the form

$$\hat{Y} = a + b_1 X_1 + b_2 X_2$$

b) Predict the average length of infants who are 75 days old and weighed 3.15 kg at birth.

c) Calculate the standard error of estimate $s_{Y.12}$.

Define the multiple correlation co-efficient and prove that

$$R^2_{1.23} = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{23}r_{31}}{1 - r_{23}^2}$$

Calculate the multiple correlation co-efficient $R_{1.23}$ of $X_1$ on $X_2$ and $X_3$ from the following data:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X_1$ | 4 | 3 | 2 | 4 | 6 | 7 | 8 |
| $X_2$ | 2 | 5 | 3 | 2 | 1 | 4 | 5 |
| $X_3$ | 8 | 10 | 13 | 5 | 17 | 16 | 20 |

(P.U., B.A. (Hons.) Part-I, 1969)

The following data represent concomitant values of three variables.

| | | | | | | |
|---|---|---|---|---|---|---|
| $X_1$ | 32 | 18 | 52 | 16 | 42 | 48 |
| $X_2$ | 3 | 2 | 5 | 1 | 4 | 6 |
| $X_3$ | 2 | 4 | 2 | 5 | 3 | 9 |

Calculate all the multiple correlation coefficients, working out the usual simple correlation co-efficients. (B.Z.U., M.A. Econ. 1991)

b) Given $\overline{X}_1 = 20$, $S_1 = 1.0$, $r_{12} = -0.20$,

$\overline{X}_2 = 36$, $S_2 = 2.0$, $r_{13} = 0.40$,

$\overline{X}_3 = 12$, $S_3 = 1.5$, $r_{23} = 0.50$.

Find the regression equation of $X_3$ on $X_1$ and $X_2$. (P.U., B.A./B.Sc. 19

11.9 a) Distinguish between the simple and the multiple correlation co-efficients.

b) If $b_{ij}$ is the regression co-efficient of $X_i$ on $X_j$, then calculate the multiple correla co-efficient of $X_2$ with $X_1$ and $X_3$, where

$b_{12} = 0.75$, $b_{13} = 0.58$, $b_{21} = 0.88$,

$b_{23} = 0.53$, $b_{31} = 1.68$, and $b_{32} = 1.30$ (P.U., B.A./B.Sc. 19

c) Three variables have in pairs simple correlation coefficients: $r_{12} = 0.60$; $r_{13} =$ 0.7 $r_{23} = 0.65$. Find the multiple correlation coefficient $R_{2.13}$ of $X_2$ on $X_1$ and $X_3$. (P.U., B.A./B.Sc. 19

11.10 a) Three variable have in pairs simple correlation coefficients given by

$r_{12} = 0.8$, $r_{13} = -0.7$, $r_{23} = -0.9$.

Find the multiple correlation co-efficient $R_{1.23}$ of $X_1$ on $X_2$ and $X_3$. (P.U., B.A./B.Sc. 19

b) Calculate the multiple correlation co-efficient $R_{2.13}$ and the partial correlation co-efficie from the values given below:

$b_{12} = -0.1$, $b_{21} = -0.4$, $b_{13} = 0.27$

$b_{31} = 0.6$, $b_{23} = 0.67$, $b_{32} = 0.38$ (P.U., B.A./B.Sc. 19



11.11 a) Explain what is meant by partial correlation. Establish a formula for the co-efficient of par correlation.

b) From the following data, determine the linear regression equations of $X_1$ on $X_3$ and $X_2$ on

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $X_1$ | 5 | 9 | 7 | 10 | 12 | 8 | 6 | 10 |
| $X_2$ | 10 | 12 | 8 | 9 | 11 | 7 | 5 | 8 |
| $X_3$ | 2 | 6 | 4 | 5 | 7 | 6 | 4 | 6 |

Find the deviations of observed values of $X_1$ from the regression equation, viz. $X_{1.3}$. Fin the same for $X_2$, *i.e.* obtain $X_{2.3}$. Determine the simple correlation co-efficient betw two sets of deviations $X_{1.3}$ and $X_{2.3}$.

11.12 The following means, standard deviations and correlations are found for

$X_1$ = Seed-hay crops in cwts. per acre,

$X_2$ = Spring rainfall in inches,

$X_3$ = Accumulated temperature above 42°F in spring in a certain district in England dur years.

$\overline{X}_1 = 28.02,\quad S_1 = 4.42,\quad r_{12} = 0.80,$

$\overline{X}_2 = 4.91,\quad S_2 = 1.10,\quad r_{13} = -0.40,$

$\overline{X}_3 = 594,\quad S_3 = 85,\quad r_{23} = -0.56.$

Find the partial correlation and the regression equation for hay-crop on spring rainfall and accumulated temperature. (P.U., B.A./B.Sc. 1974)

11.13 The following values represent sample values of 450 college students in which the three variables represent marks obtained ($X_1$), general intelligence scores ($X_2$) and hours of study ($X_3$). Find the regression equation for estimating marks obtained. Find all three partial correlations and interpret them in the light of the corresponding simple correlations.

$\overline{X}_1 = 18.5,\quad S_1 = 11.2,\quad r_{12} = 0.60,$

$\overline{X}_2 = 100.6,\quad S_2 = 15.8,\quad r_{13} = 0.32,$

$\overline{X}_3 = 24,\quad S_3 = 6.0,\quad r_{23} = 0.35.$ (P.U., M.A. Stat., 1960)

11.14 a) Prove that a variable and a residual are uncorrelated if the subscript of the variable is included among the secondary subscripts of the residual.

b) Given the equations of the three regression planes as

$x_1 = 0.41\, x_2 + 0.23\, x_3,$

$x_2 = 0.96\, x_1 - 0.025\, x_3,$

$x_3 = 1.04\, x_1 - 0.05\, x_2.$

Calculate the partial correlation co-efficients. Do we have sufficient data to determine the correlation co-efficients $r_{23}$, $r_{31}$ and $r_{12}$? (P.U., B.A. (Hons.) Part-I, 1970)

11.15 a) If $X_1 = a + b_{12.3}\, X_2 + b_{13.2}\, X_3$ and $X_3 = d + b_{32.1}\, X_2 + b_{31.2}\, X_1$ are the regression equations of $X_1$ on $X_2$ and $X_3$, and of $X_3$ on $X_2$ and $X_1$ respectively, prove that $r^2_{13.2} = b_{13.2} \times b_{31.2}$.

b) Is it possible to obtain the following from a set of data?

(i) $r_{12} = 0.6,\ r_{23} = 0.8,\ r_{31} = -0.5.$

(ii) $r_{23} = 0.7,\ r_{13} = -0.4,\ r_{12} = 0.6.$

(iii) $r_{21} = 0.01,\ r_{13} = 0.66,\ r_{23} = -0.70.$

11.16 If $X_1$, $X_2$ and $X_3$ are three correlated variables, where $S_1=1$, $S_2=1.3$, $S_3=1.9$ and $r_{12}=0.370$, $r_{13}= -0.641$, and $r_{23}= -0.736$, find $r_{13.2}$. If $X_4= X_1+ X_2$, obtain $r_{42}$, $r_{43}$ and $r_{43.2}$. Verify that the two partial correlation co-efficients are equal and explain this result.

(M.Sc. Stat., P.U., 1972, I.U., 1990, 92, 94)

11.17 a) Differentiate between multiple correlation and partial correlation.

b) If $R_{1.23} = 1$, prove that (i) $R_{2.13} = 1$ and (ii) $R_{3.12} = 1$.

c) If $R_{1.23} = 0$, does it necessarily follow that $R_{2.13} = 0$?

d) If $r_{12} = r_{23} = r_{13} = r \neq 1$, then show that $R_{1.23} = R_{2.13} = R_{3.12} = \dfrac{r\sqrt{2}}{\sqrt{1+r}}$. Discuss the case when $r=1$. (B.Sc. Eng. 1976)

11.18 a) Show that if $r_{12}$ is zero, $r_{12.3}$ will not be zero unless one at least of $r_{13}$ and $r_{23}$ is zero.

b) If the relation $aX_1 + bX_2 + cX_3 = 0$ holds true for all sets of values of $X_1$, $X_2$ and $X_3$, find out the three partial correlation co-efficients.

11.19 Show that the correlation co-efficient between the residuals $x_{1.23}$ and $x_{2.13}$ is equal and opposite to that between $x_{1.3}$ and $x_{2.3}$. (P.U., M.A. 1963)

**Solution.** The co-efficient of correlation between $x_{1.23}$ and $x_{2.13}$ is given by

$$\frac{Cov\,(x_{1.23}, x_{2.13})}{\sqrt{Var\,(x_{1.23})\,Var(x_{2.13})}} = \frac{1}{n}.\frac{\sum x_{1.23}x_{2.13}}{S_{1.23}\,S_{2.13}}$$

$$= \frac{1}{n}.\frac{\sum x_{2.13}(x_1 - b_{12.3}x_2 - b_{13.2}x_3)}{S_{1.23}\,S_{2.13}}$$

$$= \frac{1}{n}.\frac{0 - b_{12.3}\sum x_{2.13}^2 - 0}{S_{1.23}S_{2.13}} = \frac{b_{12.3}S_{2.13}}{S_{1.23}}$$

Substituting the values of $S_{1.23}$ and $S_{2.13}$ and simplifying, we get

$$Corr. = -b_{12.3}\left\{\frac{S_2\sqrt{1-r_{23}^2}}{S_1\sqrt{1-r_{13}^2}}\right\} = -b_{12.3}\frac{S_{2.3}}{S_{1.3}}$$

Again the co-efficient of correlation between $x_{1.3}$ and $x_{2.3}$ is

$$\frac{Cov\,(x_{1.3}, x_{2.3})}{\sqrt{Var\,(x_{1.3})\,Var(x_{2.3})}} = \frac{1}{n}.\frac{\sum x_{1.3}x_{2.3}}{S_{1.3}\,S_{2.3}} = b_{12.3}\frac{S_{2.3}}{S_{1.3}}$$

Hence the result.

11.20 Using the method of least-squares, fit a quadratic model $Y = \alpha + \beta_1 X + \beta_2 X^2 + \varepsilon$ to the following data:

| X | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| Y | 0.4 | 1.3 | 2.2 | 2.5 | 3.0 |

Also calculate the standard error of estimate.

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Contents

# CHAPTER 12

# CURVE FITTING BY LEAST SQUARES

# CURVE FITTING BY LEAST SQUARES

## .1 INTRODUCTION

Let us suppose that we wish to *approximate* (describe) a certain type of function that best expresses e association that exists between variables. A *scatter plot* of the set of values of the variables makes it ssible to visualize a smooth curve that effectively approximates the given data set. A more useful way represent this sort of approximating curve is by means of an equation or a formula. A term applied to e process of determining the equation and/or estimating the parameters appearing in the equation of roximating curve, is commonly called *curve fitting*.

It is relevant to point out that the relationship between the variables may be *functional* or *ressional*. In functional relationship, a variable $Y$ has a *true* value corresponding to each possible value another variable $X$, *i.e.* there is no question of random variation in the values of $Y$, and we make no babilistic assumptions in this respect. In this chapter, we shall limit our discussion to some functional tionships, *i.e.* problems of approximation and not of regression (already discussed earlier). Such tionships which are common in the natural sciences may be *linear* or *non-linear*.

## 2 APPROXIMATING CURVES AND THE PRINCIPLE OF LEAST SQUARES



The data sets encountered in practice greatly vary in nature. It is therefore necessary to decide h type of approximating curve and equation should be used. For this purpose, some of many mon types of approximating curves and their equations are given below:

| | |
|---|---|
| ight line or linear curve, | $Y = a + bX$ |
| abola of second degree or quadratic curve, | $Y = a + bX + cX^2$ |
| abola of third degree or cubic curve, | $Y = a + bX + cX^2 + dX^3$ |
| ponential curve, | $Y = ab^X$ or $Y = ae^{bX}$ |
| metric or power curve, | $Y = aX^b$ |
| erbola, | $\frac{1}{Y} = a + bX$. |

o on.

these equations, $Y$ is the *dependent* variable and $X$, the *independent* variable. In some situations, ver, the variables $X$ and $Y$ can be reversed.

We may approximate a given set of data by drawing a *free hand curve*, covering most of the points d. But it is clear that different individuals would draw different curves according to their personal ent. Therefore this procedure of fitting a curve is not satisfactory.

The *principle of least squares* is applicable to curve fitting where the purpose is simply one of hing (or approximation) of a set of observations. Accordingly, we choose to determine the values of rameters in the equations of approximating curves so as to make the sum of squares of residuals a um. A residual has been defined as the difference between the observed value and the ponding value of the approximating curve.

**12.2.1 Fitting a Straight Line.** A straight line is the simplest type of approximating curve and its n is written as

$$Y = a + bX$$

e the values of $a$ and $b$ are to be determined.

Given $n$ pairs of observations $[(X_i, Y_i), i = 1, 2, \ldots, n]$ to which we wish to fit a straight line. We e the values of $a$ and $b$ by the *principle of least squares*, which calls for the minimization of $S$,

the sum of squares of the differences between the actual $Y_i$ values and the corresponding values provided by $a + bX_i$. That is we minimize

$$S = \sum_{i=1}^{n}(Y_i - a - bX_i)^2$$

To do so, we need to solve the two equations $\dfrac{\partial S}{\partial a} = 0$ and $\dfrac{\partial S}{\partial b} = 0$.

That is $\quad \dfrac{\partial S}{\partial a} = 2\sum(Y - a - bX)(-1) = 0,$ and

$$\frac{\partial S}{\partial b} = 2\sum(Y - a - bX)(-X) = 0;$$

which on simplification become

$$\sum Y = na + b\sum X$$

$$\sum XY = a\sum X + b\sum X^2.$$

Solving these two normal equations simultaneously, we get

$$b = \frac{n\sum XY - \sum X\sum Y}{n\sum X^2 - (\sum X)^2} \text{ and } a = \overline{Y} - b\overline{X}.$$

The value of $a$ indicates that the least squares line passes through the means of observation $(\overline{X}, \overline{Y})$.

It should be noted that, when the origin is believed to lie on the curve, the straight line to be fitted is simply $Y=bX$ and the sum of squared deviations to be minimized is

$$S = \sum(Y - bX)^2.$$

For a minimum value of $S$, $\dfrac{\partial S}{\partial b}$ must be zero, that is

$$\frac{\partial S}{\partial b} = 2\sum(Y - bX)(-X) = 0, \text{ which gives } \sum XY = b\sum X^2$$

as the *normal equation* and whence $b = \dfrac{\sum XY}{\sum X^2}$.

The sum of squares of residuals for a straight line is

$$S = \sum(Y - a - bX)^2$$

$$= \sum[Y(Y - a - bX)] = \sum Y^2 - a\sum Y - b\sum XY.$$

**Example 12.1** Fit a straight line by the method of least squares to the following data:

| $X$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Y$ | 3 | 4 | 6 | 9 | 10 |

Also find the sum of squares of residuals. (P.U., B.A./B.Sc.

Let the equation of the straight line to be fitted to the data, be $Y=a+bX$, where $a$ and $b$ are to be evaluated.

The normal equations for determining $a$ and $b$ are

$$\sum Y = na + b\sum X,$$

$$\sum XY = a\sum X + b\sum X^2$$

We now calculate $\sum X, \sum X^2, \sum Y$ and $\sum XY$ as below:

| | $X$ | $Y$ | $XY$ | $X^2$ | $Y^2$ |
|---|---|---|---|---|---|
| | 1 | 3 | 3 | 1 | 9 |
| | 2 | 4 | 8 | 4 | 16 |
| | 3 | 6 | 18 | 9 | 36 |
| | 4 | 9 | 36 | 16 | 81 |
| | 5 | 10 | 50 | 25 | 100 |
| $\Sigma$ | 15 | 32 | 115 | 55 | 242 |

Thus the normal equations become

$$5a + 15b = 32$$

$$15a + 55b = 115$$

Solving these two equations simultaneously, we obtain

$$a = 0.7 \text{ and } b = 1.9$$

Hence the equation of the required straight line is

$$Y = 0.7 + 1.9X$$

The sum of squares of residuals is given by

$$S = \sum(Y_i - a - bX_i)^2$$

$$= \sum[Y(Y - a - bX)] = \sum Y^2 - a\sum Y - b\sum XY$$

Hence $\quad S = 242 - 0.7\,(32) - 1.9\,(115)$

$$= 242 - 240.9 = 1.1.$$

**12.2.2 Fitting a Second Degree Parabola.** The simplest type of a *non-linear* approximating curve is the second degree parabola that has the equation

$$Y = a + bX + cX^2$$

where the values of $a$, $b$ and $c$ are to be determined.

Let us suppose that we wish to fit this parabolic curve to n pairs of observations $[(X_i, Y_i), i = 1, 2, \ldots, n]$. Then we need to find those values of $a$, $b$ and $c$ which will minimize the sum of squares of differences between actual $Y$ values and corresponding values obtained by $a+bX+cX^2$. (the principle of least squares). That is we minimize

$$S = \sum(Y_i - a - bX_i - cX_i^2)^2$$

Minimizing $S$, we need to set its partial derivatives *w.r.t a, b* and *c* equal to zero. Thus

$$\frac{\partial S}{\partial a} = 2\sum(Y_i - a - bX_i - cX_i^2)(-1) = 0,$$

$$\frac{\partial S}{\partial b} = 2\sum(Y_i - a - bX_i - cX_i^2)(-X_i) = 0, \text{ and}$$

$$\frac{\partial S}{\partial c} = 2\sum(Y_i - a - bX_i - cX_i^2)(-X_i^2) = 0.$$

Simplifying, we get the following three normal equations

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

These equations are solved simultaneously to determine the values of *a, b* and *c*.

The sum of squares of residuals in case of second degree parabola is given by

$$S = \sum(Y - a - bX - cX^2)^2 = \sum[Y(Y - a - bX - cX^2)]$$

$$= \sum Y^2 - a\sum Y - b\sum XY - c\sum X^2Y$$

**Example 12.2** Fit a second degree parabola to the following data, taking $X$ as independent variable.

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $Y$ | 1 | 1.8 | 1.3 | 2.5 | 6.3 |

(P.U., B.A./B.Sc. 1961, ...)

Let the equation of the second degree parabola be

$$Y = a + bX + cX^2$$

The normal equations are

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

The computations involved are shown in the following table:

| | $X$ | $Y$ | $XY$ | $X^2$ | $X^2Y$ | $X^3$ | $X^4$ |
|---|---|---|---|---|---|---|---|
| | 0 | 1.0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1.8 | 1.8 | 1 | 1.8 | 1 | 1 |
| | 2 | 1.3 | 2.6 | 4 | 5.2 | 8 | 16 |
| | 3 | 2.5 | 7.5 | 9 | 22.5 | 27 | 81 |
| | 4 | 6.3 | 25.2 | 16 | 100.8 | 64 | 256 |
| Total | 10 | 12.9 | 37.1 | 30 | 130.3 | 100 | 354 |

Putting these values in the normal equations, we get

$$12.9 = 5a + 10b + 30c$$

$$37.1 = 10a + 30b + 100c$$

$$130.3 = 30a + 100b + 354c$$

Solving them as simultaneous equations in $a$, $b$ and $c$, we obtain

$$a = 1.42,\ b = -1.07, \text{ and } c = 0.55.$$

Hence the equation of the required second degree parabola is

$$Y = 1.42 - 1.07X + 0.55X^2$$

**Example 12.3** Fit an equation of the form $Y = aX^2 + bX$ to the following data:

| $X$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $Y$ | 1 | 5 | 12 | 20 | 25 | 36 |

Also find the sum of squares of residuals. (P.U., B.A./B.Sc. 1993)

The curve to be fitted is $Y = aX^2 + bX$.

The normal equations are

$$\sum X^2Y = a\sum X^4 + b\sum X^3 \text{ and } \sum XY = a\sum X^3 + b\sum X^2$$



The arithmetic is arranged in the table below:

| $X$ | $Y$ | $X^2$ | $X^3$ | $X^4$ | $XY$ | $X^2Y$ | $Y^2$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 5 | 1 | 1 | 1 | 5 | 5 | 25 |
| 2 | 12 | 4 | 8 | 16 | 24 | 48 | 144 |
| 3 | 20 | 9 | 27 | 81 | 60 | 180 | 400 |
| 4 | 25 | 16 | 64 | 256 | 100 | 400 | 625 |
| 5 | 36 | 25 | 125 | 625 | 180 | 900 | 1296 |
| 15 | 99 | 55 | 225 | 979 | 369 | 1533 | 2491 |

Substitution gives

$$979a + 225b = 1533$$
$$225a + 55b = 369$$

Solving them simultaneously, we get

$$a = 0.4006 \text{ and } b = 5.0703.$$

Hence the desired equation is $Y = 0.40X^2 + 5.07X$.

The sum of squares of residuals is given by

$$S = \sum(Y - aX^2 - bX)^2 = \sum[Y(Y - aX^2 - bX)]$$

$$= \sum Y^2 - a\sum X^2Y - b\sum XY$$
$$= 2491 - (0.40)(1533) - (5.07)(369)$$
$$= 2491 - 613.2 - 1870.83 = 6.97$$

**12.2.3 Fitting of Higher Degree Parabolic Curves.** A parabolic curve of degree $p$, approxim a set of observations $[(X_i, Y_i), i=1, 2, \ldots, n]$ has the equation

$$Y_i = a + bX_i + cX_i^2 + \ldots + kX_i^p$$

where $a, b, c, \ldots, k$ are the unknown quantities and where $k \neq 0$, and $n>p+1$..

The problem is to determine the $(p + 1)$ unknown quantities $a, b, c, \ldots, k$ in such a way th resulting values of $Y_i$ should be as close as possible to the observed values. We, therefore, take the su squares of the residuals, *i.e.*

$$S = \sum_{i=1}^{n}(Y_i - a - bX_i - cX_i^2 - \ldots - kX_i^p)^2$$

which is a function of $a, b, c, \ldots, k$ as $(X_i, Y_i)$ are certain numbers. The *principle of least-squares* ca the selection of that parabolic curve that minimizes $S$, the sum of squares of differences between th values of $Y$ and the corresponding values calculated from the curve. To minimize $S$, we calcu $\frac{\partial S}{\partial a}, \frac{\partial S}{\partial b}, \frac{\partial S}{\partial c}, \ldots, \frac{\partial S}{\partial k}$ and set them equal to zero. Simplification leads to the following $(p+1)$ equ



$$\sum Y = na + b\sum X + c\sum X^2 + \ldots + k\sum X^p$$
$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3 + \ldots + k\sum X^{p+1}$$
$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4 + \ldots + k\sum X^{p+2}$$
$$\vdots$$
$$\sum X^pY = a\sum X^p + b\sum X^{p+1} + c\sum X^{p+2} + \ldots + k\sum X^{2p}$$

These are the *normal equations* for fitting the parabolic curve of degree $p$. Solv simultaneously, we determine $a, b, c, \ldots, k$.

For the particular case, $p = 3$, the normal equations for fitting the cubic $Y_i = a + bX_i + cX_i^2 + dX_i^3$ become

$$\sum Y = na + b\sum X + c\sum X^2 + d\sum X^3$$
$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3 + d\sum X^4$$
$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4 + d\sum X^5$$
$$\sum X^3Y = a\sum X^3 + b\sum X^4 + c\sum X^5 + d\sum X^6$$

Similarly, parabolic curves of higher degree may be fitted.

The sum of squares of residuals in case of cubic parabola is given by

$$S = \sum(Y - a - bX - cX^2 - dX^3)^2$$
$$= \sum Y^2 - a\sum Y - b\sum XY - c\sum X^2Y - d\sum X^3Y$$

**A better fit.** It is important to note that the sum of squares of residuals enables us to make some ... of comparison. A simple way of judging whether a straight line, a quadratic parabola or a cubic ...abola is likely to give the better fit, is to calculate the sum of squares of residuals in each case. The ...aller the sum of squares, the better is the fit.

**12.2.4 Change of Origin and Unit.** The computational labour may be reduced by a suitable ...ice of origin and unit. If the given values of $X_i$ ($i$=1, 2, ..., $n$) are equally spaced with a common ...rval $h$ and $n$ is an *odd* number of values, say, $n$=2$k$+1, the normal equations are simplified by taking ... mid value of $X$ as the origin and the common interval $h$ as unit of measurement. That is, if $X_0$ be the ...value, then $u_i=(X_i - X_0)/h$ takes the values $-k, -(k-1), \ldots, -2, -1; 0, 1, 2, \ldots, (k-1), k$. Hence we get ... $= 0 = \sum u^3 = \ldots$ If instead, $n$ is an *even* number, say $n$=2$k$, we take the origin at the mean of the two ... values of $X$ and $h/2$ as the new unit. The values of $u_i$ then become $-(2k-1), -(2k-3), \ldots, -3, -1, 1, 3,$ ...$2k-3), (2k-1)$, so that $\sum u = 0 = \sum u^3 = \ldots$ (Also see chapter 13).

**Example 12.4** The profits, £$Y$, of a certain company in the $X$th year of its life are given by

| $X$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Y$ | 2500 | 2800 | 3300 | 3900 | 4600 |

Taking $u = X-3$ and $v = (Y-3300)/100$, find the parabolic curve of $v$ on $u$ in the form $v=a+bu+cu^2$ ...educe the curve of $Y$ on $X$. (P.U., B.A./B.Sc. (Hons) 1964)

Since $u$=$X$–3 (given), so we find that the sums of odd powers of $u$ are zero, *i.e.* $\sum u = 0 = \sum u^3$.

The normal equations are thus reduced to

$$\sum v = na + c\sum u^2,$$
$$\sum uv = b\sum u^2,$$
$$\sum u^2v = a\sum u^2 + c\sum u^4.$$

...tions are computed in the following table.

| $X$ | $u$ | $Y$ | $v$ | $u^2$ | $u^4$ | $uv$ | $u^2v$ |
|---|---|---|---|---|---|---|---|
| 1 | –2 | 2500 | –8 | 4 | 16 | 16 | –32 |
| 2 | –1 | 2800 | –5 | 1 | 1 | 5 | –5 |
| 3 | 0 | 3300 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 3900 | 6 | 1 | 1 | 6 | 6 |
| 5 | 2 | 4600 | 13 | 4 | 16 | 26 | 52 |
| $\Sigma$ | 0 | --- | 6 | 10 | 34 | 53 | 21 |

Substituting these values in the normal equations, we get

$$5a + 10c = 9,$$
$$10b = 53,$$
$$10a + 34c = 21.$$

Solving them, we find $a = -0.086$, $b = 5.3$ and $c = 0.643$.

The equation of the required parabolic curve is therefore

$$v = -0.086 + 5.3u + 0.643u^2.$$

In order to deduce the parabolic curve of $Y$ and $X$ we replace $u$ by $X-3$ and $v$ by $\dfrac{Y-3300}{100}$ in the above relation. Thus we obtain

$$\frac{Y-3300}{100} = -0.086 + 5.3(X-3) + 0.643(X-3)^2.$$

Simplifying, we get

$$Y = 2280 + 144.2X + 64.3X^2.$$

as the required parabolic curve of $Y$ on $X$.

## 12.3 EXPONENTIAL CURVES

Equations in which one of the variable quantities occurs as an exponent such as $Y=ac^{bX}$, are called *exponential equations* and graphs showing these equations as *exponential curves*. Exponential curves are used to describe a relation in which one variable forms approximately a geometric progression, while the other forms an arithmetic progression. Data of this hybrid type frequently occurs in the fields of business, banking and economics. In the equation $Y=ac^{bX}$, the letter $c$ is a fixed constant, usually either 10 or $e$ and $a$ and $b$ are determined from the data. If $a$ and $b$ are estimated by method of least-squares, we have to minimize $S$, where

$$S = \sum[Y_i - ae^{bX_i}]^2$$

Finding the partial derivatives with respect to $a$ and $b$, and equating them to zero, we get

$$\frac{\partial S}{\partial a} = 2\sum[Y_i - ae^{bX_i}][-e^{bX_i}] = 0, \text{ and}$$

$$\frac{\partial S}{\partial b} = 2\sum[Y_i - ae^{bX_i}][-ae^{bX_i} \,.\, X_i] = 0.$$

Simplifying, we get

$$\sum Y_i e^{bX_i} = a\sum e^{2bX_i}, \text{ and}$$

$$\sum X_i Y_i e^{bX_i} = a\sum X_i e^{2bX_i}.$$

It is difficult to solve these equations as the solution requires tedious numerical methods. The solution simplifies if the *non-linear* curve may be reduced to the *linear* form by some suitable transformation of one or both the variables. The equation $Y=ae^{bX}$ can be *linearized* by taking the logarithms to the base 10, of both sides. Thus the exponential curve becomes.

$$\log Y = \log a + (b \log e)\, X$$

which may be written as

$$Y' = A + BX$$

Where $Y'$ =log $Y$, $A$=log $a$ and $B$=$b$ log $e$. But this is the equation of a straight line in log $Y$ and $X$. Hence the method of fitting an exponential curve to the observed set of data is to fit a straight line to the logarithms of the $Y_i$s. It should be noted that it is the deviations of log $Y$, and not of $Y$, which are being minimized. It is relevant to point out that log form is better for calculating the values from the fitted curve.

We give some of the more common non-linear curves with suitable transformations to convert them into linear form $Y=a+bX$.

| **Non-linear Form** | **Transformation** | **Linearized Form** |
|---|---|---|
| $Y = aX^b$ | $Y' = \log Y, A = \log a,$ $X' = \log X$ | $Y' = A + bX'$ |
| $Y = ab^X$ | $Y' = \log Y, A = \log a,$ $B = \log b$ | $Y' = A + BX$ |
| $Y = \dfrac{1}{a+bX}$ or $\dfrac{1}{Y} = a + bX$ | $Y' = \dfrac{1}{Y}$ | $Y' = a + bX$ |
| $\dfrac{1}{Y} = a + \dfrac{b}{1+X}$ | $Y' = \dfrac{1}{Y}, X' = \dfrac{1}{1+X}$ | $Y' = a + bX'$ |
| $Y = a + b\sqrt{X}$ | $X' = \sqrt{X}$ | $Y = a + bX'$ |
| $Y = aX^2 + bX$ | $Y' = \dfrac{Y}{X}$ | $Y' = aX + b$ |



It is worth remaking that, if the variable $Y$ incorporates an element of random variation, we introduce a random error term e and the equations become

$$Y = a + bX + e$$

$$Y = a + bX + cX^2 + e \quad \text{etc.}$$

which will be very similar to the regression models discussed in an earlier chapter.

**Example 12.5** Fit an exponential curve $Y = ae^{bX}$ to the following data:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Y$ | 1.6 | 4.5 | 13.8 | 40.2 | 125.0 | 363.0 |

(P.U., B.A./B.Sc. (Hons.), 1962; B.Z.U., 1976)

We can write the given equation as

$$\log Y = \log a + (b \log e)\, X$$

or $\quad Y' = A + BX.$ (From of a st. line)

where $\quad Y' = \log_{10} Y,\ A = \log_{10} a$ and $B = b \log_{10} e.$

As the equation is linear in $Y' = \log Y$ and $X$, therefore the two normal equations are

$$\sum Y' = nA + B\sum X$$

$$\sum XY' = A\sum X + B\sum X^2,$$

The necessary calculations are shown in the following table:

| | $X$ | $Y$ | $X^2$ | $Y'$ (=log $Y$) | $XY'$ |
|---|---|---|---|---|---|
| | 1 | 1.6 | 1 | 0.2041 | 0.2041 |
| | 2 | 4.5 | 4 | 0.6532 | 1.3064 |
| | 3 | 13.8 | 9 | 1.1399 | 3.4197 |
| | 4 | 40.2 | 16 | 1.6042 | 6.4168 |
| | 5 | 125.0 | 25 | 2.0969 | 10.4845 |
| | 6 | 363.0 | 36 | 2.5599 | 15.3594 |
| Total | 21 | ---- | 91 | 8.2582 | 37.1909 |

Substituting these values, the normal equations become

$$6A + 21B = 8.2582$$

$$21A + 91B = 37.1909$$

Solving these equations simultaneously, we get

$A = -0.2805$, and $B = 0.4734$

$\therefore \quad a = \text{anti-log } A = \text{anti-log } (-0.2805)$

$= \text{anti-log } \bar{1}.7195 = 0.52$

and $\quad 0.4343\ b = 0.4734$ or $b = 1.09 \qquad (\because \log_{10} e = 0.4343)$

Hence the equation of the curve fitted to the data is

$$Y = 0.52\,(e)^{1.09X}$$

**Example 12.6** Fit an equation of the form $Y = aX^b$ to the following data:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Y$ | 2.98 | 4.26 | 5.21 | 6.10 | 6.80 | 7.50 |

We may reduce the given equation to a linear form by taking logs to the base 10. Thus

$\log Y = \log a + b \log X$

or $\quad Y' = A + bX'$

where $Y' = \log Y$, $A = \log a$ and $X' = \log X$.

As the equation is linear in $Y' = \log Y$ and $X' = \log X$, therefore the two normal equations are

$$\sum Y' = nA + b\sum X'$$

$$\sum X'Y' = A\sum X' + b\sum X'^2.$$

The following table contains the necessary calculations:

| $X$ | $X'$ (=log $X$) | $Y$ | $Y'$ (=log $Y$) | $X'Y'$ | $X'^2$ |
|---|---|---|---|---|---|
| 1 | 0 | 2.98 | 0.4742 | 0 | 0 |
| 2 | 0.3010 | 4.26 | 0.6294 | 0.189449 | 0.0906 |
| 3 | 0.4771 | 5.21 | 0.7168 | 0.341986 | 0.2276 |
| 4 | 0.6021 | 6.10 | 0.7853 | 0.472829 | 0.3625 |
| 5 | 0.6990 | 6.80 | 0.8325 | 0.581918 | 0.4886 |
| 6 | .0.7782 | 7.50 | 0.8751 | 0.681003 | 0.6056 |
| $\Sigma$ | 2.8574 | --- | 4.3133 | 2.267185 | 1.7749 |

Substituting these summations, we get

$$6A + 2.8574b = 4.3133,$$

$$2.8574A + 1.7749b = 2.2672.$$

Solving them simultaneously and taking antilog of $A$, we get

$$a = 2.978 \text{ and } b = 0.5144$$

Hence the required equation is

$$Y = 2.978\,(X)^{0.5144}$$

$$= 3X^{1/2} \text{ approximately.}$$

## OTHER TYPES OF CURVES

Some other types of curves frequently encountered in applied statistics are the following:

**12.4.1 Modified Exponential Curve.** A *modified exponential* curve, which is obtained by adding ...nt $k$ to an exponential curve, is defined by the relation

$$Y = k + ab^X.$$

...describes a set of data, the absolute growth of which decreases by a constant proportion when ...gative and "$b$" is less than one.

...e first method to fit this curve is to transform it into a linear form by taking logarithms of both ... then to use the least-squares method. But this method is difficult for practical use. In the second ... we need three equations, because there are three constants $k$, $a$ and $b$ which are to be determined. ...rved data are therefore divided into three equal parts, leaving one or two values at the beginning,

if necessary, to obtain the three equations, the criterion of fit being that the three partial totals of the trend values must equal those of the original data.

Let $n$ denote the number of values in each third of the data.

Then the first equation is

$$\sum_1 Y = nk + a + ab + ab^2 + ab^3 + \ldots + ab^{n-1}$$

$$= nk + a[1 + b + b^2 + b^3 + \ldots + b^{n-1}]$$

$$= nk + a\left[\frac{b^n - 1}{b - 1}\right] \qquad (\because \frac{b^n - 1}{b - 1} = 1 + b + b^2 + \ldots + b^{n-1})$$

In a similar way. the other two equations are obtained as

$$\sum_2 Y = nk + ab^n\left(\frac{b^n - 1}{b - 1}\right), \text{ and}$$

$$\sum_3 Y = nk + ab^{2n}\left(\frac{b^n - 1}{b - 1}\right).$$

Now we find the constant $k$, $a$ and $b$.

Subtracting the first equation from the second one, we get

$$\sum_2 Y - \sum_1 Y = a\left(\frac{b^n - 1}{b - 1}\right)(b^n - 1) = a.\frac{(b^n - 1)^2}{b - 1}$$

Again, subtracting the second equation from the third one, we get

$$\sum_3 Y - \sum_2 Y = ab^n\frac{(b^n - 1)^2}{b - 1}.$$

Dividing, we have

$$\frac{\sum_3 Y - \sum_2 Y}{\sum_2 Y - \sum_1 Y} = \left[ab^n.\frac{(b^n - 1)^2}{b - 1}\right] \div \left[a\frac{(b^n - 1)^2}{b - 1}\right] = b^n$$

which gives $$b = \sqrt[n]{\frac{\sum_3 Y - \sum_2 Y}{\sum_2 Y - \sum_1 Y}}.$$

Finally, $$a = (\sum_2 Y - \sum_1 Y)\frac{b - 1}{(b^n - 1)^2}, \text{ and}$$

$$k = \frac{1}{n}\left[\sum_1 Y - \left(\frac{b^n - 1}{b - 1}\right)a\right]$$

**12.4.2 The Compertz Curve,** named after Benjamin Gompertz, is given by the equation

$$Y = ka^{b^X},$$

where $k$, $a$ and $b$ are constants. The equation is changed to *modified exponential equation* by taking logarithms of both sides. Thus

$$\log Y = \log k + b^X \log a$$

or $$Y' = k' + a' b^X$$

where $Y' = \log Y$, $k' = \log k$ and $a' = \log a$.

The Compertz curve, which increases first at an increasing rate, then increases at a decreasing rate until it reaches a maximum level, is frequently used in business and actuarial work.

**12.4.3 The Logistic Curve,** which is widely used to represent growth, is defined by the relation

$$Y = \frac{k}{1+bc^X},$$

or inverting, we get $$\frac{1}{Y} = \frac{1}{k} + \frac{b}{k}c^X$$

$$= k' + ac^X,$$

where $k' = \frac{1}{k}$ and $a = \frac{b}{k}$. This is similar in form to the *modified* exponential curve if $\frac{1}{Y}$ is expressed as a function of $X$ and the same method of fitting may therefore be applied with the reciprocals $\frac{1}{Y_i}$ instead of $Y_i$. The use of this curve to analyse population and biological growth was advocated by Raymond Pearl and L.J. Reed. It should be noted that the *logistic curve* has four different stages, viz., (i) a period of relatively slow growth, (ii) then a period of accelerated growth (iii) then a period of decelerated growth and (iv) finally a period of stability, when the curve does not go up at all. The growth of human population and that of economic variables are appropriately described by the curve as they conform to these stages.

**12.4.4 The Makeham Curve** is defined as

$$Y = ks^X b^{c^X}$$

or in the logarithmic form;

$$\log Y = \log k + X \log s + c^X \log b$$

$$= A + CX + Bc^X$$

where $A = \log k$, $C = \log s$ and $B = \log b$.

This type of curve, which is actually a combination of a straight line with a Compertz curve, is used in actuarial and insurance work.

## 12.5 CRITERIA FOR A SUITABLE CURVE

Frequently, we are required to choose a suitable form of curve to obtain a reasonable fit to the observed sets of data in two variables. The suitability of several curves may be determined by examining the *differences* in the values of the dependent variable *Y*. The first difference, denoted by $\Delta Y$ read as *delta Y*) is defined by $\Delta Y_i = Y_{i+1} - Y_i$, the second difference is defined by $\Delta^2 Y_i = \Delta Y_{i+1} - \Delta Y_i$, and so on. A straight line has the property that its first difference is equal to b (a constant), a second degree parabola has the property that its second difference is equal to $2c$ (a constant) and, in general, a parabolic curve of the *nth* degree has the property that its *nth* differences are constant. Thus we fit

i) a straight line, if the first differences between successive values are approximately constant;
ii) a second degree parabola, if the second differences are approximately constant;
iii) a third degree parabola, if the third differences prove to be constant;
iv) an exponential curve, if the first differences of the logarithms are approximately constant;
v) a log parabola $Y = ab^X c^{X^2}$, if the second differences of the logarithms of the *Y*-values tend to be constant;
vi) a modified exponential curve, if each first difference is a constant percentage of the preceding first difference;
vii) a Gompertz curve, if the first differences of logarithms are changing by a constant percentage;
viii) a logistic curve, if the first differences of the reciprocals are changing by a constant percentage; and
ix) a reciprocal line $\frac{1}{Y} = a + bX$, if the reciprocals of the data show a straight line when plotted on a graph.

## 12.6 FINDING PLAUSIBLE VALUES BY THE PRINCIPLE OF LEAST-SQUARES

The *principle of least squares* can also be applied to find the most satisfactory values of the unknown quantities from a number of independent liner equations in the unknowns when the number of equations is greater than the number of unknowns.

Suppose there are $k$ unknown quantities $X_1, X_2, \ldots, X_k$ and let the $n$ observed relations where $n > k$, be

$$\begin{aligned}
a_1X_1 + b_1X_2 + \ldots + f_1X_k &= l_1 \\
a_2X_1 + b_2X_2 + \ldots + f_2X_k &= l_2 \\
\cdot \qquad \cdot \qquad & \cdot \\
\cdot \qquad \cdot \qquad & \cdot \\
\cdot \qquad \cdot \qquad & \cdot \\
a_nX_1 + b_nX_2 + \ldots + f_nX_k &= l_n
\end{aligned}$$

where *a's*, *b's*, …, *l's* are constants.

When $n>k$, *i.e.* the number of equations is greater than the number of unknowns, there may not exist a unique solution. In such cases, we therefore try to find those values of $X_1, X_2, \ldots, X_k$ which will simultaneously satisfy the given set of independent linear equations as nearly as possible. Such values are obtained by the least-squares method and are called the *best* or *most plausible* values.

The least-squares criterion calls for the selection of those values of $X_1, X_2, \ldots, X_k$ which make the sum of squares of the discrepancies $D_i$'s, also called *errors* or *residuals*, a minimum, where

$$D_i = a_iX_1 + b_iX_2 + \ldots + f_iX_k - l_i, \qquad i = 1, 2, \ldots, n$$

In other words, we have to select those values of $X_1, X_2, \ldots, X_k$ which minimize

$$S = \sum_{i=1}^{n} D_i^2 = \sum_{i=1}^{n}(a_iX_1 + b_iX_2 + \ldots + f_iX_k - l_i)^2$$

It is obvious that $S = f(X_1, X_2, \ldots, X_k)$, that is, the sum of squares of residuals is some function of $X_1, X_2, \ldots, X_k$. If $S$ is to have a minimum value, it is necessary that its partial derivatives with respect to $X_1, X_2, \ldots, X_k$, if they exist, vanish there; hence $X_1, X_2, \ldots, X_k$ must satisfy the equations

$$\frac{\partial S}{\partial X_1} = 2\sum a_i(a_iX_1 + b_iX_2 + \ldots + f_iX_k - l_i) = 0$$

$$\frac{\partial S}{\partial X_2} = 2\sum b_i(a_iX_1 + b_iX_2 + \ldots + f_iX_k - l_i) = 0$$

$$\vdots$$

$$\frac{\partial S}{\partial X_k} = 2\sum f_i(a_iX_1 + b_iX_2 + \ldots + f_iX_k - l_i) = 0$$

The equations given above may be written in the standard form as

$$X_1\sum a_i^2 + X_2\sum a_ib_i + \ldots + X_k\sum a_if_i = \sum a_il_i$$

$$X_1\sum a_ib_i + X_2\sum b_i^2 + \ldots + X_k\sum b_if_i = \sum b_il_i$$

$$\vdots$$

$$X_1\sum a_if_i + X_2\sum b_if_i + \ldots + X_k\sum f_i^2 = \sum f_il_i$$

These simultaneous equations obtained by minimizing process, are the normal equations which are simultaneously solved to obtain the best or the *most plausible values* of $X_1, X_2, \ldots, X_k$.

It should be noted that the normal equations for a set of variables are obtained by multiplying each equation by the co-efficient of the respective variable in the equations and adding them together. This is a convenient way for remembering the normal equations.

**Example 12.7** Apply the principle of least-squares to solve

$$2X + Y = 0,\ 3X - 2Y = 0,\ -X + Y = -2. \qquad \text{(P.U., B.A./B.Sc. 1971, 75)}$$

There are 3 linear equations and 2 unknown variables $X$ and $Y$, therefore we apply the least-squares method to get the most plausible values of $X$ and $Y$.

Now $\quad S = (2X + Y - 0)^2 + (3X - 2Y - 0)^2 + (-X + Y + 2)^2$

The normal equations are $\frac{\partial S}{\partial X} = 0$ and $\frac{\partial S}{\partial Y} = 0$,

i.e. $2(2X + Y) + 3(3X - 2Y) - (-X + Y + 2) = 0$

and $(2X + Y) - 2(3X - 2Y) + (-X + Y + 2) = 0$

or $14X - 5Y = 2$ and $-5X + 6Y = -2$.

Solving these equations simultaneously, we get

$X = 0.034$, and $Y = -0.305$.

**Example 12.8** Find the most plausible values of $X$ and $Y$ from the following equations:

$$X - Y - 3 = 0$$
$$3X + 2Y - 4 = 0$$
$$2X - 3Y + 1 = 0$$

(P.U., B.A. (Hons.) Part-I, 1963, B.A./B.Sc. 197)

We first find the normal equation for $X$. Multiplying each equation by the co-efficient of $X$ in it, we have

$$X - Y = 3$$
$$9X + 6Y = 12$$
$$4X - 6Y = -2$$

Adding, we get $14X - Y = 13$, which is the normal equation for $X$.

We then find the normal equation for $Y$. Again multiplying each equation by the co-efficient of $Y$ in it, we get

$$-X + Y = -3$$
$$6X + 4Y = 8$$
$$-6X + 9Y = 3$$

Adding them together, we have $-X + 14Y = 8$ as the normal equation for $Y$.

Thus the two normal equations are

$$14X - Y = 13$$
$$-X + 14Y = 8$$

Solving them simultaneously, we obtain

$X = 0.97$ and $Y = 0.64$

which is the required solution.



## EXERCISES

12.1 a) What is meant by Curve Fitting? (P.U., B.A./B.Sc.

b) Explain the principle of Least Squares with particular reference to a straight line fit. In what sense, does it give the "best" solution? (P.U., B.A./B.Sc.

c) Fit a straight line to the following data:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Y$ | 2 | 6 | 7 | 8 | 10 | 11 |

Calculate the values of $Y$ for each value of $X$, obtain the values of residuals $e_i$'s and check that $\sum e_i = 0$.

2.2 a) By means of Least Squares, show how a straight line can be fitted to a set of given observations, and obtain the normal equations.

b) Prove that a least squares line always passes through the point ( $\overline{X}$, $\overline{Y}$ ).

(P.U., B.A./B.Sc. 1978)

c) Fit a straight line to the following data and plot on the graph paper the actual and calculated values.

| $X$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 5 | 11 | 8 | 14 | 10 | 16 | 2 | 20 | 15 |

3 a) Write down the equation of a straight line through the origin and derive an expression for finding its slope by the principle of least squares. (P.U., B.A./B.Sc. 1991)

b) Fit a least-squares line to the following data:

| Year ($X$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Output ($Y$) | 1 | 3 | 2 | 4 | 3 | 5 | 4 | 6 | 5 |

Measure the deviations from the fitted line and find the sum of squared deviations.

4 a) Find the normal equations which determine the values of $a$ and $b$ in least squares line $Y = a + bX$; and show that the sum of squares of residuals from the least squares line is given by

$$S = \sum Y^2 - a\sum Y - b\sum XY$$

b) Fit a straight line to the following data:

| $X$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $Y$ | 12 | 15 | 17 | 22 | 24 | 30 |

(P.U., B.A./B.Sc. 1962; 80)

5 a) Fit the least squares line for 20 pairs of observations having $\overline{X} = 2$, $\overline{Y} = 8$, $\sum X^2 = 180$ and $\sum XY = 404$. (P.U., B.A./B.Sc. 1986)

b) Given

| $X$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Y$ | 8 | 9 | 13 | 18 | 27 |

Fit $Y=a+bX$ by least-squares and estimate $Y$ for $X$=6. Also fit $X=c+dY$ and use this equation to estimate $Y$ for $X$=6. Account for the difference in two estimates.

(P.U., B.A. (Hons.) Part-II, 1963-S)

12.6 a) Find the normal equations for $a$, $b$ and $c$ that will minimize

$$S = \sum[Y - (a + bX + cX^2)]^2$$

b) Show that the sum of squares of residuals for a second degree parabola is

$$S = \sum Y^2 - a\sum Y - b\sum XY - c\sum X^2 Y$$

c) Fit a parabola of the form $Y=a+bX+cX^2$ to the data:

| $X$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $Y$ | −5 | −2 | 1 | 2 | 1 |

12.7 a) By means of the principle of least squares, show how a parabola of second order can b fitted to a set of $n$ observation $(X_i, Y_i)$ and obtain the normal equations.

b) For 5 pairs of observations, it is given that A.M. of $X$ series is 2 and A.M. of $Y$ series is 1 is also known that

$$\sum X^2 = 30, \sum X^3 = 100, \sum X^4 = 354, \sum XY = 242, \sum X^2 Y = 850$$

Fit a second degree parabola, taking X as the independent variable.

(P.U., B.A./B.Sc.

c) Fit a second degree parabola to the following data:

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $Y$ | 1 | 5 | 10 | 22 | 38 |

(P.U., B.A. (Part-I),

12.8 Fit a second degree parabola to the following seven pairs of values:

| $X$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|---|
| $Y$ | 1.1 | 1.3 | 1.6 | 2.0 | 2.7 | 3.4 | 4.1 |

(P.U., B.A./B.Sc.

12.9 Fit a second degree equation to the following data:

| $X$ | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 2.4 | 4.8 | 8.3 | 9.5 | 11.2 | 24.3 | 22.2 | 21.2 | 25.4 |

(P.U., B.A. (Hons.) Part-I,

12.10 The profits, £$Y$, of a certain company in the $X$th year of its life are given by:

| $X$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Y$ | 1250 | 1400 | 1650 | 1950 | 2300 |

Taking $u = X - 3$, $v = (Y - 1650)/50$, show that the parabolic curve of $v$ on $u$ is

$$v + 0.086 = 5.30\,u + 0.643\,u^2,$$

and deduce that the parabolic curve of $Y$ on $X$ is

$$Y = 1140 + 72.14X + 32.14X^2.$$

(P.U., B.A./B.Sc.

2.11 Fit, by the method of least-squares,

i) the straight line of best fit,

ii) the 2nd degree parabola of best fit, to the following data:

| $X$ | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 |
|---|---|---|---|---|---|---|---|---|
| $Y$ | 240 | 315 | 403 | 450 | 488 | 520 | 525 | 532 |

Also calculate the sum of squares of residuals in the two cases.

12 Fit a straight line and parabolas of the second and third degrees to the following data, taking $X$ to be the independent variable;

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $Y$ | 1 | 1.8 | 1.3 | 2.5 | 6.3 |

and calculate the sum of squares of residuals in the three cases.

13 a) You are given data in two variables $X$ and $Y$ and you have to take a decision about fitting a suitable trend. How will you proceed?

(P.U., B.A./B.Sc. 1987)

b) Given the following pairs of values of $X$ and $Y$.

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $Y$ | 10 | 17 | 28 | 43 | 62 |

Fit a suitable curve. (P.U., B.A./B.Sc. 1976)

4 a) Explain the principle of least squares and use it to obtain the normal equations when a cubic parabola is fitted to $n$ pairs of observations.

b) Fit a curve of the form $Y=ab^X$ to the following data in which $Y$ represents the number of bacteria per unit volume existing in a culture at the end of $X$ hours:

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $Y$ | 73 | 91 | 112 | 131 | 162 |

Estimate the value of $Y$ when $X$=5 and 6.

(P.C.S. 1972; P.U., B.A./B.Sc. 1978)

5 The number ($Y$) of bacteria per unit volume present in a culture after $X$ hours is given in the following table:

| No. of hours ($X$) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| No. of bacteria per unit volume ($Y$) | 32 | 47 | 65 | 92 | 132 | 190 | 275 |

Fit a least-squares curve having the form $Y=ab^X$ to the data. Estimate the value of $Y$ when $X$=7.

(P.U., B.A./B.Sc. 1969, 79, 80)

12.16 Fit a simple exponential to the following data for a growing plant by taking the logarithms of the exponential equation.

| Day | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| Height | 0.75 | 1.20 | 1.75 | 2.50 | 3.45 | 4.70 | 6.20 | 8.25 | 11.50 |

12.17 Fit a curve of the type $Y=ab^X$ to the following data:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $Y$ | 10 | 12.2 | 14.5 | 17.3 | 21.0 | 25.0 | 29.0 |

12.18 The following data represent the enrolments at a small liberal arts college during the past seven years:

| $X$ (years) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $Y$ (enrolments) | 304 | 341 | 393 | 457 | 548 | 670 | 882 |

Use the method of least-squares to estimate a curve of the form $Y=ab^X$ and predict the enrolment years from now. (P.U., B.A./B.Sc. 19

12.19 a) Given $n$=8, $\sum X=16$, $\sum X^2=204$, $\sum X^3=582$, $\sum \log Y$=23, $\sum X \log Y$=104. Fit a suitable curve. (P.U., B.A./B.Sc. Hons. 19

b) Fit a curve of form $Y=a+b\sqrt{X}$ to the following data:

| $X$ | 1.20 | 2.50 | 3.40 | 4.70 | 5.30 |
|---|---|---|---|---|---|
| $Y$ | 6.33 | 8.03 | 8.95 | 10.09 | 10.56 |

12.20 Fit a curve $Y=aX^b$ to the following data:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Y$ | 1200 | 900 | 600 | 200 | 110 | 50 |

12.21 Fit a curve of the form $Y=aX^b$ to the following data on the unit cost in dollars of producing certain electronic components and the number of units produced.

| Lot size ($X$) | 50 | 100 | 250 | 500 | 1000 |
|---|---|---|---|---|---|
| Unit cost ($Y$) | 108 | 53 | 24 | 9 | 5 |

Use the result to estimate the unit cost for a lot of 400 components.

(P.U., B.A./B.Sc.

12:22 It is thought that two physical quantities $X$ and $Y$ should be connected by a relation of the form $Y=aX^n$. The experimental values are:

| $X$ | 0.5 | 1.5 | 2.5 | 5.0 | 10.0 |
|---|---|---|---|---|---|
| $Y$ | 3.4 | 7.0 | 12.8 | 29.8 | 68.2 |

Find the best values of $a$ and $n$. (P.U., B.A./B.Sc.

2.23 The discharge of a capacitor through a resistance gave he following results:

| $t$ (seconds) | 0.5 | 0.8 | 1.4 | 2.0 | 2.5 |
|---|---|---|---|---|---|
| $v$ (volts) | 9.1 | 8.5 | 7.5 | 6.7 | 6.1 |

Fit a curve of the type $v = ae^{bt}$ to these data.

2.24 a) Fit a curve of the type $Y=ae^{bX}$ to the following data:

| $X$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Y$ | 27 | 73 | 200 | 545 | 1484 |

where $e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

b) Obtain the values of $Y$ from the approximating line for various values of $X$. Do the deviations of the observed values of $Y$ from the corresponding calculated values add to zero? Explain your result.

(P.U., B.A./B.Sc. 1977)

25 Estimate the constant of Pareto Curve, $n = AX^{-a}$, which fits the data below:

| Income (£$X$) | Number ($n$) |
|---|---|
| 150 | 14,000,000 |
| 500 | 825,000 |
| 1,000 | 173,000 |
| 2,000 | 35,500 |



26 The pressure ($p$) of a gas and its volume ($v$) are known to be related by an equation of the form $pv^{\gamma}$ = constant. From the following data, find the value of $\gamma$ by fitting a straight line to the logarithms of $p$ and $v$, taking $p$ to be the independent variable.

| $p$ (kg. per sq. cm) | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|
| $v$ (litres) | 1.62 | 1.00 | 0.75 | 0.62 | 0.52 | 0.46 |

27 a) Derive the *least-squares* equations for fitting a curve of the type $\dfrac{1}{Y} = a + bX$ to a set of $n$ observations. Also find the values of $a$ and $b$.

b) Fit a reciprocal curve $\dfrac{1}{Y} = a + bX$ to the following data:

| $X$ | 0 | 1 | 4 | 6 | 12 | 16 |
|---|---|---|---|---|---|---|
| $Y$ | 10 | 8 | 5 | 4 | 2.5 | 2 |

28 a) Find the normal equations for determining $a$, $b$ and $c$ from the linear equation $Y = a + bX_1 + cX_2$.

b) Find the least-squares fit $Y = a + bX_1 + cX_2$, given

| | | | | | |
|---|---|---|---|---|---|
| $Y$ | 2 | 5 | 7 | 8 | 5 |
| $X_1$ | 8 | 8 | 6 | 5 | 3 |
| $X_2$ | 0 | 1 | 1 | 3 | 4 |

12.29 a) What is the modified exponential curve? Describe the method of fitting it.
(P.U., B.A. (Hons.) Part-II, 196

b) Derive the least-squares equations for fitting a modified exponential, $Y = c + ae^{bX}$ to a set of $n$ observations, and indicate why these equations would be difficult to solve.

12.30 Write a critical note on the law of growth as portrayed by the logistic curve and the Gomper curve. (P.U., B.A./B.Sc. 196

12.31 Use the "principle of least-squares" to find the normal equations when the number of equations is greater than the number of unknown quantities.
(P.U., B.A./B.Sc. 1981, 84, 86,

12.32 a) Explain the method of *least-squares*. Apply it to solve the equations

$X + 7Y = 17,\ 2X - Y = 0,\ 3X - 2Y = -1$ (P.U., B.A./B.Sc. 19

b) Find the most plausible values of $X$ and $Y$ from the following equations. Also compute the sum of squares of residuals.

$2X + Y = 4.8,$ $-X + 3Y = 6.3$

$3X - 2Y = -2.1,$ $3X + 2Y = 8.0,$ (P.U., B.A./B.Sc. 1981,

12.33 a) Find the most plausible values of $X$ and $Y$ from the following equations:

$X + Y = 3.01,$ $2X - Y = 0.03$

$X + 3Y = 7.02,$ $3X + Y = 4.97$

b) Obtain the best possible values of $X$ and $Y$ from

$2X + Y = 4,$ $3X - Y = 10.02,$

$X + 2Y = 5.02,$ $3X + 2Y = 0.97.$ (P.U., B.A./B.Sc.

12.34 Form normal equations and solve

$X + 2Y + Z = 1,$ $2X + Y + Z = 4,$

$-X + Y + 2Z = 3,$ $4X + 2Y - 5Z = -7.$ (P.U., B.A./B.Sc. 1962, 6

12.35 Find the most plausible values of $X$, $Y$ and $Z$ from the following equations:

$X - Y + 2Z = 3,$ $3X + 2Y - 5Z = 5,$

$4X + Y + 4Z = 21,$ $-X + 3Y + 3Z = 14.$

(P.U., B.A./B.Sc.

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Contents

# CHAPTER 13

# TIME SERIES ANALYSIS

# TIME SERIES ANALYSIS

## 13.1 INTRODUCTION

A *time series* consists of numerical data collected, observed or recorded at more or less regular intervals of time each hour, day, month, quarter or year. More specifically, it is any set of data in which observations are arranged in a chronological order. Examples of time series are the hourly temperature recorded at a locality for a period of years, the weekly prices of wheat in Lahore, the monthly consumption of electricity in a certain town, the monthly total of passengers carried by rail, the quarterly sales of a certain fertilizer, the annual rainfall at Karachi for a number of years, the enrolment of students in a college or university over a number of years and so forth.

The *analysis of a time* series is a process by which a set of observations in a time series is analysed. Time series analysis is rather a difficult topic but we shall limit our discussion to the basics of time series analysis.

The observations in a time series, denoted by $Y_1, Y_2, \ldots, Y_t, \ldots$, are usually made at equally spaced points of time or they are associated with equal intervals of time ($t$). Given an observed time series, the first step in analyzing a time series is to plot the given series on a graph taking time intervals ($t$) along the $X$-axis, as the independent variable, and the observed values ($Y_t$) on the $Y$-axis, as the dependent variable. Such a graph will show various types of fluctuations and other points of interest.

It is worthwhile to note that the middle of the period is taken to represent the data for that period. For example, the yearly data corresponds to June 30 or July 1, the middle of a calendar year and monthly data to the middle of the month, *i.e.* the 15th day of the month.

**Example 13.1** The following table shows the number of bags (hundreds) of fertilizer sold by a certain dealer. Plot these data as a time series and comment on the graph.

| Year | Quarters | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| 2001 | 72 | 98 | 79 | 106 |
| 2002 | 79 | 122 | 101 | 143 |
| 2003 | 94 | 141 | 128 | 160 |
| 2004 | 125 | 143 | 135 | 187 |

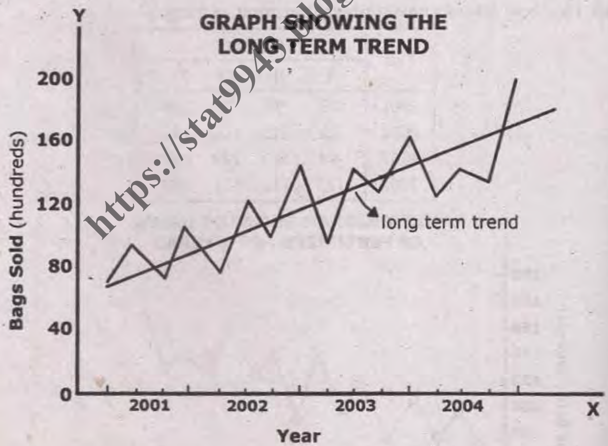


The *historigram* obtained by plotting the given time series, shows that the sales have risen for the second quarter, have fallen for the third quarter and then have risen to a higher point for the fourth quarter every year. The graph also reveals that the sales, on the whole, have risen over 4 years. The graph further suggests that by smoothing out the irregularities, the annual rate at which the sales have increased, may be ascertained.

## 13.2 COMPONENTS OF A TIME SERIES

A *typical* time series may be regarded as composed of four basic types of movements, usually called *components* of a time series. The four components are: *secular trend* ($T$), *seasonal variations* ($S$), *cyclical fluctuations* ($C$) and *irregular or random variations* ($I$). These components are assumed to be the outcomes of distinct causes of variation. All four of these components are not necessarily present in all time series occurring in practice. Let us discuss each of these components in turn.

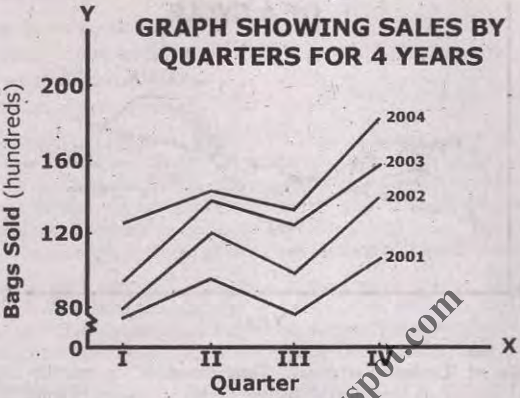
**13.2.1 Secular Trend.** A *secular trend* ($T$) is a long-term movement that persists for many years and indicates the general direction of the change of observed values. In other words, it refers to a smooth broad movement of a time series in the same direction, showing a gradual rise or fall within the data. The secular trend generally dominates other variations in the long run and covers a fairly long period of time, not less than 10 years. The long-term trend is a peculiar characteristic of most of the economic variables such as sales (see figure),





prices, industrial production, capital formation, etc. Analysing the trend component helps is ascertaining the rate of change to be used for further estimates. It also helps in business planning and in studying other variations.

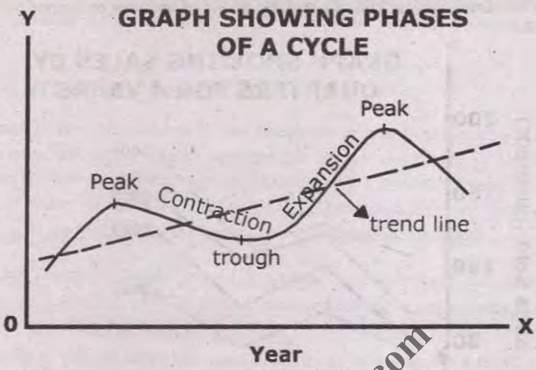
**13.2.2 Seasonal Variations.** The *seasonal variations* (*S*), which are mainly, caused by the change in seasons, are short-term movements occurring in a periodic manner. These fluctuations are repeated with more or less the same intensity within a specific period of one year or shorter (see figure).




e main causes for seasonal variations are the weather condition, the religious festivals and the social stoms. Examples of seasonal variations are the prices of wheat which fall after the harvesting season d rise before the sowing time, the sales of soft drinks which are high in the summer and low in the nter, investments in Savings Certificates which are high in the months of May and June and low in er months, and so forth. The concept of *seasonal* variation is customarily broadened to include the re or less regular fluctuations of shorter duration occurring within a day, a week, a month, a quarter d so forth. Examples of such variations are the daily variations in temperature or the monthly variations Bank deposits.

**13.2.3 Cyclical Fluctuations.** The *cyclical fluctuations* (*C*) are the long-period oscillations about ong-tem trend, which tend to occur in a more or less regular pattern over a period of certain number ears. The so-called *business cycles* which represent alternating period of prosperity and depression, de an important example of cyclical movements. A *cycle,* as it is known, is said to be completed beginning with a *peak* (a *peak* is a value which is greater than the two-neighbouring values), the ing curve reaches a low point, called a *trough* (a trough is a value which is lower than the two bouring values) and then rising again reaches the next peak. The period *either* from peak to peak *or* rough to trough, is usually referred to as the *duration* of a cycle. Cycles have a duration of ere from two to ten years or even a longer period. In general, a complete business cycle has the ing four *phases:* (i) the period of *prosperity*, (ii) the period of *contraction,* (iii) the period of

*recession or depression* and (iv) the period of *recovery or expansion*, which finally develops into a period of prosperity.




**13.2.4 Irregular or Random Variations.** These variations are irregular and unsystematic in nature. They occur in a completely unpredictable manner as they are caused by some unusual events such as floods, droughts, strikes, fires, earthquakes, wars, locusts, political events and the like. These variations are also known as the *accidental, residual or erratic* variations. It is difficult to make a study of such non-recurring variations, though they can be easily identified.



## 13.3 TIME SERIES DECOMPOSITION

A time series analysis is mainly concerned with the *decomposition* of the observed series data into its components so as to estimate their separate effects. To do this, we must make assumptions about the relationship existing among the various components. Accordingly, these components are assumed to have either the *multiplicative relationship* (also called *multiplicative model*) or the *additive relationship* or *model*.

In multiplicative (decomposition) model, we assume that each observed value $Y_t$ at any time can be determined by the product of the measures of all the four components $T$, $C$, $S$ and $I$. Symbolically,

$$Y_t = T \times S \times C \times I, = TSCI,$$

where the observed values and $T$-values are stated in original units but the other components, *i.e.* $S$, $C$ and $I$ are expressed in percentages. On the other hand, in the additive (decomposition) model, each value that we observe, is thought of as being a sum of all the four components, *i.e.*

$$Y_t = T + S + C + I,$$

where the components $T$, $S$, $C$ and $I$ are assumed to be mutually independent. Before analyzing a time series, it is often desirable to adjust the data for calendar variations, for holidays, for price changes and so forth. Such adjustments help in removing the effects of certain false differences.

## 13.4 ANALYSING THE SECULAR TREND

The analysis of a trend component involves its measurement and/or elimination from an observed time series data. To measure a trend which can be represented as a straight line or some type of smooth curve, the following methods are used.

i) The method of freehand curve.
ii) The method of semi-averages.
iii) The method of moving averages.
iv) The method of least-squares.

**13.4.1 The Method of Freehand Curve.** Plot the given data on a graph paper and join the plotted points by segments of straight line. Observe the up and down movements on the graph and draw a smooth curve or a straight line freehand passing through the plotted points in a way such that the general direction of change in values is indicated. This line may also be drawn by a transparent ruler or by stretching a piece of thread through the central region of the plotted points. The line smoothes out short-term fluctuations. The trend values for the given periods can be read from the graph.

This method is simple and quick. It will be a close approximation to a mathematically based trend if drawn with care. It has certain disadvantages. It is a rough and subjective method. As the drawing of trends depends on individual judgment and experience, different persons will draw the graphical trend at different positions with different slopes. Moreover, considerable practice is needed to make a good fit.



**13.4.2 The Method of Semi-Averages.** Divide the values in the series into two equal parts, excluding the middle value in each half or omitting it altogether when the number of values is odd. Find the average of the values in each part and place the average values against the respective midpoints of the two parts. Plot these two average values on the graph of the original values, draw a straight line connecting the two points and extend the line to cover the whole series. This is the *semi-average* trend line which is to be used to read or to compute the trend values.

This method is simple and quick. It gives an entirely objective result when the trend is a straight line. It has two disadvantages. The arithmetic mean which is used to average the observed values, is greatly affected by abnormally small or large values. The method is only suitable when the trend is linear or nearly linear.

**Example 13.2** . Compute and insert on graphs the semi-average trends for the following series:

a) Annual profits in thousands of rupees in a certain business

| Year | 1973 | 1974 | 1975 | 1976 | 1977 | 1978 | 1979 | 1980 |
|---|---|---|---|---|---|---|---|---|
| Profit | 85 | 97 | 100 | 90 | 83 | 105 | 112 | 120 |

b) Indoor patients ('000) treated in hospitals in the Punjab

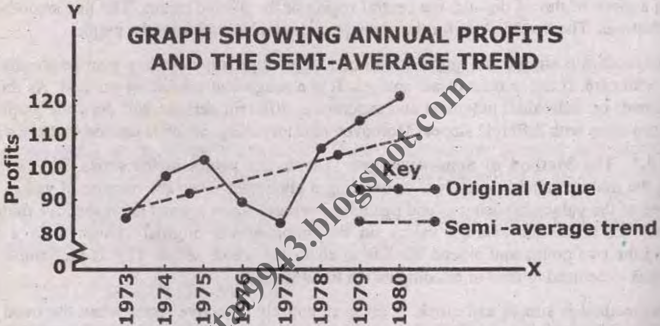
| Year | 1967 | 1968 | 1969 | 1970 | 1971 | 1972 | 1973 | 1974 | 1975 |
|---|---|---|---|---|---|---|---|---|---|
| Patients | 276 | 270 | 260 | 286 | 302 | 321 | 351 | 348 | 346 |

(*Source*: Bureau of Statistics, Govt. of the Punjab, 1977)

a) The data are divided into two equal parts, each consisting of 4 years. Having found the simple averages of the two parts, the first average value is placed opposite the middle of 1974 and 1975, and the second average value is placed opposite the middle of 1978 and 1979. These

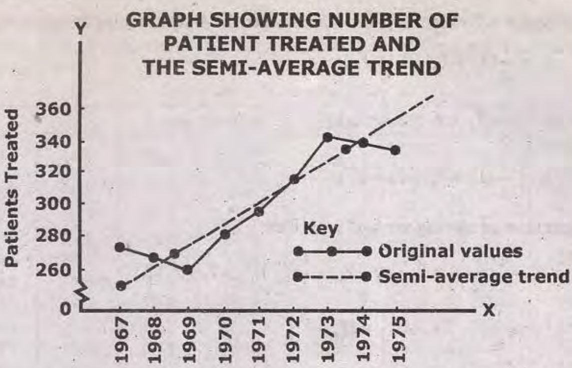
values are then plotted on the graph of the observed series. The line joining these points give the semi-average trend.

| Year | Profit | Total | Averages |
|---|---|---|---|
| 1973 | 85 | | |
| 1974 | 97 | 372 | $372 \div 4 = 93$ |
| 1975 | 100 | | |
| 1976 | 90 | | |
| 1977 | 83 | | |
| 1978 | 105 | 420 | $420 \div 4 = 105$ |
| 1979 | 112 | | |
| 1980 | 120 | | |




b) Here the number of years is *odd*, therefore the middle value is omitted in order to divide the values in the series into two equal parts. Having computed the simple averages of the two parts, they are placed opposite the respective midpoints of the two parts. They are placed on the graph of the original values. The line connecting these points gives the semi-average trend.

| Year | Profit | Total | Averages |
|---|---|---|---|
| 1967 | 276 | | |
| 1968 | 270 | 1092 | $1092 \div 4 = 273$ |
| 1969 | 260 | | |
| 1970 | 286 | | |
| 1971 | --- | | |
| 1972 | 321 | | |
| 1973 | 351 | 1366 | $1366 \div 4 = 341.5$ |
| 1974 | 348 | | |
| 1975 | 346 | | |




**13.4.3 The Method of Moving Averages.** The *k-period moving averages* are defined as the erages calculated by using the $k$ consecutive values of the observed series, then repeating the operation dropping one value at the beginning and including the first value after the preceding total, and so on, *ving on* one value to calculate each successive average. This process is continued till the last $k$ secutive values have been averaged.

Symbolically, the values of the $k$-period *moving averages*, denoted by $a$'s, will be as given below:

$$a_1 = \frac{1}{k}\sum_{t=1}^{k} Y_t,\ a_2 = \frac{1}{k}\sum_{t=2}^{k+1} Y_t,\ a_3 = \frac{1}{k}\sum_{t=3}^{k+2} Y_t, \text{ and so on.}$$

In practice, the moving averages may also be obtained by the relations:

$$a_2 = a_1 + \frac{Y_{k+1} - Y_1}{k},\ a_3 = a_2 + \frac{Y_{k+2} - Y_2}{k}, \quad \text{and so on.}$$

$k$-period moving average is placed against the middle of its time period. It is relevant to note that the age will correspond directly to the observed value in the series when $k$ is odd and when $k$ is even, the ng average will be placed at points which will be located between two periods. It is then necessary to these averages so that they should coincide in time with the observed values in the series. To shift, ically speaking to *centre*, each average, a 2-period moving average of the already computed $k$-period ng average is calculated. Then it is called a *k-period centred moving* average.

For the purposes of illustration, let us first choose $k$ to be odd, say, $k$=3 years. Then we compute the moving averages as

$$a_1 = \frac{1}{3}(Y_1 + Y_2 + Y_3),$$

$$a_2 = \frac{1}{3}(Y_2 + Y_3 + Y_4),$$

$$a_3 = \frac{1}{3}(Y_3 + Y_4 + Y_5),$$

on. The averages so obtained are placed opposite the middle year of each group *i.e.* opposite the ear, 3rd year, 4th year, and so on. This process is continued till the last 3 consecutive values have eraged.

Next, we choose $k$ to be even, say, $k = 4$ years. Then the 4-year moving averages are computed as

$$a_1 = \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4),$$

$$a_2 = \frac{1}{4}(Y_2 + Y_3 + Y_4 + Y_5),$$

$$a_3 = \frac{1}{4}(Y_3 + Y_4 + Y_5 + Y_6),$$

and so on. A 4-year *centred* moving average, $a'$, is then

$$a_1' = \frac{1}{2}\left[\frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4) + \frac{1}{4}(Y_2 + Y_3 + Y_4 + Y_5)\right]$$

$$= \frac{1}{8}\left[Y_1 + 2Y_2 + 2Y_3 + 2Y_4 + Y_5\right]$$

That is, a 4-year *centred* moving average is clearly equivalent to a 5-year *weighted* moving average with weights 1, 2, 2, 2, 1 respectively. Similarly, the 12-month *centred* moving average can be obtained by adding the observations for the 13 consecutive months with the 11 central months being counted twice and then dividing the weighted sum by the weights, *i.e.* 24. In general, a $k$-period ($k$ is even) centred moving average is equivalent to a $(k + 1)$-period weighted moving average where the $(k - 1)$ central periods are given double weights.



These average values are plotted on the graph of the original values and the line connecting these points is the moving average trend, which smoothes out periodic fluctuations of the seasonal and cyclical types present in the series. The line may be continued in the general direction indicated by the original plotting for the purposes of future estimates.

The period of the moving averages is chosen in a way that the period over which fluctuations occur, is covered. This is usually equal to the period of at least one cycle. For example, when the yearly data are made up of four quarters or 12 months, 4-quarter or 12-months moving averages should be computed. When the trend is of the exponential type, the moving average are to be computed by using the geometric mean instead of the arithmetic mean.

The method of moving average is easy and simple. The moving averages of appropriate period estimate the combined effects of trend and cyclical components and give a smooth version of the series by removing the seasonal and other effects. It has a number of disadvantages. The moving average trend does not provide values at the end or the beginning of the original series by half the period. The simple moving averages are unduly affected by large $Y$-values. This disadvantage may be reduced by using the geometric mean. In the absence of an appropriate period, moving averages may make the resulting series more cyclical than the observed series. It also has the drawback that the method does not provide a mathematical expression for the trend, therefore the estimation by extrapolation is subjective.

**Example 13.3** Compute (i) 3-year moving averages, (ii) 5-year moving averages and (iii) 7-year moving averages for the following data and show the moving-average trends on the graph.

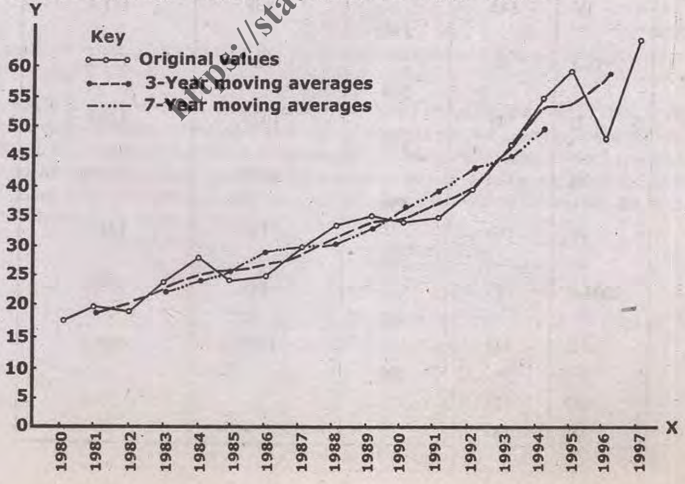
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Year: | 1980 | 1981 | 1982 | 1983 | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 |
| Values: | 18.0 | 20.5 | 19.6 | 24.2 | 27.8 | 25.1 | 25.9 | 30.2 | 34.0 | 36.0 |
| Year: | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | | |
| Values: | 35.0 | 35.8 | 40.9 | 48.4 | 55.6 | 60.4 | 48.6 | 68.7 | | |

The computation of the 3, 5 or 7-year simple moving averages consists of two steps; (i) computation of a 3, 5, or 7-year moving totals and (ii) division of these moving totals by 3, 5, or 7 to obtain moving averages.

*Computation of Moving Averages*

| Year ($t$) | Values ($Y_t$) | 3-year moving | | 5-year moving | | 7-year moving | |
|---|---|---|---|---|---|---|---|
| | | Total | Average (trend) | Total | Average (trend) | Total | Average (trend) |
| 1980 | 18.0 | --- | --- | --- | --- | --- | --- |
| 1981 | 20.5 | 58.1 | 19.4 | --- | --- | --- | --- |
| 1982 | 19.6 | 64.3 | 21.4 | 110.1 | 22.0 | --- | --- |
| 1983 | 24.2 | 71.6 | 23.9 | 117.2 | 23.4 | 161.1 | 23.0 |
| 1984 | 27.8 | 77.1 | 25.7 | 122.6 | 24.5 | 173.3 | 24.8 |
| 1985 | 25.1 | 78.8 | 26.3 | 133.2 | 26.6 | 186.8 | 26.7 |
| 1986 | 25.9 | 81.2 | 27.1 | 143.0 | 28.6 | 203.2 | 29.0 |
| 1987 | 30.2 | 90.1 | 30.0 | 151.2 | 30.2 | 214.0 | 30.6 |
| 1988 | 34.0 | 100.2 | 33.4 | 161.1 | 32.2 | 222.2 | 31.7 |
| 1989 | 36.0 | 105.0 | 35.0 | 171.0 | 34.2 | 237.8 | 34.0 |
| 1990 | 35.0 | 106.8 | 35.6 | 181.7 | 36.3 | 260.3 | 37.2 |
| 1991 | 35.8 | 111.7 | 37.2 | 196.1 | 39.2 | 385.7 | 40.8 |
| 1992 | 40.9 | 125.1 | 41.7 | 215.7 | 43.1 | 312.1 | 44.6 |
| 1993 | 48.4 | 144.9 | 48.3 | 241.1 | 48.2 | 324.7 | 46.4 |
| 1994 | 55.6 | 164.4 | 54.8 | 253.9 | 50.8 | 358.4 | 51.2 |
| 1995 | 60.4 | 164.6 | 54.9 | 281.7 | 56.3 | --- | --- |
| 1996 | 48.6 | 177.7 | 59.2 | --- | --- | --- | --- |
| 1997 | 68.7 | --- | --- | --- | --- | --- | --- |

The data and the moving averages trends are plotted on the graph as below:



**Example 13.4** Compute the 4-qurater *centred* moving averages for the time series given in Example 13.1, and show them and the data on a graph.

The 4-quarter *centred* moving averages appear in the last column of the following table. These are the trend values. They are plotted on the graph of original observations and the trend is shown dashed (see page 487).

| Year and quarter (1) | *Y*-values (2) | 4-quarter moving totals (3) | 4-quarter *centred* moving totals (4) | 4-quarter *centred* moving averages (5) = col(4) ÷ 8 |
|---|---|---|---|---|
| 2001–I | 72 | | -- | -- |
| II | 98 | | -- | -- |
| | | 355 | | |
| III | 79 | | 717 | 89.6 |
| | | 362 | | |
| IV | 106 | | 748 | 93.5 |
| | | 386 | | |
| 2002–I | 79 | | 794 | 99.2 |
| | | 408 | | |
| II | 122 | | 853 | 106.6 |
| | | 445 | | |
| III | 101 | | 905 | 113.1 |
| | | 460 | | |
| IV | 143 | | 939 | 117.4 |
| | | 479 | | |
| 2003–I | 94 | | 985 | 123.1 |
| | | 506 | | |
| II | 141 | | 1029 | 128.6 |
| | | 523 | | |
| III | 128 | | 1077 | 134.6 |
| | | 554 | | |
| IV | 160 | | 1110 | 138.8 |
| | | 556 | | |
| 2004–I | 125 | | 1119 | 139.9 |
| | | 563 | | |
| II | 143 | | 1153 | 144.1 |
| | | 590 | | |
| III | 135 | | -- | -- |
| IV | 187 | | -- | -- |

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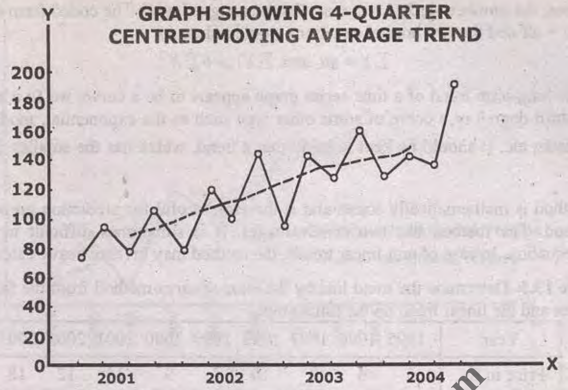
**GRAPH SHOWING 4-QUARTER CENTRED MOVING AVERAGE TREND**




**13.4.4 The Method of Least-Squares.** A trend line, which can be described by a mathematical equation of the form of a straight line, a parabola or an exponential, can be measured using the method of *least-squares*, by letting time to be the independent variable. Let us suppose that the long-term trend appears to be linear. Then the equation of the *least squares* linear trend would be

$$\hat{Y}_t = a + bt.$$

The values of $a$ and $b$, the two constants, are determined by solving the following two normal equations simultaneously:

$$\sum Y = na + b\sum t,$$
$$\sum tY = a\sum t + b\sum t^2.$$

The trend values are computed from this equation by substituting the values of $t$ and are plotted on the graph of the original values.

To simplify computations, the time variable ($t$) is *coded* by taking the time deviations from the middle point of the periods and the coded time period is denoted by $X$. For example, when the number of years is odd, the middle year is taken as 0 or the origin, and when the number of years is even, $X$=0 is taken at the middle of the two middle years. In the latter case, for convenience, the fractional values of $X$ may be multiplied by 2 to express the time units in half year units. For an illustration, the coded year numbers $X$ are shown in the table:

| Odd number of years | | Even number of years | | |
|---|---|---|---|---|
| Year ($t$) | Coded year ($X$) | Year ($t$) | Coded year (X) | $X$ in half-year units |
| 1978 | –2 | 1978 | –2.5 | –5 |
| 1979 | –1 | 1979 | –1.5 | –3 |
| 1980 | 0 | 1980 | –0.5 | –1 |
| 1981 | 1 | 1981 | +0.5 | +1 |
| 1982 | 2 | 1982 | 1.5 | 3 |
| | | 1983 | 2.5 | 5 |

Sometimes, the numbers 1, 2, 3, ..., may also be assigned to $X$. The coded form of the linear trend would be $Y_t = a + bX$ and the two normal equations would reduce to

$$\sum Y = na \text{ and } \sum XY = b\sum X^2$$

When the long-term trend of a time series graph appears to be a curve, we fit a parabolic curve of the second or third degree or a curve of some other type such as the exponential, modified exponential, Gompertz, logistic, etc. It should be kept in mind that a trend, which has the smaller $\sum(Y - Y_t)^2$, is the better fit.

This method is mathematically sound and is therefore useful for prediction purposes. It is also an objective method. The method has two disadvantages. It is sometimes difficult to choose a proper mathematical equation. In case of non-linear trends, the method may involve heavy calculations.

**Example 13.5** Determine the trend line by the *least-squares* method from the following data. Plot the actual values and the linear trend on the same graph.

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 |
|---|---|---|---|---|---|---|---|---|---|
| Price in Rs. | 3 | 6 | 2 | 10 | 7 | 9 | 14 | 12 | 18 |

Let the equation of the linear trend be $Y_t = a + bX$. Since the number of years in the data is odd, we can assign $X$=0 to the middle year 1999, $X$ = 1, 2, 3, 4 to the successive years and $X$ = –1, –2, –3, –4 to the preceding years. The normal equations then reduce to

$$\sum Y = na \quad \text{and} \quad \sum XY = b\sum X^2 \qquad (\because \sum X = 0)$$

The arithmetic can be arranged as in the table below:

| Year ($t$) | Coded year ($X$) | $Y$ | $XY$ | $X^2$ | Trend $Y_t$=9+1.7$X$ |
|---|---|---|---|---|---|
| 1995 | –4 | 3 | –12 | 16 | 2.2 |
| 1996 | –3 | 6 | –18 | 9 | 3.9 |
| 1997 | –2 | 2 | –4 | 4 | 5.6 |
| 1998 | –1 | 10 | –10 | 1 | 7.3 |
| 1999 | 0 | 7 | 0 | 0 | 9.0 |
| 2000 | 1 | 9 | 9 | 1 | 10.7 |
| 2001 | 2 | 14 | 28 | 4 | 12.4 |
| 2002 | 3 | 12 | 36 | 9 | 14.1 |
| 2003 | 4 | 18 | 72 | 16 | 15.8 |
| $\sum$ | 0 | 81 | 101 | 60 | 81.0 |

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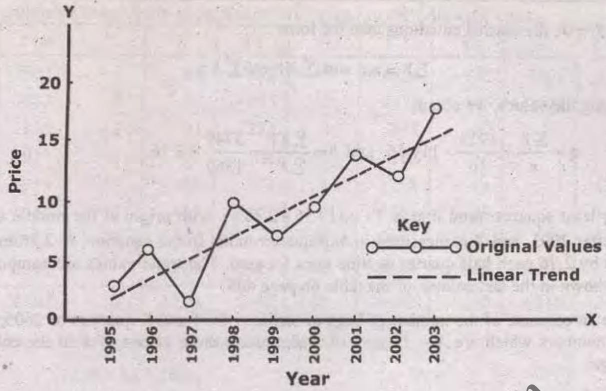
Substituting, we get $a = \frac{\sum Y}{n} = \frac{81}{9} = 9$ and $b = \frac{\sum XY}{\sum X^2} = \frac{101}{60} = 1.7.$

Hence the required equation of the linear trend is $\hat{Y}_t = 9 + 1.7X$,

where the middle year 1999 is taken as $X$=0 and units of $X$ are 1 year. The trend values are computed by substituting the values of $X$ corresponding to various years into the equation, and are shown in the last column of the table shown above. The total of the trend values agree with the total of the original values. The original values and the trend line are graphed on next page:




**Example 13.6** Use the method of least-squares to determine the linear trend line for the data in mple 13.1. Compute the trend values and make an estimate of the number of bags of fertilizer sold in e 1st and 2nd quarters of 2005.

Let the equation of the linear trend line by $\hat{Y}_t = a + bX$. Since the number of quarters in the erved series is *even*, therefore the middle point of the two middle quarters, *i.e.* the middle of rter IV of 2002 and quarter I of 2003, is taken as $X = 0$. We then assign $X = -1, -3, -5, \ldots$, to the eding quarters and $X = 1, 3, 5, \ldots$, to the following quarters. For computing the values of $a$ and $b$ in rend line, the necessary calculations are shown in the table below:

"Computation for linear trend with even number of values".

| Year by quarter | Coded quarter ($X$) | $Y$ | $XY$ | $X^2$ | Trend $\hat{Y}_t = 119.56 + 2.76X$ |
|---|---|---|---|---|---|
| 2001–I | –15 | 72 | –1080 | 225 | 78.2 |
| II | –13 | 98 | –1274 | 169 | 83.7 |
| III | –11 | 79 | –869 | 121 | 89.2 |
| IV | –9 | 106 | –954 | 81 | 94.7 |
| 2002–I | –7 | 79 | –553 | 49 | 100.2 |
| II | –5 | 122 | –610 | 25 | 105.8 |
| III | –3 | 101 | –303 | 9 | 111.3 |
| IV | –1 | 143 | –143 | 1 | 116.8 |
| 2003–I | +1 | 94 | +94 | 1 | 122.3 |
| II | 3 | 141 | 423 | 9 | 127.8 |
| III | 5 | 128 | 640 | 25 | 133.4 |
| IV | 7 | 160 | 1120 | 49 | 138.9 |
| 2004–I | 9 | 125 | 1125 | 81 | 144.4 |
| II | 11 | 143 | 1573 | 121 | 149.4 |
| III | 13 | 135 | 1755 | 169 | 155.4 |
| IV | 15 | 187 | 2805 | 225 | 161.0 |
| $\Sigma$ | 0 | 1913 | 3749 | 1360 | 1913.0 |

Since $\sum X = 0$, the normal equations take the form

$$\sum Y = na \text{ and } \sum XY = b\sum X^2$$

Substituting the values, we obtain

$$a = \frac{\sum Y}{n} = \frac{1913}{16} = 119.56, \text{ and } b = \frac{\sum XY}{\sum X^2} = \frac{3749}{1360} = 2.76.$$

Thus the least squares trend line is $\hat{Y}_t = 119.56 + 2.76X$, with origin at the middle of IV 2002 and I quarter 2003, and $X$ is measured in *half* quarter units. In the equation, $b$=2.76 means trend line rises by 2.76 each half quarter as time goes forward. The trend values are computed trend line and shown in the last column of the table on page 489.

To make an estimate of the number of bags of fertilizer for I and II quarters of 2005, we coded quarter numbers which are $X$ = 17 and 19. Substituting these values of $X$ in the equation trend line, we get

I quarter 2005; $\hat{Y}_t = 119.56 + 2.76\,(17)$

$= 166.5$ (hundred bags) and

II quarter 2005; $\hat{Y}_t = 119.56 + 2.76\,(19)$

$= 172.0$ (hundred bags).



**Example 13.7** Fit a second degree trend curve (parabola) to the following data and comp trend values.

| Year | 1931 | 1933 | 1935 | 1937 | 1939 | 1941 | 1943 | 1945 |
|---|---|---|---|---|---|---|---|---|
| Index of Wholesale Prices | 96 | 87 | 91 | 102 | 108 | 139 | 307 | 289 |

(P.U., B.A./B.Sc

Let $X$ and $Y$ denote respectively the coded year number and the index of wholesale prices. even number of years is given and units of $X$ are 2, therefore, we can assign $X = 0$ to the mi 1938, $X$ = 1, 3, 5, 7 to the following years and $X$ = −1, −3, −5, −7 to the preceding years. Let the of the trend curve of second degree fitting the data be

$$\hat{Y}_t = a + bX + cX^2,$$

where $a$, $b$ and $c$ are to be computed from the data by the *least-squares* method.

Since $\sum X = 0 = \sum X^3$, the normal equation obtained by the method of least-squares, red

$$\left.\begin{aligned} \sum Y &= na + c\sum X^2, \\ \sum XY &= b\sum X^2, \\ \sum X^2 Y &= a\sum X^2 + c\sum X^4. \end{aligned}\right\}$$

The arithmetic involved in computation is arranged in the following table;

| Year | $X$ | $Y$ | $XY$ | $X^2$ | $X^2Y$ | $X^3$ | $X^4$ | Trend |
|---|---|---|---|---|---|---|---|---|
| 1931 | – 7 | 96 | – 672 | 49 | 4704 | –343 | 2401 | 100.3 |
| 1933 | – 5 | 87 | – 435 | 25 | 2175 | –125 | 625 | 83.0 |
| 1935 | – 3 | 91 | – 273 | 9 | 819 | –27 | 81 | 81.8 |
| 1937 | – 1 | 102 | – 102 | 1 | 102 | –1 | 1 | 96.7 |
| 1939 | + 1 | 108 | + 108 | 1 | 108 | +1 | 1 | 127.7 |
| 1941 | + 3 | 139 | 417 | 9 | 1251 | 27 | 81 | 174.7 |
| 1943 | + 5 | 307 | 1535 | 25 | 7675 | 125 | 625 | 237.8 |
| 1945 | + 7 | 289 | 2023 | 49 | 14161 | 343 | 2401 | 317.0 |
| Total | 0 | 1219 | 2601 | 168 | 30995 | 0 | 6216 | 1219.0 |

Substituting these values in the normal equations, we get

$$1219 = 8a + 168c,$$

$$2601 = 168b,$$

$$30995 = 168a + 6216c.$$

Solving these questions, we obtain

$$a = 110.16,\ b = 15.48, \text{ and } c = 2.01,$$

Hence the equation of the second degree trend or the *quadratic* parabola is

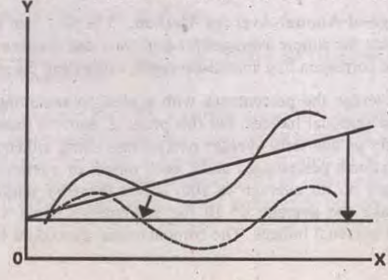
$$\hat{Y}_t = 110.16 + 15.48X + 2.01X^2,$$

where the origin $X = 0$ is the year 1938, and $X$ is measured in 1 year units.

The computed trend values are shown in the last column of the table shown above. The total of the values agree with the total of the original values.

## DETRENDING

After the determination of the trend by any method, the next step in the analysis of a time series, is to remove its effect from the observed time series. The process of removing the trend component is called detrending. A *detrended* time series is also known as a *stationary* time series. The data in a detrended series would fluctuate around the horizontal line as displayed by the dashed line in the following figure:



The trend component is removed by computing *either* the deviations from the trend *or* the ratio trend depending on whether the components follow the additive or the multiplicative model. When relationship is additive, we subtract the trend values from the corresponding original observations. If the other hand, the model is multiplicative, we divide each of the original observations by corresponding trend value. The ratios so obtained, are usually expressed as percentages.

## 13.6 ANALYSING THE SEASONAL VARIATIONS

Having removed the trend component from a time series, we are left with deviations or which are averaged for each season or month to measure the seasonal variation. The deviation usually called the *seasonal differences* with the ratios expressed as percentages, may be called *seasonal relatives.* That is

$$\text{a seasonal relative} = \frac{\text{original Y - observations}}{\text{corresponding trend value}} \times 100.$$

A measure of variation which is usually computed in index form, is called a *seasonal index*. compute seasonal indices, the components of the time series are assumed to follow the multipl (decomposition) model.

It is relevant to note that simple averages of the monthly or quarterly values over a period of are known as the *seasonal averages.* When these seasonal averages are expressed as percentages of average of all the seasonal averages, *i.e.* grand average, they are called the *seasonal indices.* In symb



$$\text{Seasonal Index} = \frac{\text{seasonal average}}{\text{grand average}} \times 100.$$

A seasonal index may also be computed from weekly or daily data. Assuming that the time components follow the multiplicative (decomposition) model $Y = TSCI$, discussed earlier, we give the various methods available for computing a seasonal index.

i) The Percentage-of-Annual-Average Method.

ii) The Ratio-to-Moving-average method.

iii) The Ratio-to-Trend Method.

iv) The Link-Relative Method.

Let us now discuss each of these methods in turn.

**13.6.1 The Percentage-of-Annual-Average Method.** The *first step* is to eliminate the eff the trend. To this end, compute the simple averages for each year and divide each of the given mon quarterly observations by the corresponding annual-averages, expressing the result as a percentage.

The *next step* is to average the percentages with a view to removing the cyclical and variations and computing the seasonal indices. For this purpose, sort out these percentages by quarters and find the monthly or quarterly average percentages using either the mean or the case of mean, discard the extreme percentages under each month or quarter. If the 12 monthly quarterly average percentages do no average to 100, adjust them by multiplying each of them suitable factor that will make the average of all the percentages equal 100. The resulting percentages are the required seasonal indices. The computational procedure is illustrated by the 13.8 on the next page.

**Example 13.8** Compute the seasonal indices for the data in Example 13.1, using the percentage-of-annual-average method.

We find compute the annual-average as below:

| Year | Quarters | | | | Annual or yearly | |
|---|---|---|---|---|---|---|
| | I | II | III | IV | Total | Average |
| 2001 | 72 | 98 | 79 | 106 | 355 | 88.75 |
| 2002 | 79 | 122 | 101 | 143 | 445 | 111.25 |
| 2003 | 94 | 141 | 128 | 160 | 523 | 130.75 |
| 2004 | 125 | 143 | 135 | 187 | 590 | 147.50 |

We then divide each of the given quarterly observations by the corresponding annual-average and express the result as a percentage. The percentages so obtained appear below:

| Year | Quarters | | | | Total |
|---|---|---|---|---|---|
| | I | II | III | IV | |
| 2001 | 81.13 | 110.42 | 89.01 | 119.44 | |
| 2002 | 71.01 | 109.66 | 90.79 | 128.54 | |
| 2003 | 71.89 | 107.84 | 97.90 | 122.37 | |
| 2004 | 84.75 | 96.65 | 91.53 | 126.78 | |
| Total | 308.78 | 424.87 | 369.23 | 497.13 | |
| Mean | 77.20 | 106.22 | 92.31 | 124.28 | 400.01 |



As the total of the average percentages is almost equal to the desired total of 400, no adjustment is ade. Hence the 4 mean percentages are the desired seasonal indices.

**13.6.2 The Ratio-to-Moving-Average Method.** This is the most frequently used method for the mputation of seasonal index numbers. The *first step* is to eliminate the trend component. To this end, ide the original observations for each month or quarter by the corresponding 12-month or 4-quarter *tred* moving average and express the result as a percentage, *i.e.* compute the seasonal relative for each nth or quarter. It is worthwhile to note that each monthly or quarterly value is assumed to consist of product of the effects of $T$, $C$, $S$ and $I$ components, and each moving average is a measure of the bined effect of trend and cyclical components, i.e. $T\times C$. Thus dividing the original data by the esponding moving averages and then multiplying by 100, an estimate of the effect of seasonal and gular components combined is obtained; that is

$$\frac{\text{original data}}{\text{moving average}}\times 100=\frac{TCSI}{TC}\times 100=SI \text{ (seasonal relative)}$$

The *next step* is to remove the effects of the irregular component in order to obtain seasonal es. To achieve this end, arrange the seasonal relatives by months or quarters and find the monthly or erly averages, using either the mean or the median. If mean is to be used, compute a *modified mean* scarding the unusually large or small seasonal relative under each month or quarter so that the age is not distorted. If these monthly or quarterly averages do not average to 100, then adjust them by plying each median or modified mean estimate of seasonal index by the correction factor that will

make the average of all the indices equal 100. The resulting averages give the desired indices of seasonal variations.

**Example 13.9** Compute the seasonal indices by using the ratio-to-moving-average method for the data in Example 13.4.

The original data and the 4-quarter *centred* moving averages which measure the combined effects of trend and cyclical components, appear in columns (2) and (3) respectively in the table below. We then divide each of the original quarterly values by the corresponding *centred* moving average and express the result as a percentage, *i.e.* we compute the seasonal relatives, shown in column (4).

| Year and quarter (1) | *Y*-values *TCSI* (2) | 4-quarter centred moving average *TC* (3) | Seasonal relatives $(TCSI \div TC)\times 100 = SI$ (4) |
|---|---|---|---|
| 2001–I | 72 | --- | --- |
| II | 98 | --- | --- |
| III | 79 | 89.6 | 88.2 |
| IV | 106 | 93.5 | 113.4 |
| 2002–I | 79 | 99.2 | 79.6 |
| II | 122 | 106.6 | 114.4 |
| III | 101 | 113.1 | 89.3 |
| IV | 143 | 117.4 | 121.8 |
| 2003–I | 94 | 123.1 | 76.4 |
| II | 141 | 128.6 | 109.6 |
| III | 128 | 134.6 | 95.1 |
| IV | 160 | 138.8 | 115.3 |
| 2004–I | 125 | 139.9 | 89.3 |
| II | 143 | 144.1 | 99.2 |
| III | 135 | --- | --- |
| IV | 187 | --- | --- |

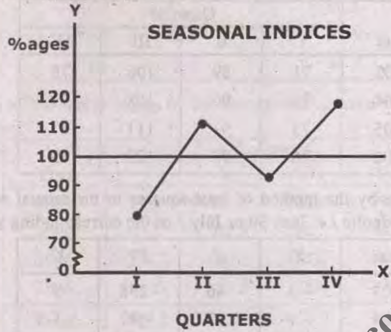
In order to remove the irregular effects and to compute the seasonal index for each quarter, the seasonal relatives are arranged in the following table.

*Computation of Seasonal Indices*

| Year | Quarters I | II | III | IV | Total |
|---|---|---|---|---|---|
| 2001 | --- | --- | 88.2 | 113.4 | |
| 2002 | 79.6 | 114.4 | 89.3 | 121.8 | |
| 2003 | 76.4 | 109.6 | 95.1 | 115.3 | |
| 2004 | *89.3 | *99.2 | --- | --- | |
| Total | 156.0 | 224.0 | 272.6 | 350.5 | |
| Mean | 78.0 | 112.0 | 90.87 | 116.83 | 397.70 |
| Seasonal Index | 78.45 | 112.65 | 91.40 | 117.51 | 400.01 |

*Discard these extreme relatives before computing total.

Since the total of the 4 mean percentages is 397.70, we therefore adjust them by multiplying each of the 4 quarterly mean percentages by 400/397.70 as the sum of the 4-quarterly measures should be 400. Thus the desired seasonal indices are obtained as 78.45 112.65, 91.40 and 117.51. These indices of seasonal variation are shown graphically below.




**13.6.3 The Ratio-to-Trend Method.** In this method, the trend values are obtained for each time period by fitting a *least-squares* trend line either to the observed time series data or to the annual averages. The rest of the computational procedure is the same as that of the ratio to-moving average method. But this method is inferior to the ratio-to-moving average method as the seasonal index computed by it includes cyclical and irregular variations.

**Example 13.10** Compute the indices of seasonal variation by the ratio-to-trend method by fitting a *least-squares* straight line trend to the observed time series data in Example 13.1.

We obtain the trend values by fitting a linear trend to the observed time series data (Example 13.6). We then divide each observation in the original data by the corresponding trend value and multiply it by 100. The percentages so obtained are arranged by quarters as shown in the following table in order to compute the indices of seasonal variation.

*Computation of Seasonal Indices*

| Year | Quarters I | II | III | IV | Total |
|---|---|---|---|---|---|
| 2001 | *92.1 | 117.1 | 88.6 | 111.9 | |
| 2002 | 78.8 | 115.3 | 90.7 | *122.4 | |
| 2003 | 76.9 | 110.3 | *96.0 | 115.2 | |
| 2004 | 86.6 | *95.4 | 86.9 | 116.1 | |
| Total | 242.3 | 342.7 | 266.2 | 343.2 | |
| Mean | 80.77 | 114.23 | 88.73 | 114.40 | 398.13 |
| Seasonal Index | 81.15 | 114.77 | 89.15 | 114.94 | 400.01 |

*Discard these extreme relatives before computing totals.

The sum of the 4 mean percentages is 398.13. The mean percentages are therefore adjusted by multiplying each of them by the correction factor 400/398.13 to get the desired sum of 400.

**Example 13.11** Obtain the seasonal indices by the ratio-to-trend method and by fitting the least-squares trend line to the annual averages of the following data, showing the amount of the money (£ 000,000) spent upon passenger travel in the United Kingdom at levels of fares, and charges current during the periods given.

| Year | Quarters I | II | III | IV |
|---|---|---|---|---|
| 2003 | 71 | 89 | 106 | 78 |
| 2004 | 71 | 90 | 108 | 79 |
| 2005 | 73 | 91 | 111 | 81 |
| 2006 | 76 | 97 | 122 | 89 |

We first fit a straight line by the method of least-squares to the annual averages ($Y$), which are assumed to correspond to the midpoint *i.e.* June 30 or July 1 of the corresponding year.

| Year | $X$ | $Y$ | $XY$ | $X^2$ |
|---|---|---|---|---|
| 2003 | – 3 | 86 | – 258 | 9 |
| 2004 | – 1 | 87 | – 87 | 1 |
| 2005 | + 1 | 89 | 89 | 1 |
| 2006 | + 3 | 96 | 288 | 9 |
| $\sum$ | --- | 358 | + 32 | 20 |



The equation of the straight line (linear trend) is $\hat{Y} = a + bX$. Since there is an even number of years, the origin is taken at December 31, 2004 or January 1, 2005. The normal equations then reduce to

$$\sum Y = na \text{ and } \sum XY = b\sum X^2.$$

Substituting the values in the normal equations and solving them, we get $a = 89.5$ and $b = 1.6$.

Thus the required trend line is $\hat{Y} = 89.5 + 1.6X$, where $X$ is measured in half years.

This line shows that the values of $Y$ increase by 1.6 after every half year or $\frac{1.6}{2} = 0.8$ after every quarter. Assuming that the given quarterly data correspond to the middle of the quarter, we calculate the trend values as below:

When $X = 0$ which corresponds to January 1, 2005; $Y = 89.5$

But we need the values of $Y$ a half quarter later. Thus

$$Y = 89.5 + \frac{1}{2}(0.8) = 89.9$$

This is the trend value corresponding to first quarter of 2005. Now, by successive addition of 0.8 to 89.9, the trend values for the 2nd, 3rd and 4th quarters of 2005 and the quarters of 2006 are worked out, while by successively subtracting 0.8 from 89.9, the trend values for the preceding quarters are found. The quarterly trend values thus found are given on the next page:

| Year | Quarters I | II | III | IV |
|---|---|---|---|---|
| 2003 | 83.5 | 84.3 | 85.1 | 85.9 |
| 2004 | 86.7 | 87.5 | 88.3 | 89.1 |
| 2005 | 89.9 | 90.7 | 91.5 | 92.3 |
| 2006 | 93.1 | 93.9 | 94.7 | 95.5 |

Dividing each of the actual values by the corresponding trend value and expressing the result as a percentage, we obtain:

| Year | Quarters I | II | III | IV | Total |
|---|---|---|---|---|---|
| 2003 | 85.0 | 105.6 | 124.6 | 90.8 | |
| 2004 | 81.9 | 102.8 | 122.3 | 88.6 | |
| 2005 | 81.2 | 100.3 | 121.3 | 87.8 | |
| 2006 | 81.6 | 103.3 | 128.8 | 93.2 | |
| Total | 329.7 | 412.0 | 497.0 | 360.4 | |
| Mean | 82.4 | 103.0 | 124.2 | 90.1 | 399.7 |

The sum of these mean percentages is 399.7, which is so close to the desired 400 that no adjustment is necessary. Hence the desired seasonal indices are 82.4, 103.0, 124.2 and 90.1. Median can also be used to get the seasonal indices.

**13.6.4 The Link-Relative Method.** This method was at one time the most widely used method as the data for each month or quarter were utilized more completely. But nowadays it is seldom used as its disadvantages outweigh its advantages.

To eliminate the trend component, the computational steps are as follows:

i) Compute the *link-relatives* by expressing each monthly or quarterly value as a percentage of the preceding monthly or quarterly value.

ii) Arrange the link-relatives by months or quarters and find an appropriate average of these relatives for each month or quarter. Usually median is used.

iii) Convert the average (median or mean) relatives into a series of *chain relatives* by setting the value of January or the first quarter as 100, and carrying the process to include the first unit of the next period

iv) A discrepancy due to trend increment (positive or negative) exists between the chain relative for the first January or quarter and that for the next period. Adjust the chain relatives for the trend component by subtracting one-twelfth of the discrepancy from the value of February, two-twelfth from the value of March and so on or by subtracting one-fourth of the discrepancy from the relative of second quarter, two-fourth from the third quarter relative and three fourth from the fourth quarter relative

To obtain seasonal indices, reduce the adjusted chain relatives to the same level as January or the first quarter by multiplying each of the adjusted chain relatives by the correction factor that will make the average of all the indices equal 100. These final figures are the desired indices of seasonal variation.

**Example 13.12** Obtain the seasonal indices for the data in Example 13.11, using the link-relative method.

Expressing the data for each quarter as a percentage of the data for the preceding quarter, we get the link-relatives as below:

| Year | Quarters I | II | III | IV |
|---|---|---|---|---|
| 2003 | -- | 125.4 | 119.1 | 73.6 |
| 2004 | 91.0 | 126.8 | 120.0 | 73.1 |
| 2005 | 92.4 | 124.7 | 122.0 | 73.0 |
| 2006 | 93.8 | 127.6 | 125.8 | 73.0 |
| Median | 92.4 | 126.1 | 121.0 | 73.1 |

Next, we calculate the chain relatives for the four quarterly averages, setting the value of the first quarterly average equal to 100%. The chain relatives are:

| Quarter | I | II | III | IV | I |
|---|---|---|---|---|---|
| Chain Relative | 100 | 126.1 | 152.6 | 111.6 | 103.1 |

Continuing the process, the chain relative for the first quarter works out to be 103.1 which as a matter of fact, ought to have been 100. This increase of 3.1 is due to the trend component present in the data. An adjustment for the trend therefore becomes necessary. Since the difference is positive so we subtract one-fourth of this from the second quarter figure, two-fourth from the third quarter figure and three-fourth from the fourth quarter. The sum of these adjusted chain relatives is 485.65. The quarterly figures are further adjusted by multiplying each figure by $\frac{400}{485.65}$ so as to get a total of 400. The adjusted figures are given below:

| Quarter | I | II | III | IV | Total |
|---|---|---|---|---|---|
| Adjusted Chain Relatives | 100 | 125.32 | 151.05 | 109.28 | 485.65 |
| Seasonal Index | 82.4 | 103.2 | 124.4 | 90.0 | 400.0 |



## 13.7 DESEASONALIZATION OF DATA

We remove the seasonal effect from an observed time series data to see how things might have been, if there had been no seasonal component. The process of removing the seasonal component from a time series is known as *deseasonalization* or *seasonal adjustment* of data and the time series thus obtained is called the *deseasonalized* or *seasonally adjusted* time series. To get deseasonalized data, we divide (considering multiplicative model) each value in the original data by the corresponding value of seasonal index and multiply the result by 100. Thus

$$\text{Deseasonalized data} = \frac{\text{period's original value}}{\text{period's seasonal index}} \times 100$$

$$= \frac{TCSI}{S} \times 100 = TCI \times 100$$

Thus the deseasonalized data contain the effects of trend, cyclical and irregular components. For example, the deseasonalized data for 2002 of Example 13.9 are shown below:

| Quarter | Number of bags of fertilizer (00) | Seasonal Index | Deseasonalized Data |
|---|---|---|---|
| I | 79 | 78.45 | 101 |
| II | 122 | 112.65 | 108 |
| III | 101 | 91.40 | 111 |
| IV | 143 | 117.51 | 122 |

We find that the increase from the first quarter to the second quarter of 2002, expected on the basis of seasonal pattern is $\left(79\times\dfrac{112.65}{78.45}-79\right)$, *i.e.* 34 hundred bags which is less than actual increase of 122–79, *i.e.* 43 hundred bags. From the second quarter to the third quarter, there is a decrease of 122–101, *i.e.* 21 hundred bags which is less than the decrease expected on the basis of seasonal pattern amounting to $\left(122-122\times\dfrac{91.40}{112.65}\right)$, *i.e.* 23 hundred bags of fertilizer. If there had been no seasonal effect, sales for the first quarter of 2002, would have been 101 (hundred) bags of fertilizer.

When the time series components follow the additive model, *i.e.* $Y=T+C+S+I$, the data are deseasonalized by *subtracting* the seasonal effects from the corresponding original values.

## 13.8 ANALYSING THE CYCLICAL VARIATIONS

The cyclical variations can be measured by *first* moving the trend and seasonal components by division and *then* averaging out irregular variations. The simplest method of obtaining cyclical movement is called the *residual method*. This method consists of removing the effects of trend, seasonal and irregular components from the observed time series data in any order. Any one of the following *three* procedures can be used to estimate cyclical and irregular movements:

*First Procedure:*

i) Deseasonalize the data, *i.e.* divide each value of the original data by the corresponding seasonal index to remove the seasonal component: $TCSI \div S=TCI$.

ii) Divide the results just obtained, *i.e.* the deseasonalized data by the corresponding trend value to eliminate the trend component: $TCI \div T=CI$.

*Second Procedure:*

i) Remove the trend component by dividing each value of the original data by the corresponding trend value: $TCSI \div T=CSI$.

ii) Eliminate seasonal variation by dividing the results, *i.e.* detrended data by the corresponding seasonal index: $CSI \div S=CI$.

*Third Procedure:*

i) Multiply each trend value by the corresponding value of the seasonal index to get $T \times S$ values.

ii) Eliminate trend and seasonal components by dividing each value of original data by the corresponding $T \times S$ value obtained in (i): $TCSI \div TS=CI$.

All these three procedures give the same results. In order to remove the irregular variations, if any, take an appropriate moving average of a few months or quarters duration. The resulting quantities in percentage form are called *cyclical relatives or percentages*. Do not divide by $I$.

Another method for isolating cyclical movements is the *Harmonic analysis*, which is beyond the scope of this text and hence is not discussed. A study of cyclical relatives is useful for economic forecasting.

**Example 13.13** Compute the cyclical relatives for the data in Example 13.1.

The process of computing the cyclical-irregular movements and cyclical relatives is shown in the table below. To remove the irregular variations, a three-quarter moving average has been thought appropriate.

| Year and quarter (1) | *Y*-values TSCI (2) | Trend Values *T* (3) | Seasonal Index (*S*)% (4) | Trend x Seasonal *TS*+100 (5) | Cyclical-Irregular-percentages *CI*(%) (6) | 3-quarter moving Total (7) | Cyclical Relatives *C*(%) (8) |
|---|---|---|---|---|---|---|---|
| 2001–I | 72 | 78.2 | 81.15 | 63.46 | 113.46 | --- | --- |
| II | 98 | 83.7 | 114.77 | 96.06 | 102.02 | 314.83 | 104.94 |
| III | 79 | 89.2 | 89.15 | 79.52 | 99.35 | 298.75 | 99.58 |
| IV | 106 | 94.7 | 114.94 | 108.85 | 97.38 | 293.89 | 97.96 |
| 2002–I | 79 | 100.2 | 81.15 | 81.30 | 97.16 | 295.01 | 98.34 |
| II | 122 | 105.8 | 114.77 | 121.43 | 100.47 | 299.42 | 99.81 |
| III | 101 | 111.3 | 89.15 | 99.22 | 101.79 | 308.78 | 102.93 |
| IV | 143 | 116.8 | 114.77 | 134.25 | 106.52 | 303.02 | 101.01 |
| 2003–I | 94 | 122.3 | 81.15 | 99.25 | 94.71 | 297.36 | 99.12 |
| II | 141 | 127.8 | 114.94 | 146.68 | 96.13 | 298.47 | 99.49 |
| III | 128 | 133.4 | 89.15 | 118.93 | 107.63 | 303.98 | 101.33 |
| IV | 160 | 138.9 | 114.77 | 159.65 | 100.22 | 314.52 | 104.84 |
| 2004–I | 125 | 144.4 | 81.15 | 117.18 | 106.67 | 290.01 | 96.67 |
| II | 143 | 149.9 | 114.77 | 172.08 | 83.12 | 287.23 | 95.74 |
| III | 135 | 155.4 | 89.15 | 138.54 | 97.44 | 281.61 | 93.87 |
| IV | 187 | 161.0 | 114.94 | 185.05 | 101.05 | --- | --- |

## 13.9 ANALYSING THE IRREGULAR VARIATIONS

The *irregular movements* of a time series are estimated by dividing the combined cyclical-irregular variations by the corresponding values of the cyclical relatives; that is

$$I = \frac{C \times I}{C}$$

The irregular movements can be shown graphically.

## 13.10 FORECASTING

In time series analysis, *forecasting* is a process of assessing the magnitude of a time series variable which it will assume at some future point of time. Forecasting is based on the assumption that the past pattern and behaviour of a variable will continue in the future. The simple and elementary technique of *short-term* forecasting involves the components of trend and seasonal index.

Forecasts for a period of 1 year or less can be made by projecting *either* the least-squares equation to obtain forecasts of trend values ($T$) or the centred moving averages to obtain forecasts of moving average values ($T\times C$), then multiplying these projected values by the seasonal index for that period and dividing by 100 to obtain a $T\times S$ or $T\times C\times S$ forecast. Thus

$$\text{period's forecast} = \frac{(\text{period's projected trend value}) \times (\text{seasonal index})}{100}$$

It is relevant to note that it is very difficult to forecast cyclical and irregular movements.

**13.10.1 Forecasting by Exponential Smoothing.** The exponential smoothing is a method of forecasting that assigns positive weights to past and current values only. This technique often provides good short-term forecasts. The exponentially smoothed series $\hat{Y}_t$ from the original time series $Y_t$ is calculated a follows:



$$\hat{Y}_1 = Y_1$$

$$\hat{Y}_2 = wY_2 + (1-w)\hat{Y}_1,$$

$$\hat{Y}_3 = wY_3 + (1-w)\hat{Y}_2,$$

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$$\hat{Y}_t = wY_t + (1-w)\hat{Y}_{t-1},$$

where the weight $w$, called the *exponential smoothing constant*, is selected so that $w$ is between 0 and 1. The most commonly used value of $w$ is between 0.01 and 0.3. The exponential smoothing has an advantage that no values are lost at either end of the smoothed series.

## 13.11 SERIAL CORRELATION

While analyzing a time series data, there is a possibility of dependence (or association) between the successive observations. In case, the successive observations are dependent, one measure of this effect is the simple correlation between successive observations. Such a correlation is called a *serial correlation*.

Generally, a serial correlation is defined as correlation between observations ordered in time periods. Given $n$ observations $Y_1, Y_2, \ldots, Y_t, \ldots, Y_n$ made over successive time periods, we change the observations into $(n-1)$ pairs such as $(Y_1, Y_2), (Y_2, Y_3), \ldots, (Y_{n-1}, Y_n)$. If we regard the first observation in

each pair as $Y_t$, then the other observation is $Y_{t+k}$, where $k$ can be 1, 2, 3, etc. The correlation between $Y_t$ and $Y_{t+1}$, i.e. the correlation between successive overlapping pairs is called the *serial correlation of first order*. The co-efficient of first serial correlation, denoted by $r_1$, is generally calculated by the following slightly modified formula

$$r_1 = \frac{\sum_{t=1}^{n-1}(Y_t - \overline{Y})(Y_{t+1} - \overline{Y})}{\sum_{t=1}^{n}(Y_t - \overline{Y})^2}$$

where $\overline{Y} = \sum_{t=1}^{n} Y_t / n.$

This is also called the co-efficient of auto-correlation at lag 1. The terms serial correlation and autocorrelation are used interchangeably.

Likewise, we can find the correlation coefficient between observations, separated by a lag of $k$ time periods, which is given by

$$r_k = \frac{\sum_{t=1}^{n-k}(Y_t - \overline{Y})(Y_{t+k} - \overline{Y})}{\sum_{t=1}^{n}(Y_t - \overline{Y})^2}$$

This is called the coefficient of autocorrelation at lag $k$ or serial correlation of order $k$.

We can also draw a scatter diagram by plotting the pairs $(Y_t, Y_{t+k})$ on a graph paper to see whether the successive observations appear to be correlated.



**Example 13.14** Sixteen successive observations on a stationary time series are as follows:

1.6, 0.8, 1.2, 0.5, 0.9, 1.1, 1.1, 0.6, 1.5, 0.8, 0.9, 1.2, 0.5, 1.3, 0.8, 1.2

Calculate $r_1$, the first serial correlation co-efficient.

The co-efficient of serial correlation of order 1 is given by

$$r_1 = \frac{\sum_{t=1}^{n-1}(Y_t - \overline{Y})(Y_{t+1} - \overline{Y})}{\sum_{t=1}^{n}(Y_t - \overline{Y})^2}$$

where $\overline{Y} = \sum_{t=1}^{n} Y_t / n = \frac{16.0}{16} = 1.0.$

The calculations needed to computer $r_1$ are shown below:

| $Y_t$ | $Y_{t+1}$ | $Y_t - \overline{Y}$ | $Y_{t+1} - \overline{Y}$ | $(Y_t - \overline{Y})(Y_{t+1} - \overline{Y})$ | $(Y_t - \overline{Y})^2$ |
|---|---|---|---|---|---|
| 1.6 | 0.8 | 0.6 | −0.2 | −0.12 | 0.36 |
| 0.8 | 1.2 | −0.2 | 0.2 | −0.04 | 0.04 |
| 1.2 | 0.5 | 0.2 | −0.5 | −0.10 | 0.04 |
| 0.5 | 0.9 | −0.5 | −0.1 | +0.05 | 0.25 |
| 0.9 | 1.1 | −0.1 | 0.1 | −0.01 | 0.01 |
| 1.1 | 1.1 | 0.1 | 0.1 | +0.01 | 0.01 |
| 1.1 | 0.6 | 0.1 | −0.4 | −0.04 | 0.01 |
| 0.6 | 1.5 | −0.4 | 0.5 | −0.20 | 0.16 |
| 1.5 | 0.8 | 0.5 | −0.2 | −0.10 | 0.25 |
| 0.8 | 0.9 | −0.2 | −0.1 | +0.02 | 0.04 |
| 0.9 | 1.2 | −0.1 | 0.2 | −0.02 | 0.01 |
| 1.2 | 0.5 | 0.2 | −0.5 | −0.10 | 0.04 |
| 0.5 | 1.3 | −0.5 | 0.3 | −0.15 | 0.25 |
| 1.3 | 0.8 | 0.3 | −0.2 | −0.06 | 0.09 |
| 0.8 | 1.2 | −0.2 | 0.2 | −0.04 | 0.04 |
| 1.2 | --- | 0.2 | --- | −0.98+0.08 | 0.04 |
| 16.0 | --- | --- | --- | −0.90 | 1.64 |

Substituting the values in the formula, we get

$$r_1 = \frac{-0.09}{1.64} = -0.55$$

Hence the serial correlation of first order between the successive observations is found to be = −0.55.



## EXERCISES

### OBJECTIVE

Answer 'True' or 'False'. If the statement is not true then replace the underlined words with words that make the statement true:

i) Secular trend measures the short-term variation of a time series.

ii) A typical time series may be regarded as composed of five components.

iii) A secular trend is mainly caused by the change in seasons.

iv) Irregular variations can be predicted in time.

v) A histogram is a graph of time series.

vi) The addition model of a time series is Y = TSCI.

vii) A main objective of fitting trend lines is to forecast cyclical turning points.

viii) Seasonal indexes can be calculated from weekly, monthly and quarterly data only.

ix) If the yearly sales for year 2007 is Rs.5,00,000/- and the sales index for year 2007 is 60, then the seasonally adjusted sales figure is Rs.4,00,000/-.

x) Yearly time series contain the following four components: trend, cyclical, seasonal and irregular.

## b) MULTIPLE CHOICE QUESTIONS

i) Decomposing a time series means that past data is distributed into components of:

a) Trend, cycles, seasonal and random
b) Long term, medium term and short term variations
c) Constants and variations
d) All of above

ii) The seasonal variation in the time series is computed by

a) Ratio to moving average method
b) Ratio to trend method
c) Link relative method
d) All of above



iii) The seasonal variation in the time series is computed by

a) Trend
b) Cyclical
c) Seasonal
d) Irregular

iv) After detrending the data, the time series (multiplicative model) consists of

a) Y = TSCI
b) Y = TSI
c) Y = CSI
d) None of above

v) If a time series changes at exact constant percentage then

a) A good fitted trend line cannot be obtained
b) A linear line fitted to the data gives a perfect fit
c) A linear line fitted to the logarithms data gives a perfect fit
d) A nonlinear is required to be fitted

vi) Dividing the original time series by moving average, the time series (multiplicative model) consists of

a) Y = SI

b) Y = CS

c) Y = TS

d) Y = TC

vii) A company's trend figure for sales for December 2007 is Rs.2,00,000/-. Actual sales during that period were Rs.1,60,000/- and C x I = 0.80. The value for seasonal index is

a) 100

b) 125

c) 60

d) 64

viii) A second degree trend line is $\hat{Y} = 15 - 0.1t + 0.05t^2$ where Y is sales (in thousands) and t is time (in years) and t = 1 for 1995. What is the predicted figure for year 2007?

a) 20,000

b) 22,150

c) 18,000

d) 15,000



ix) If a 4-quarter moving average is projected to obtain short-term forecasts, it contains the following components

a) TC

b) TS

c) CSI

d) C only

x) Exponential smoothing is a forecasting method which

a) uses the actual data, not the forecast data

b) requires to fit a mathematical model to the data

c) gives equal weight to all the periods

d) all of above

## BJECTIVE

a) Define a time series. What are its various components? Describe each carefully.

b) Associate the following phenomena of business with the components of time series they belong to:

(i) The prosperity in a Business.

(ii) The production of sugar recorded for 1956, 1957, …, 1962.

(iii) The weekly statement of the sale of pens.

(iv) The festival sale.

(v) The fire in a factory. (P.U., B.A./B.Sc. 197

13.2 a) Define the following terms:

(i) Time Series Analysis, (ii) Secular trend, (iii) Seasonal variations, (iv) Cyclical fluctuations (v) Irregular movements.

b) With which characteristic movement of a time series would you mainly associate each of the following?

i) a fire in a factory delaying production for 3 weeks,

ii) an era of prosperity,

iii) an after Eid sale in a departmental store,

iv) a need for increased wheat production due to a constant increase in population.

v) the monthly number of inches of rainfall in a city over a 5-year period,

vi) a recession,

vii) an increase in employment during summer months.

viii) the decline in the death rate due to advance in science,

ix) a steel strike,

x) a continually increasing demand for smaller automobiles.

(P.U., B.A. (Part II), 1966, 19



13.3 Describe the different components of a time series. Describe various methods of measuring Secular Trend in a time series, giving the advantages and disadvantages for each.

(P.U., B.A./B.Sc. 1961,

13.4 Describe the different components of time series. Discuss the measurement techniques of any one of them. (P.U., M.A. (Econ), 19

13.5 What do you understand by Time Series Analysis? Discuss how you would analyse a time series to determine the trend and the seasonal variation.

13.6 What is a time series? What are its main components and how will you isolate them?

(P.U., B.A./B.Sc. 1961,

13.7 What is meant by seasonal variation? Explain how seasonal variations are measured and removed from the time series data?

13.8 a) Distinguish between the Additive and Multiplicative models in time series analysis.

b) When do you compute the deviations from trend and when ratios to trend? Explain how you eliminate the average seasonal variations from the observed values of the time series.

13.9 Critically comment on various methods of eliminating seasonal variations from a time series.

(P.U., M.A. (Stat.), 1968)

13.10 Plot the following data showing average wages in rupees of some workers during 1990–2001 and find the trend by the method of (i) free-hand curve and (ii) semi-averages.

| Year | 1990, | 1991, | 1992, | 1993, | 1994, | 1995, | 1996, | 1997, | 1998, | 1999, | 2000, | 2001 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wages | 140, | 148 | 180 | 195, | 200, | 235, | 260, | 280, | 290, | 330, | 325, | 340 |

13.11 Name the methods used to measure trends. Determine a trend line by a simple moving average of 5 years from the following data:

| Year | 1921, | 1922, | 1923, | 1924, | 1925, | 1926, | 1927, | 1928, | 1929, | 1930 |
|---|---|---|---|---|---|---|---|---|---|---|
| Value | 102 | 108 | 130 | 140 | 158 | 180 | 196 | 210 | 220 | 230 |

(P.U., B.A./B.Sc. 1963)

13.12 i) Explain briefly the meaning and purpose of moving averages.

ii) The number of items of certain product imported into the United Kingdom is given below in thousands of units:

| Year | Number | Year | Number | Year | Number |
|---|---|---|---|---|---|
| 1951 | 170 | 1957 | 205 | 1963 | 135 |
| 1952 | 210 | 1958 | 182 | 1964 | 80 |
| 1953 | 188 | 1959 | 90 | 1965 | 60 |
| 1954 | 98 | 1960 | 92 | 1966 | 107 |
| 1955 | 83 | 1961 | 141 | 1967 | 140 |
| 1956 | 131 | 1962 | 183 | 1968 | 124 |

a) Calculate 5-year moving averages for the above data. Using these moving averages, determine the trend line.

b) Estimate the number of items imported in 1969. (P.C.S., 1971)

13.13 a) What is a Moving Average? Calculate a seven-day moving average for the following record of attendances:

| Week | Sun | Mon | Tues | Wed | Thurs | Fri | Sat |
|---|---|---|---|---|---|---|---|
| 1 | 24 | 55 | 29 | 48 | 52 | 55 | 61 |
| 2 | 27 | 52 | 32 | 43 | 53 | 56 | 65 |

Plot the given attendances and the moving averages on the same graph.

(B.Z.U., B.A./B.Sc. 1988)

b) Fit a linear trend to the data given in (a) by least squares. Find trend values as well.

(P.U., B.A./B.Sc. 1990)

13.14 Explain the use of moving averages in determining the trend line in a time series. Determine such a line in the following series of values by the use of a simple average of seven consecutive terms:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Year:** | 1980 | 1981 | 1982 | 1983 | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1990 |
| **Index:** | 187 | 161 | 149 | 142 | 125 | 129 | 133 | 127 | 130 | 129 | 129 |
| **Year:** | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | |
| **Index:** | 130 | 136 | 152 | 171 | 169 | 218 | 258 | 279 | 295 | 314 | |

Would you say that this was a satisfactory line of a trend? If not, why not?

13.15 a) Find out and plot the nine-year moving average for the following series:

8, 7, 5, 2, 4, 9, 10, 9, 8, 6, 4, 7, 11, 13, 11, 9, 8, 5, 10, 13, 15, 12, 10, 8, 6, 11, 12, 16.

b) Compute 4-month centred moving average from the following:

23, 26, 28, 30, 31, 35, 37, 32, 34, 38. (P.U., B.A. (Hons.), 1967)

13.16 The following are the quarterly index numbers of wholesale prices in the U.K. for the years 1951–55:

86, 80, 83, 84, 85, 80, 80, 78, 77, 80, 81, 80, 82, 81, 83, 82, 83, 84, 85, 86.

By a centred moving average of 4, calculate the trend.

(P.U., M.A. Econ. 1969; B.A./B.Sc. 1973)

13.17 Plot the following data as a time series. Compute 4-quarter centred moving average trend and show it on the graph.

| Year | Quarters | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| 2000 | 102 | 71 | 47 | 98 |
| 2001 | 125 | 106 | 73 | 231 |
| 2002 | 281 | 229 | 209 | 488 |
| 2003 | 484 | 447 | 457 | 966 |

13.18 The estimated number of visitors ('00s) at a holiday resort were as follows:

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 |
|---|---|---|---|---|---|---|---|---|---|---|
| Visitors | 31 | 49 | 74 | 62 | 65 | 73 | 70 | 84 | 86 | 79 |

a) Show by direct numerical calculation that the 2-year *centred* moving average is equivalent to a 3 year weighted moving average with weights 1, 2, 1 respectively.

b) Determine a 3 year weighted average if the weights 1, 4, 1 are used.

13.19 For the following time series, determine the trend by using the method of (i) semi-averages, (ii) 3-year moving averages, and (iii) least-squares for fitting a straight line:

| Year | 1968 | 1969 | 1970 | 1971 | 1972 | 1973 | 1974 | 1975 | 1976 |
|---|---|---|---|---|---|---|---|---|---|
| Values of series | 2 | 4 | 6 | 8 | 7 | 6 | 8 | 10 | 12 |

Which of the trend do you prefer, and why? (P.U., B.A./B.Sc. 19[illegible])

13.20 a) Define the following terms:

Secular Trend; Time series Decomposition; Centred Moving Average; Irregular Movements.

b) For the following time series, determine the trend by using the method of (i) three year moving averages, and (ii) least-squares for fitting a straight line.

| Year | 1970 | 1971 | 1972 | 1973 | 1974 | 1975 | 1976 | 1977 | 1978 |
|---|---|---|---|---|---|---|---|---|---|
| Values | 12 | 23 | 37 | 48 | 41 | 37 | 48 | 61 | 70 |

Which trend do you prefer, and why? (P.U., B.A./B.Sc. 1978)

13.21 a) The following are the quarterly index numbers of wholesales prices.

| Year | 1995 | 1996 | 1997 | 1998 | 1999 |
|---|---|---|---|---|---|
| Quarterly Index | 125 | 114 | 99 | 80 | 80 |

Fit a linear trend to these data and add the trend to the original chart.

b) Fit a straight line $Y=a+bX$ from the following results, for the year 1948–58 (both inclusive).

$$\sum X = 0, \sum Y = 438.9, \sum X^2 = 110, \sum XY = -84.4.$$

Find out the trend values of $Y$ as well. (P.U., M.A. (Econ.), 1968)

13.22 The following are the annual profits is thousands of rupees in a certain business:

| Year | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 |
|---|---|---|---|---|---|---|---|
| Profit | 88 | 101 | 108 | 91 | 113 | 120 | 132 |

i) Use the method of least-squares to fit a straight line trend and make an estimate of the profits in 2005.

ii) Fit a parabolic trend.

iii) Determine which is the better fitting trend.

13.23 The production of vegetables ghee ('000s tons) in Pakistan is given below:

| Year | Production | Year | Production |
|---|---|---|---|
| 1970–71 | 136 | 1974–75 | 272 |
| 1971–72 | 162 | 1975–76 | 277 |
| 1972–73 | 187 | 1976–77 | 322 |
| 1973–74 | 225 | | |

Fit a second degree parabola to the data and estimate the production for 1978–79.

(P.U., B.A./B.Sc. 1979)

13.24 Fit a parabola of second order to the following data and find out the tend values.

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Production (in 000 mds) | 17 | 20 | 19 | 26 | 24 | 40 | 35 | 35 | 51 | 74 | 69 |

13.25 Fit a second degree curve $Y=a_0+ a_1t+ a_2t^2$ to the following data:

| Year | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 |
|---|---|---|---|---|---|---|---|---|
| Profit | 273.7 | 293.5 | 315.0 | 336.8 | 364.4 | 394.8 | 424.2 | 458.7 |

Compute the trend values.

13.26 Fit a quadratic parabola to the following series of observations, taking the year as the indepe variable.

| Year | 1924 | 1927 | 1930 | 1933 | 1936 | 1939 | 1942 |
|---|---|---|---|---|---|---|---|
| Index of coal price | 187 | 142 | 133 | 129 | 136 | 169 | 279 |

Use your results to estimate the value of the index for 1935.

(P.U., B.A. (Hons.) Part-I,

13.27 The population of a country for the years 1911 to 1971 in ten yearly intervals in millions, is

5.38, 7.22, 9.64, 12.70, 17.80, 24.02, and 31.34.

Fit a curve of the type $Y=ab^X$ to these data and forecast the population for the year 1991.

(P.U., B.A./B.Sc.

13.28 a) Define the following terms:

(i) Detrending, (ii) Seasonal variation, (iii) Seasonal Index, (iv) Deseasonalization

b) Compute the seasonal indices for the four quarters by the method of ratio to-moving averages.

| Year | Quarters 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1949 | 105 | 77 | 68 | 95 |
| 1950 | 107 | 83 | 74 | 106 |
| 1951 | 117 | 99 | 86 | 112 |

(P.U., B.A./B.Sc.

13.29 Compute the seasonal indices for the four quarters by the ratio-to-moving average method from the following data of wholesale prices:

| Year | Quarters I | II | III | IV |
|---|---|---|---|---|
| 1961 | 122 | 125 | 118 | 117 |
| 1962 | 119 | 114 | 114 | 109 |
| 1963 | 105 | 99 | 93 | 89 |
| 1964 | 86 | 80 | 83 | 84 |

(P.U., B.A./B.Sc. 1973, 80-S)

13.30 A merchant's sale ('00s tons) of ordinary coal over a period were as shown below:

| Year | Quarters I | II | III | IV |
|---|---|---|---|---|
| 1996 | 118 | 87 | 47 | 83 |
| 1997 | 94 | 73 | 41 | 68 |
| 1998 | 73 | 61 | 36 | 56 |

a) Construct seasonal indices, using the percentage of annual-average method.

b) Construct quarterly seasonal index values using ratio-to-moving average method and use them to deseasonalize the 1997 values.

13.31 Quarterly sales of a certain fertilizer over the period 1994–98 in thousand of tons, were as follows:

| Quarter | Year 1994 | 1995 | 1996 | 1997 | 1998 |
|---|---|---|---|---|---|
| 1st | 48 | 50 | 68 | 93 | 84 |
| 2nd | 52 | 46 | 34 | 56 | 61 |
| 3rd | 16 | 22 | 26 | 16 | 29 |
| 4th | 35 | 40 | 35 | 45 | 48 |

By means of centred moving-averages, compute the trend and estimate the seasonal indices, and hence forecast sales for each quarter of 1999.

13.32 Compute seasonal indices for the four quarters by the ratio to-trend method for the data in question 13.29.

13.33 Compute the indices of seasonal variation by ratio-to-trend method from the following data:

| Year | Summer | Autumn | Winter | Spring |
|---|---|---|---|---|
| 1982 | 70 | 81 | 89 | 115 |
| 1983 | 75 | 93 | 108 | 152 |
| 1984 | 80 | 105 | 150 | 195 |

Use the seasonal indices to deseasonalize the 1984 values.

13.34 Construct seasonal indices by ratio-to-trend method from the following data:

| Quarter | Year | | | |
|---|---|---|---|---|
| | 2000 | 2001 | 2002 | 2003 |
| I | 107 | 118 | 126 | 121 |
| II | 115 | 122 | 129 | 122 |
| III | 103 | 115 | 118 | 116 |
| IV | 98 | 104 | 107 | 103 |

13.35 For the time series data in exercise 13.28(b),

a) determine the trend line by the least-squares method;

b) assuming the multiplicative model, compute the following:

i) seasonal indices for the four quarters;

ii) deseasonalized values.

13.36 Calculate the indices of seasonal variation by *link-relatives* method from the following data:

| Year | Quarters | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| 1970 | 112 | 125 | 129 | 110 |
| 1971 | 119 | 132 | 147 | 115 |
| 1972 | 120 | 142 | 150 | 118 |
| 1973 | 128 | 151 | 162 | 125 |

(P.U., B.A./B.Sc. 19[illegible])

13.37 a) What is *forecasting?* What are the different methods of forecasting?

b) The following table gives the numbers of deep freezers sold by a certain company:

| Year | Quarters | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| 2001 | 42 | 45 | 33 | 36 |
| 2002 | 30 | 45 | 40 | 65 |
| 2003 | 51 | 69 | 65 | 94 |

Use the trend equation and the seasonal index to forecast the sales for each quarter of 2004

c) Calculate the exponentially smoothed series, using w=0.1 and w=0.3. Which will produce a smoother trend?

13.38 a) Describe the *residual method* in time series analysis.

b) Use the data of exercise 13.29 to calculate the cyclical irregulars and cyclical relatives.

13.39 The following data pertain to the rainfall in inches in England and Wales:

| Months / Year | Jan | Feb | Mar | April | May | June | July | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2000 | 6.2 | 1.8 | 0.9 | 1.4 | 3.2 | 2.3 | 2.2 | 3.2 | 3.4 | 3.4 | 2.7 | 2.1 |
| 2001 | 3.3 | 1.7 | 0.5 | 2.2 | 1.5 | 2.5 | 2.8 | 3.2 | 4.2 | 4.5 | 6.1 | 2.8 |
| 2002 | 3.4 | 3.1 | 1.3 | 1.7 | 3.2 | 3.2 | 2.6 | 2.8 | 2.5 | 4.1 | 0.8 | 4.1 |
| 2003 | 3.4 | 3.4 | 1.5 | 1.8 | 3.0 | 3.4 | 3.1 | 5.4 | 4.9 | 1.5 | 6.2 | 4.0 |

a) Determine the trend by the least-squares method.

b) On the basis of the least-squares line and the multiplicative model, compute the following:
   i) seasonal indices for the twelve months;
   ii) deseasonalized values;
   iii) cyclical and irregular variations.

13.40 a) Explain what you understand by serial correlation.

b) The following noise measurements were recorded at an intersection in time order they were observed:

65, 64, 63, 61, 60, 58, 63, 64, 62, 64, 63, 63, 62, 60, 62, 64, 66, 68, 68, 69.

   i) Plot the scatter diagram for the pairs $(Y_t, Y_{t+1})$.
   ii) Calculate the first serial correlation coefficient $r_1$, and the coefficient of auto-correlation of lag 2.

♦♦♦♦♦♦♦♦♦♦♦

# ANSWERS TO EXERCISES

# ANSWERS TO EXERCISES

## Chapter 1, Pp. 11–14

### OBJECTIVE

(i) F (are), (ii) F (parameter), (iii) F (statistic), (iv) F (inferential),
(v) F (descriptive), (vi) F (population), (vii) F (continuous), (viii) F (discrete),
(ix) F (attribute), (x) T.

### SUBJECTIVE

1.9 (b) (i) Discrete, (ii) Continuous, (iii) Discrete, (iv) Discrete,
(v) Continuous, (vi) Continuous, (vii) Discrete, (viii) Continuous,
(ix) Continuous, (x) Discrete.

1.10 (i) Qualitative, (ii) Quantitative, (iii) Qualitative, (iv) Quantitative,
(v) Quantitative, (vi) Quantitative, (vii) Qualitative.

1.11 (i) ratio-level, (ii) ordinal-level, (iii) internal-level,
(iv) ratio-level, (v) nominal-level, (vi) ratio-level,
(vii) ordinal-level, (viii) Ratio instead of interval. (ix) ordinal-level, (x) ratio-level.

1.12 (i) 32.22, (ii) 937.1, (iii) 0.003599, (iv) 1.004,
(v) 0.07000, (vi) 22.26.



## Chapter 2, Pp. 44–45

### BJECTIVE

(a) (i) F (time series data), (ii) T, (iii) F (does),
(iv) F (mutually exclusive), (v) F (can), (vi) F (graphically),
(vii) F (histogram), (viii) F (cannot), (ix) F (one dimensional),
(x) F (height), (xi) F (mid points), (xii) F (lowest),
(xiii) F (1), (xiv) F (qualitative), (xv) F (same).

(b) (i) b, (ii) d, (iii) c, (iv) b,
(v) d, (vi) c, (vii) d, (viii) d,
(ix) a, (x) b.

## Chapter 3, Pp. 77–86

### BJECTIVE

(a) (i) T, (ii) F (median), (iii) T,
(iv) F (is), (v) T, (vi) F (mode),
(vii) F (mode), (viii) F (second), (ix) T,
(x) F (geometric mean), (xi) T, (xii) T,
(xiii) F (negatively), (xiv) F (right), (xv) F (negatively).

(b) (i) c, (ii) b, (iii) c, (iv) c,
(v) c, (vi) a, (vii) c, (viii) b,
(ix) b, (x) d.

**SUBJECTIVE**

3.14 (i) Median or Mode, (ii) Mode, (iii) Mean.

3.15 (c) 6.5

3.16 (c) 2.13

3.17 (i) 18; (ii) 17; (iii) 17.07

3.18 $\overline{X} = 111.60$; G.M. = 55.35; H.M. = 28.82. Here G.M. is the best average.

3.19 $\overline{X} = 1037.73$; G.M. = 377.2; H.M. = 186.7.

3.20 Rs.2.75 per hour; (b) 69.25% marks.

3.21 (i) $\overline{X} = \text{Rs.}10.41$; (ii) $\overline{X}_w = \text{Rs.}10.08$

3.22 Rs.24.50

3.23 (a) 14.07 years; (b) 13.82 years

3.24 $\overline{X} = \dfrac{1}{n+1}(2^{n+1}-1)$; G.M. = $2^{n/2}$; H.M. = $\dfrac{n+1}{2\left(1-\dfrac{1}{2^{n+1}}\right)}$

3.25 $\overline{X} = \dfrac{1}{2(n+1)}(3^{n+1}-1)$; G.M. = $3^{n/2}$; H.M. = $\dfrac{n+1}{\dfrac{3}{2}\left(1-\dfrac{1}{3^{n+1}}\right)}$

3.27 18.1%

3.28 (b) H.M. = 40 miles per hour; (c) H.M. = 6.8 miles per hour.

3.29 (a) 22.56 k.p.h; (b) (i) H.M. = 7.66; (ii) G.M.= 29.3%

3.30 G.M. = 49.18; H.M. = 48.47.

3.31 G.M. = 19.80; H.M. = 16.27.

3.32 (i) Median = Rs.1500; (ii) $\overline{X} = 65.5$; (iii) Median = 18.

3.33 Median = 7; $Q_1 = 6\frac{1}{2}$; $Q_3 = 7\frac{1}{2}$; $D_7 = 7\frac{1}{2}$; $P_{64} = 7\frac{1}{2}$.

3.34 Mean = 3.78; Median = 3; Mode = 3.

3.35 (ii) Median = Rs.9.04.

3.36 Median = 48.44; $Q_1 = 34.34$; $Q_3 = 61.45$.

3.37 Median = 67.72 inches; $Q_1 = 65.4"$; $Q_3 = 69.4"$.

3.38 Median = 138.63.

3.39 Mean = 24.96 years: Median = 23.36 years

3.40 Median = 41.4; $Q_1 = 32.46$; $Q_3 = 50.72$.

3.41 Mean = 79.57; Median = 76.77; $Q_1 = 57.0$; $Q_3 = 99.14$.

3.43 (a) Mean = 91.27 mg; Median = 96.58; Mode = 103.33.

(b) $Q_1 = 77.04$ mg; $Q_3 = 109.06$; $D_3 = 84.96$; $P_{45} = 93.68$.

3.44 Assuming a range of Rs.55.00 to Rs.105.00, the frequency distribution would be:

| Groups | $f$(%) | $f$ |
|---|---|---|
| Rs.55.00 and under Rs.60.00 | 4 | 20 |
| Rs.60.00 and under Rs.62.50 | 11 | 55 |
| Rs.62.50 and under Rs.72.75 | 10 | 50 |
| Rs.72.75 and under Rs.78.75 | 15 | 75 |
| Rs.78.75 and under Rs.82.25 | 10 | 50 |
| Rs.82.25 and under Rs.85.25 | 10 | 50 |
| Rs.85.25 and under Rs.90.50 | 15 | 75 |
| Rs.90.50 and under Rs.95.00 | 10 | 50 |
| Rs.95.00 and under Rs.100.00 | 10 | 50 |
| Rs.100.00 and under Rs.105.00 | 5 | 25 |
| Total | 100 | 500 |

The mean is approximately Rs.80.92.



3.45 (b) Mean = 146.975; Median = 146.75; Mode = 147.20

3.46 Mean = 11.10; Median = 11.07; Mode = 11.06.

3.47 Mode = Rs.32.48; Median = 32.49.

3.48 (c) Mode = 27

3.49 (a) (i) Median or Mode, (ii) Mean, (iii) Mean or Median, (iv) Weighted Mean, (v) Mean, (vi) Median or Mean (vii) Mode, (viii) Mean.

## Chapter 4, Pp. 116–129

### OBJECTIVE

(a) (i) F (Dispersion), (ii) F (zero), (iii) T, (iv) T, (v) F (range), (vi) F (square root), (vii) F (relative), (viii) T, (ix) F (range), (x) F (standard deviation).

| | | | |
|---|---|---|---|
| (b) (i) c, | (ii) c, | (iii) c, | (iv) d, |
| (v) b, | (vi) b, | (vii) b, | (viii) a, |
| (ix) a, | (x) a, | (xi) c, | (xii) a, |
| (xiii) c, | (xiv) b, | (xv) d. | |

**SUBJECTIVE**

4.4 (b) Q.D. = Rs.9.78

4.5 Median $\simeq$ 152 lb; SIQR $\simeq$ 7.5 lb; Mean=152.75 lb; S.D.=10.52 lb Mean of origi data=152.917 lb, S.D. of original data=10.32 lb.

4.6 (b) $\overline{X}$ = 46.17; M.D. = 11.28

4.7 Group A: Q.D.=1.285; M.D.=1.45, Co-efficient of Q.D.=0.02: Co-efficient of M.D. = 0.024.

Group B: Q.D.=1.435; M.D.=1.60, Co-efficient of Q.D.=0.012; Co-efficient of M.D. = 0.026.

4.9 (b) $\sigma^2$ = 6.85, $\sigma$ = 2.62.

4.11 (b) Mean = 32; S = 5; (c) Mean = 74.1, S = 1.36.

4.12 (c) (i) S = 2; (ii) S = 2; answers of (i) and (ii) coincide because standard deviations rem unaffected if a constant is added.

4.15 $\overline{X}$ = 33.9 in ; S = 1.507 in.

4.16 30.886 and 36.914; Contains 190 observations.

4.17 Place A: $\overline{X}$ = Rs.106.32; s = Rs.30.6.

Place B: $\overline{X}$ = Rs.106.46; s = Rs.32.7.

4.18 $\overline{X}$ = Rs.12.006: s = Rs.2.626.

4.19 $\overline{X}$ = Rs.90.15; S = 15.99; 65%; 95%; 100%.

4.20 $\overline{X}$ = 14.23; s = 0.72 in. smallest size = 12.82 in.; largest size = 17.14 in.

4.21 Actual class-intervals are: 109.5 – 115.5; 115.5 – 121.5, 121.5 – 127.5; 127.5 – 133.5 – 139.5; 139.5 – 145.5, 145.5 – 151.5; 151.5 – 157.5 (h = 6, P.M. = 136.5).

4.22 Source A: $\overline{X}$ = 1060 hours; s = 21.1 hours

Source B: $\overline{X}$ = 1060 hours; s = 22.2 hours

These data give a false impression as the distribution of U-shaped.

4.24 (b) 13.87%.

4.25 (b) s = 8.3; C.V. (A) = 16.58%; C.V. (B) = 9.50%.

Locality *A* has a greater relative dispersion.

https://stat9943.blogspot.com

4.26 C.V. for $X = 19.25$: C.V. for $Y = 25.58$.
Candidate $X$ showed more consistent performance.

4.27 (a) s (corrected) = 2.79; C.V. = 4.14%.
(b) C.V. for batsman $A$=117.67%; C.V. for batsman $B$=70.45%.
Batsman $A$ is better as a run getter but batsman $B$ is the more consistent players.

4.28 (b) C.V. = 9.33, 9.17.
(c) Tube B has a greater absolute dispersion. Tube A has a greater relative dispersion.

4.29 Town A: s = 11.88, C.V. = 21.15%.
Town B: s = 12.95, C.V. = 23.59%.

4.30 (A): C.V. = 24.99; (B): C.V. = 23.54; (C): C.V. = 22.80.

4.32 (a) $\overline{X} = 57.06$, S = 8.75.
(b) $\overline{X} = 16$, S = 7.2, C.V. = 45.

4.33 (c) 78, 15.

4.34 Average score of student $A$=59; Average score of student $B$=64.

4.35 (b) Trimmed: mean = 71.33, s.d. = 5.83
Winsorized: mean = 71.2, s.d. = 6.61.

4.39 (a) $m_1 = 0$, $m_2 = 1.5$, $m_3 = 0$, $m_4 = 6$; $m_1' = 3$,
$m_2' = 10.5$, $m_3' = 40.5$, $m_4' = 168$.
(b) $\overline{X}$=8.16; $m_2 = 57.825$; $b_1 = 0.04$; $b_2 = 2.76$.

4.40 $m_1 = 0$; $m_2 = 2.49$; $m_3 = 0.7$; $m_4 = 18.33$.

4.41 $m_1 = 0$; $m_2 = 6.314$; $m_3 = -5.125$; $m_4 = 82.58$;
$b_1 = 0.104$; $b_2 = 2.071$

4.42 $m_1 = 0$, $m_2 = 13.76$, $m_3 = 3.16$, $m_4 = 528.06$;
$b_1 = 0.004$; $b_2 = 2.79$.

4.43 (b) $b_1 = 0.0003$; $b_2 = 2.75$.

4.44 $b_1 = 0.0002$; $b_2 = 2.97$; $b_1$(corrected) = 0.002;
$b_2$(corrected) = 2.66.

4.45 (b) (i) Symmetrical; (ii) Negatively skewed; (iii) Positively skewed.

4.47 (i) 0.32; (ii) 0.06.

4.48 $m_1 = 0$; $m_2 = 2.2081$; $m_3 = 0.1949$; $m_4 = 12.9646$; $b_1 = 0.0035$; $b_2 = 2.66$. Hen the distribution is slightly positively skewed and is Platy-kurtic.

4.49 (i) (b) is more consistent. (ii) (b) is negatively skewed.
(iii) None of the distribution is mesokurtic.

4.50 (a) 3; (b) $b_1 = 0.49$; $b_2 = 0.65$; Platy-kurtic.

4.51 (a) (i) Second, (ii) Neither, (iii) First;
(b) (i) Greater than 1875, (ii) 1875, (iii) Less than 1875.

4.54 (i) mean = 80, s.d. = 8.944 (ii) mean = 75, s.d. = 11.18

4.55 (i) sk = 0.413, C.V. = 25.43%
(ii) Mean = 3633.33, S.D. = 796.70
(iii) Mean = 3446.6630, S.D. = 876.37

## Chapter 5, Pp. 174–184



### OBJECTIVE

(a) (i) F (un-weighted), (ii) F (laspeyres), (iii) F (two), (iv) F (geometric mean), (v) F (geometric mean), (vi) F (weighted price index), (vii) F (33.3%), (viii) F (183), (ix) T, (x) T.

(b) (i) a, (ii) b, (iii) d, (iv) c, (v) b, (vi) a, (vii) a, (viii) b, (ix) c, (x) a, (xi) d, (xii) d, (xiii) b, (xiv) d, (xv) d.

### SUBJECTIVE

5.20 (i) 100, 99.9, 101.0, 104.7, 108.9, 110.6.
(ii) 95.5, 95.5, 96.5, 100, 104.0, 105.6.

5.21 (i) 100, 137.9, 155.3, 166.7, 181.2, 196.8, 200.0, 223.4, 234.8, 241.5.

5.22 100, 100.6, 89.3, 92.9, 118.3, 114.7; 100, 91.6, 82.1, 81.7, 110.7, 113.4.

5.23 (i) 94.38, 97.67; (ii) 94.65, 98.60; (iii) 94.14, 97.59.

5.24 100, 99.2, 74.4, 53.9.

5.25 88.5, 88.9, 94.2, 94.7.

5.26 100, 107.7, 109.5, 115.0, 114.4, 118.6.

5.27 (i) 100, 101.8, 109.8, 125.8, 130.0 (ii) 100, 101.8, 115.8, 132.6, 138.2

5.28 116.6.

5.29 (a) 84.14, 84.13; (b) 85.22, 85.22.

5.30 (i) 101.6, (ii) 106.4.

5.31 (i) 116.30, (ii) 116.30.

5.32 (i) 115.16, 110.70, (ii) 115.37, 107.28.

5.33 $P_{01}$ (Marshall-Edgeworth) = 86.5, $P_{01}$ (Fisher's) = 86.8

5.34 (i) 49.4; (ii) 202.5.

5.35 Index for 2004 = 92.68; Index for 2005 = 99.01

5.36 (i) 99.06; 103.84 (ii) 99.06, 103.92; (iii) 99.06, 103.88; (iv) 99.06, 103.89; (v) 99.06, 103.88; (vi) 99.06, 104.02.

5.37 (a) (i) 100, 97, 94, 82; (ii) 100, 101, 104, 161.

(b) Index for 1961 = 118.19; Index for 1962 = 120.00.

5.38 126.72.

5.39 Quantity Index for 2007 on 1997 = 129.8, Quantity Index for 1997 on 2007 = 76.3
Price index for 2007 with 1997 = 118.0 and price index for 1997 with 2007 = 83.9

5.40 100.33 (in both cases).

5.41 (i) 173.8; (ii) 70.0.

5.42 (ii) 98.15. The prices in 1929 as compared with the prices in 1928 have fallen down.

5.43 (i) 121.23; (ii) 121.22

5.44 (i) 116.4; (ii) 116.4.

5.45 124.35.

5.46(b) (i) 130.48; (ii) 165.34

5.47 (i) 70.24, (ii) 114.18, 161.09, 205.97



## Chapter 6, Pp. 233–243

### BJECTIVE

(a) (i) F (a fraction), (ii) F (dependent), (iii) F (equally likely),
(iv) T, (v) F (not mutually exclusive),
(vi) F, (vii) T, (viii) T,
(ix) F (are not equal), (x) F ($P(A \cap B) = P(A)\,P(B)$).

(b) (i) c, (ii) c, (iii) c, (iv) b,
(v) a, (vi) c, (vii) a, (viii) b,
(ix) a, (x) b, (xi) d, (xii) a,
(xiii) a, (xiv) b, (xv) c.

### BJECTIVE

6.1 {chair, student}, {chair, pen}, {student, pen}, {chair}, {student}, {pen}, $\phi$.

6.3 (i) {(1, 1), (1, 2), (2, 1), (2, 2)}; (ii) {(1, 2), (1, 3), (2, 2), (2, 3)}, (iii) {(2, 1), (2, 2), (3, (3, 2)}; (iv) {(1, 2), (1, 3), (2, 2), (2, 3), (3, 3)} (v) {2, 3)}.

6.4 (b) (i) {5}; (ii) {1, 3, 4, 5, 6, 7, 8, 9, 10}; (iv) {2, 3, 4, 5}; (iv) {1, 2, 5, 6, 7, 8, 9, 10}.

6.5 (i) {0, 2, 3, 4, 5, 6, 8}; (ii) $\phi$; (iii) {0, 1, 6, 7, 8, 9}; (iv) {1, 3, 5, 6, 7, 9}; (v) {0, 1, 6, 7, 8, (vi) {2, 4}.

6.6 (a) $A$ = {(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (2, 2)}

$B$ = {(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}

6.7 (a) (i) 4; (ii) 24; (b) (i) 2730; (ii) 455.

6.8 635, 013, 559, 600.

6.9 2520.

6.12 (a) The investment counsellor's claim is wrong, as the sum of the given three mutually exclusive events cannot exceed unity.

(b) The given statement is wrong, as the probability of each of the outcomes is not 1/3.

(c) The given statement is wrong, as the sum of the given three mutually exclusive events cannot exceed unity.

(d) Same remarks as for (c) above.

6.13 (i) $\frac{1}{2}$; (ii) $\frac{5}{36}$; (iii) $\frac{7}{8}$; (iv) $\frac{5}{26}$.

6.14 (b) (i) $\frac{1}{3}$; (ii) $\frac{3}{5}$; (iii) $\frac{11}{15}$; (iv) $\frac{2}{5}$; (v) $\frac{4}{5}$.

6.15 (a) $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}; \frac{11}{12}$.

(b) $\frac{1}{6}$.

6.16 (a) $\frac{1}{6}, \frac{23}{36}, \frac{1}{3}$. (b) $\frac{5}{36}, \frac{1}{6}, \frac{1}{36}, \frac{5}{18}$.

6.17 $\frac{1}{15}, \frac{1}{15}, \frac{2}{15}, \frac{2}{15}, \frac{3}{15}, \frac{2}{15}, \frac{2}{15}, \frac{1}{15}, \frac{1}{15}$.

6.18 (i) $\frac{1}{3}$; (ii) $\frac{15}{36}$.

6.19 (b) 0.5177; 0.4914 (c) $\frac{25}{216}; \frac{1}{8}$.

6.20 (a) $\frac{8}{663}$; (b) $\frac{5}{8}$.

6.21 (a) (i) $\frac{16}{33}$; (ii) $\frac{19}{33}$; (b) (i) 0.38, (ii) 0.62, (iii) 0.12

6.22 (a) $\frac{1}{2};\frac{1}{5}$ (b) (i) $\frac{1}{2}$; (ii) $\frac{17}{19}$ (c) (i) $\frac{40}{143}$; (ii) $\frac{9}{143}$

6.23 (a) (i) 0.598, (ii) $1-\left(\frac{5}{6}\right)^n$ (b) 26; (c) 5.

6.24 (i) $\frac{160}{1001}$; (ii) $\frac{109}{143}$

6.25 (a) (i) $\frac{1}{6}$; (ii) $\frac{1}{30}$; (iii) $\frac{5}{6}$ (b) $\frac{49}{143}$.

6.26 (i) $\frac{1}{30}$; (ii) $\frac{1}{6}$; (iii) $\frac{5}{6}$.

6.27 (b) $\frac{2}{3}$ (c) $\frac{9}{20}$

6.28 (b) $\frac{6}{26}$

6.29 (c) $\frac{5}{8}$

6.30 (b) $\frac{2}{9}$; (c) 0.9; (ii) 0.6.

6.31 (b) (i) $\frac{1}{3}$; (ii) $\frac{2}{3}$; (iii) $\frac{1}{12}$ (c) (i) $\frac{1}{8}$; (ii) $\frac{1}{4}$; (iii) $\frac{7}{8}$; (iv) $\frac{1}{4}$

6.33 (b) (i) $\frac{2}{15}$; (ii) $\frac{2}{15}$

6.34 (a) $\frac{5}{9}$; (b) $\frac{1}{3}$; (c) 0.25.

6.35 (i) 0.0779, (ii) 0.4062; (iii) 0.5.

6.36 (c) 0.60, 0.76, 0.60, 0.40, 0.60.

6.37 (b) (i) 0.3; (ii) 0.5; (c) (i) 0.10, (ii) 0.20; (iii) 0.17.

6.38 (a) True; (b) False, if $A$ and $B$ are independent, then $P(A/B) = P(A)$; (c) True. (d) False, independent does not mean that two events have equal probabilities.

6.39 (b) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$ and $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$



6.40 (i) not independent; (ii) not independent; (iii) not independent.

6.41 (a) $\frac{37}{40}$; (b) $\frac{8}{5525}$

6.42 (i) $\frac{4}{63}$; (ii) $\frac{10}{63}$; (iii) $\frac{2}{9}$; (iv) $\frac{5}{9}$

6.43 (a) $\frac{65}{176}$ (b) $\frac{49}{80}$

6.44 $\frac{7}{429}$

6.45 (b) $\frac{25}{1218}$

6.47 (i) $\frac{3}{32}$; (ii) $\frac{13}{32}$

6.48 $\frac{2}{5}$ or $\frac{94}{175}$ (In this case all three also favour).

6.49 (a) $\frac{5}{6}$ (b) 0.012



6.50 (i) $\frac{31}{72}$; (ii) $\frac{6}{31}$

6.51 (i) 0.12; (ii) 0.88; (iii) 0.38; (iv) 0.38

6.52 (a) $\frac{1}{8}$; (b) $\frac{5}{72}$; (c) $\frac{5}{36}$; (d) $\frac{19}{27}$

6.53 $\frac{901}{1680}$.

6.54 0.586.

6.55 (a) 2 : 1.

6.56 (a) $\frac{15}{64}$; (b) (i) 0.196; (ii) $\frac{1}{15}$

6.57 (i) 0.31; (ii) 0.04



6.58 (b) $\frac{1}{3}$.

6.59 $\frac{9}{29}$.

6.60 $\frac{3}{11}$

6.61 $\frac{1000}{29}\%$

6.62 (i) 0.256 (ii) 0.415 (iii) 0.328

6.63 0.0020; 0.0096; 0.9883.

## Chapter 7, Pp. 294–302



### BJECTIVE

(a) (i) F (any), (ii) F (discrete), (iii) F (continuous),
(iv) F ($\sum xP(x)$), (v) F ($\sum(x-\mu)^2 P(x)$), (vi) F (fixed set of values),
(vii) F (constant), (viii) F (independent), (ix) F (one),
(x) F (at least).

(b) (i) d, (ii) a, (iii) d, (iv) a,
(v) b, (vi) a, (vii) b, (viii) b,
(ix) a, (x) c.

### BJECTIVE

7.2 (b) (i) $\frac{1}{12}$; (ii) $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$

(c) 0.3038, 0.4389, 0.2135, 0.0412, 0.00264.

7.3 (b) $\frac{15}{210}, \frac{80}{210}, \frac{90}{210}, \frac{24}{210}, \frac{1}{210}$.

7.4 (a) $\frac{120}{455}, \frac{225}{455}, \frac{100}{455}, \frac{10}{455}$ (b) $\frac{1}{56}, \frac{15}{56}, \frac{30}{56}, \frac{10}{56}$

7.5 (b) $\frac{3}{8}$.

7.6 $\frac{1}{8}, 0, \frac{15}{32}, \frac{5}{8}$.



7.7 (i) $\frac{3}{16}$ (ii) 0, (iii) $\frac{11}{16}$ (iv) $\frac{0}{16}$ (v) $\frac{5}{16}$

7.8 (ii) $3x^2 - 2x^3$; (iii) $\frac{13}{27}, \frac{1}{2}$; (iv) $b = 0.6130$.

7.9 (i) $\frac{1}{4}$; (ii) $\frac{3}{64}$.

7.10 (b) $g(x) = \frac{4}{15}, \frac{1}{3}, \frac{2}{5}$; $h(y) = \frac{2}{5}, \frac{1}{3}, \frac{4}{15}$

Conditional p.d. of $X$ given that $Y = 1$; $f(x/1) = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

$$f(y/2) = \frac{2}{5}, \frac{1}{2}, \frac{1}{10}.$$

7.11 (a) $g(x) = \frac{5}{36}, \frac{19}{36}, \frac{1}{3}$; $h(y) = \frac{1}{4}, \frac{14}{45}, \frac{79}{180}$.

$$f(x/1) = \frac{1}{3}, \frac{2}{3}, 0; \; f(y/3) = 0, \frac{3}{5}, \frac{2}{5}.$$

(b) $X$ and $Y$ are not independent.



7.12 (i) $g(x) = \frac{x}{11}$, $x = 2, 4, 5$; $h(y) = \frac{y}{6}, y = 1, 2, 3$.

$X$ and $Y$ are independent.

(ii) $g(x) = \frac{x}{6}$, $x = 1, 2, 3$; $h(y) = \frac{y^2}{5}, y = 1, 2$.

$X$ and $Y$ are independent.

7.13 (b) $g(x) = \frac{3}{2}x^2 + x$; $h(y) = y + \frac{3}{2}y^2$; $f(x/y) = \frac{3x(x+y)}{1 + \frac{3}{2}y}$,

$$f(y/x) = \frac{3y(x+y)}{1 + \frac{3}{2}x}.$$

7.14 (a) It is a *p.d.f.* (b) (i) $\frac{5}{6}$ (ii) $\frac{7}{24}$ (iii) $\frac{65}{72}$, (iv) $\frac{5}{32}$.



7.15 (b) $g(x)=\frac{3}{2}(x+1), h(y)=\frac{1}{3}+\frac{4}{3}y; f(x/y)=\frac{2x+4y}{1+4y}$.

$f(y/x)=\frac{x+2y}{x+1};\frac{5}{12}$.

7.16 $g(x)=\frac{3}{2}x^2+x;\ h(y)=y+\frac{3}{2}y^2;\ f(x/y)=\frac{6x(x+y)}{2+3y}$:

$f(y/x)=\frac{6y(x+y)}{2+3x};\frac{769}{2528}$.

7.17 $g(x)=12x^2(1-x);\ f(y/x)=2y$. The variables $X$ and $Y$ are independent.

7.19 (c) 7.

7.20 (a) $\frac{105}{56}$ (b) $E(X)$ does not exist.

7.22 (a) $\frac{3}{4};\frac{9}{8}$ (b) (i) 0.55; 1.35; (ii) 2.1; 5.4.

7.23 (a) 20 (b) $\mu=\frac{2}{3}$, $\sigma^2=\frac{2}{63}$ (c) $\frac{11}{64}$

7.24 (b) 7 (c) 2.75, $\frac{15}{16}$



7.25 (a) 7, $\frac{35}{6}$; (b) $\frac{n+1}{2},\frac{n^2-1}{12}$.

7.26 (a) Rs.6 and Rs.5 (b) £64, £48, £36, £27 (c) Rs.9, Rs.6, Rs.3

7.27 (a) $\mu=0.6,\ \sigma=0.2$.

(b) (i) $\frac{2}{3}$; (ii) $\frac{14}{9},\frac{13}{162},\sqrt{\frac{13}{162}}$; (c) $\frac{1}{2}$;0.39

7.28 (a) $\mu=\frac{2}{3};\sigma^2=\frac{2}{63}$; (b) $A=\frac{4}{81},\mu=\frac{8}{5},\ \sigma=0.66$

7.29 $\mu=0;\ \sigma=1$.

7.30 $k = \frac{2}{27}$; mode $= \frac{1}{2}; \mu = \frac{7}{6}; \sigma^2 = \frac{97}{180}$.

7.31 $k = \frac{2}{9}$; $\mu = \frac{7}{2}$; $\sigma^2 = \frac{9}{20}$, mode $= \frac{7}{2}$, median $= \frac{7}{2}$



7.33 $\mu_3 = 0.001$.

7.34 (a) $\mu = 0$, $\sigma^2 = \frac{a^2}{3}$, $M.D. = \frac{a}{2}$.

(b) $\mu_1 = 0$, $\mu_2 = \frac{1}{12}$, $\mu_3 = 0$, $\mu_4 = \frac{1}{80}$, $M.D. = \frac{1}{4}$

7.35 $\mu = \frac{3}{2}$, $\sigma^2 = \frac{5}{12}$, $\beta_2 = 2.184$

7.36 $\mu_1 = 0$, $\mu_2 = 129$, $\mu_3 = 0$, $\mu_4 = 3.86$, $\beta_2 = 2.33$.

7.37 $g(x) = \frac{3}{5}, \frac{2}{5}$; $h(y) = \frac{1}{5}, \frac{1}{3}, \frac{7}{15}, \rho = 0.25$

7.38 −0.091

**7.41** (a)

| $x_i$ | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| $f(x_i)$ | 10/30 | 8/30 | 6/30 | 4/30 | 2/30 | 1 |

(b) Rs.4, Rs.3, Rs.3 and Rs.1.

**7.42** (b) Mean = 12; variance = 36

(c) Expected value = 6,00,000, S.D. = 12,000



## Chapter 8, Pp. 329–340

### OBJECTIVE

(a) (i) F, (ii) F (two), (iii) F (fixed specified), (iv) T, (v) F (not equal), (vi) T, (vii) F (equal), (viii) F (small, large), (ix) T, (x) F (one), (xi) F (independent), (xii) F (less), (xiii) F (positively skewed), (xiv) T, (xv) T.

(b) (i) d, (ii) b, (iii) c, (iv) b, (v) a, (vi) a, (vii) b, (viii) b, (ix) d.

### SUBJECTIVE

8.2 (b) 0, 0.29, 0.936, 0, 0.352.

8.3 (a) (i) 0.1317, $\frac{131}{243}, \frac{200}{243}$ (b) (i) 0.28, (ii) 0.31, (iii) 0.23.

8.4 (a) (i) $\frac{15}{64}$, (ii) $\frac{57}{64}$ (b) (i) $\frac{57}{64}$, (ii) $\frac{21}{32}$.

8.5 (a) 0.512; (b) 0.227; (c) 0.124



8.6 (a) (i) $\frac{2816}{3125}$; (ii) $\frac{53}{3125}$; (b) (i) 0.0055; (ii) 0.2903

8.7 (i) $\frac{32}{243}$; (ii) $\frac{192}{243}$; (iii) $\frac{40}{243}$; (iv) $\frac{11}{243}$

8.8 (a) 0.3134, (b) 0.10737, 0.99363

8.9 (a) 0.4374, 6.124, 35.72, 111.132, 194.48, 181.5, 70.6.

(b) Expected frequencies are: 3, 15, 30, 30, 15, 3.

8.10 (a) 27.31% : (b) 1 approximately.

8.11 52.08, 41.67, 12.50, 1.67, 0.08; Mean = $\frac{2}{3}$.

8.12 (c) 0.0073;

8.13 (a) $p = 0.36$, $n = 100$ : (b) $p = 0.3021$, $n = 41$; (c) No, it makes $q = 1.8$ which is wrong.

8.17 (b) Median = 5, Mode = 5.

8.18 (a) 0.65, (b) Theoretical frequencies are 8, 56, 155, 192, 89.

8.19 $p = 0.25$ and the expected frequencies are, 35.60, 71.19, 59.33, 26.37, 6.59, 0.88, 0.05.

8.20 $p = 0.32$ and the expected frequencies are: 32, 60, 43, 13 and 2.

8.21 $\overline{X} = 5.42$ and $s = 1.70$.

8.23 (b) $E(X) = 9$; $\text{Var}(X) = 2.25$; 0.3907.

8.26 $\frac{840}{462}$; $\frac{84}{121}$.

8.27 (a) $\frac{4}{20}, \frac{12}{20}, \frac{4}{20}$; (b) $\frac{3}{14}$.

8.28 0.004, 0.50.

8.29 (a) 0.3179; (b) 0.82.

8.31 (a) $\dfrac{\binom{30}{3}\binom{120}{1}}{\binom{150}{4}}=0.0240$ (b) $b(3; 4, 0.2) = 0.0256$.



8.32 (e) 0.2019, 0.3230, 0.2584, 0.2167

8.33 (a) 0.04918, 0.1494, 0.2241, 0.2241, 0.1681.

(b) 0.2644, 0.1037 (c) 0.135, 0.857.

8.34 (b) (i) 0.99994 (ii) 0.9442

8.35 (b) (i) 0.1937 (ii) 0.1839

8.36 (a) (i) 0.9513 (ii) 0.9989; (b) 0.221.

8.37 (b) 0.2231, 0.1913.

8.38 0.5620

8.39 (b) 0.1839.

8.40 (a) The statement is wrong. (b) 51 and 11.

8.44 Expected frequencies are: 202.16, 137.96, 47.08, 10.72, 1.84, 0.24.

8.45 (b) (i) 0.642 (ii) 0.073.

8.46 123, 110, 49, 14, 3, 1, 0.

8.47 90.3, 108.4, 65.0, 26.0, 7.8, 1.9, 0.4, 0.1; 0.0341

8.48 (b) 0.3679.

8.49 (a) 0.5272, 0.8646; (b) 0.2231

8.50 (i) 0.2682; (ii) 0.0614

8.51 (i) 0.4380; (ii) 0.5620

8.54 (b) 0.10033

8.55 (b) no, (c) 0.1172.

8.56 (b) 0.0515.

8.58 (c) 1/8.

8.59 (b) 0.0129

8.60 (a) 0.09; (b) (i) 0.135, (ii) 0.081, (iii) 0.081.

**8.61** (b) (i) 0.1432; (ii) 0.0682

(c) 0.09

**8.62** (b) (i) 0.5217, (ii) 0.0059, (iii) 0,0000001.

## Chapter 9, Pp. 411–420

### OBJECTIVE



(a) (i) T, (ii) F (most), (iii) F (all),
(iv) T, (v) F (0.5), (vi) F (mean),
(vii) F (equal), (viii) F (zero, one), (ix) F (one),
(x) F (equal to), (xi) F (1), (xii) F (whole & fraction),
(xiii) F (equal to), (xiv) F (will), (xv) F (not same).

(b) (i) c, (ii) d, (iii) d, (iv) b,
(v) c, (vi) d, (vii) b, (viii) d,
(ix) c, (x) b.

### SUBJECTIVE

9.1 (b) $\frac{1}{t}(e^{t/2}-e^{-t/2})$; $\mu_2=\frac{1}{12}$; $\mu_3=0$, $\mu_4=\frac{1}{80}$.

9.2 (b) 0.2231.

9.3 (a) $\mu=2$; $\sigma^2=4$; $\frac{1}{1-2t}$ : 0.2231; 0.2232.

(b) 18.1%, 49.8%.

9.4 (b) m.g.f. = $\frac{1}{1-at}$; $\mu_1=0$, $\mu_2=a^2$, $\mu_3=2a^3$, $\mu_4=9a^4$.

9.5 $\mu_1=0, \mu_2=2, \mu_3=4, \mu_4=24$.

9.7 (a) $\mu=0, \sigma^2=2$ (b) $a=\frac{2}{\pi}$; 0.6014.

9.11 (a) The distribution is $\beta_2(m,n)$ variate.

9.15 $k=\frac{1}{\sqrt{24\pi}}$; $\mu=3$; $\sigma=\sqrt{12}=3.464$.

9.18 (b) (i) 43.82%; (ii) 67.05

9.19 (a) 0.5000; 0.3694 (b) (i) 0.1672; (ii) 0.7492.

9.20 (a) 0.6147; (b) 0.4822; (c) 0.9973.

9.21 (a) 0.3085; (b) 0.6915; (c) 0.0548; (d) 0.1832; (e) 0.6898; (f) 0.9452.

9.22 (b) (i) 0.4052; (ii) 0.3745; (iii) 0.2358

9.23 (a) (i) 0.0663; (ii) 0.0062; (iii) 0.9198 (b) 69.15%.

9.24 (a) (i) 0.0918; (ii) 0.5375 (b) 0.3983; 0.4649.

9.25 (i) 6; (ii) 131; (iii) 880; (iv) 24.

9.26 (a) (i) 26,600; 22,660 (b) (i) 2 days (ii) 58 days (iii) 228 days.



9.27 (a) (i) 2.28; (ii) 15.87; (iii) 68.26; (iv) 13.36 (b) 3.59.

9.28 (i) The old route is better; (ii) the new route is better.

9.29 (a) 99.36; (b) 0.263

9.30 (i) 0.4404; (ii) 12.43, 82.77; (iii) 26.86, 39.11, 85.28.

9.31 (a) (i) 570; (ii) 0.4052.

9.32 (a) 0.38% (b) 0.4514, (c) 23, (d) 189.95.

9.33 (a) 168 cm, (b) 6.24 years.

9.34 87 inches.

9.35 (a) $\mu = 12.5$; $\sigma = 6.67$ (b) 50; 10 (c) $\mu = 71.46$, $\sigma^2 = (13.70)^2$

9.36 40.37; 12.32.

9.37 $\mu = 1.7905$m, $\sigma = 0.0706$m, 2.009m.

9.38 (b) 0.0143.

9.39 (b) (i) 0.9962; (ii) 0.0681; (iii) 0.0558.

9.40 (a) (i) 0.1925; (ii) 0.2177 (b) (i) 0.6982; (ii) 0.6970.

9.41 (a) 0.599 (b) 0.7469, (i) 0.6246, (ii) 0.7462.

9.42 (a) (i) 0.3251, (ii) 0.1781, (iii) 0.2803, (iv) 0.0459.

9.43 (a) $\overline{X} = 67.9$; $s = 2.36$ approx. Proportions are 69%, 95% and 100%. The distribution is nearly normal.

(b) Fitted frequencies by the area method are:

2.55, 5.70, 15.30, 32.62, 59.25, 92.78, 116.18, 124.65, 114.82, 84.38, 54.52, 28.05, 12.52, 6.68.



9.44 The equation to the fitted normal distribution is

$$f(x) = \frac{280}{9.50\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-64.961}{9.50}\right)^2\right]$$

Frequencies: 4.42, 10.00, 23.58, 41.61, 54.82, 57.20, 43.93, 26.80, 12.12, 5.52.

Ordinates: 3.14, 9.86, 23.31, 41.31, 56.03, 57.50, 44.71, 26.25, 11.85, 3.98.

9.45 Frequencies: 6.8, 30.7, 100.4, 210.4, 269.6, 221,0, 114.6, 37.6, 8.9.

## Chapter 10, Pp. 449–456

### OBJECTIVE



(a) (i) F (does not indicate), (ii) F (regression analysis),
(iii) F (−1 and +1), (iv) T (is),
(v) F (correlation analysis), (vi) F (must not both),
(vii) T, (viii) T,
(ix) F (is always positive), (x) F (zero).

(b) (i) a, (ii) a, (iii) a, (iv) d,
(v) c, (vi) b, (vii) a, (viii) a,
(ix) b, (x) c.

### SUBJECTIVE

10.6 (a) $\hat{Y} = 27.63 - 1.063X$ (b) 17.00, 15.94, 11.69, 9.56, 7.43, 6.37.

(c) $\sum(Y-\hat{Y})^2 = 12.995,\ s_{Y.X} = 1.802$

10.7 (b) $\hat{Y} = 2.00 + 0.387\,X$ (d) 5.87

10.8 (a) $\hat{Y} = 30 - X$; (b) $\hat{Y} = 364.68 + 0.095X$;

(c) $\hat{Y} = -3.17 + 0.32X$, (d) $\hat{Y} = -75.9 + 0.89X$

(e) $\hat{Y} = 53.68 + 3.53X$.

10.9 (b) $\hat{Y} = 130.62 - 0.57X$; (c) 1.22.

10.10 (a) $\hat{Y} = -0.43 + 2.06X$; (b) 0.41; (c) $\hat{X} = 0.27 + 0.47Y$; (d) 0.20.

10.11 (b) $r^2 = 0.95$.

10.12 (b) (i) 1234; (ii) 1168.45; (iii) 65.55; 95% variability is explained by regression model.

10.14 (b) $r = -0.92$ (c) $r = -0.92$.

10.15 (b) $r = 0.93$.

10.16 (a) (i) 0.7; (ii) −0.7 (b) $r = 0.70$.

10.17 (i) $r = 0.91$ (ii) $\hat{Y} = 33.3 + 1.43X$.

10.18 $r = 0.94$.

10.19 $r = -0.61$.

10.20 $r = -0.98$; $\hat{Y} = 31.55 - 1.94X$ and $\hat{X} = 16 - 0.5Y$.

10.21 (a) $r = 0.876$ (b) $r = 0.97$

10.22 $r = 0.898$.

10.23 $r = 0.67$.

10.24 (a) $r = -0.415$; (b) $r = 0.75$ (c) $r = 0.54$; (d) $r = -0.94$.



10.25 $r = 0.58$; $\hat{Y} = 11.32 + 0.54X$.

10.26 $r = 0.39$.

10.27 (b) $r = 0.9883$; $\hat{Y} = 0.33X - 54.09$; $\hat{X} = 3.03Y - 192.16$

10.28 $r = \dfrac{1}{2}$.

10.29 (b) $r_s = 0.80$.

10.30 (b) $r = -0.6$.

10.31 (b) $r_s = 0.8545$.

10.32 Denoting the judges by 1, 2, 3; $r_{12} = -0.21$; $r_{23} = -0.30$, $r_{13} = 0.64$. This suggests that judges 1 and 3 have the nearest approach to common tastes.

10.33 $r_{xy} = 0.625$; $r_{zy} = 0.503$; $r_{xz} = 0.673$. Pair $(Z, X)$ has the nearest approach to common tastes.

10.34 $r_s = 0.33$.

10.35 $W = 0.21$.



## Chapter 11, Pp. 478–484

### OBJECTIVE

(a) (i) F (co-efficient of Multiple correlations), (ii) F (Partial correla
(iii) F (sq-root), (iv) F (positive), (v) T,
(vi) F (0.6), (vii) F (n – k – 1), (viii) T,
(ix) T, (x) F (change).

(b) (i) c, (ii) b, (iii) a, (iv) c,
(v) b, (vi) d, (vii) c, (viii) c,
(ix) a, (x) b.

### SUBJECTIVE

11.2 $a = 3.88$, $b_1 = 2.09$, $b_2 = 2.65$.

11.3 (a) $\hat{Y} = 4.49 - 0.04X_1 + 0.64X_2$.

11.4 (a) $\hat{X}_1 = 0.04 + 0.21X_2 + 0.28X_3$; (b) 0.21; (c) 0.9863; 0.99.

11.5 (a) $\hat{X}_3 = 61.40 - 3.65X_1 + 2.54X_2$ (b) 40; (c) 0.9927.

11.6 (a) $\hat{Y} = 20.1084 + 0.4136X_1 + 2.0253X_2$ (b) $\hat{Y} = 57.5$ kg

11.7 (b) $R_{1.23} = 0.77$



11.8 (a) $r_{12} = 0.952$, $r_{13} = 0.056$, $r_{23} = 0.304$; $R_{1.23} = 0.98$,
$R_{2.31} = 0.98$; $R_{3.12} = 0.82$.

11.9 (b) $R_{2.13} = 0.83$ (c) $R_{2.13} = 0.68$

11.10 (a) $R_{1.23} = 0.80$ (b) $R_{2.13} = 0.66$; $r_{23.1} = 0.64$.

11.11 (b) 0.48.

11.12 $r_{12.3} = 0.759$; $r_{23.1} = -0.436$; $r_{13.2} = 0.097$.
$X_1 = 9.22 + 3.37X_2 + 0.0038X_3$.

11.13 $\hat{X}_1 = -26.77 + 0.39X_2 + 0.23X_3$;
$r_{12.3} = 0.55$; $r_{13.2} = 0.15$; $r_{23.1} = 0.21$.

11.14 $r_{12.3} = 0.63$; $r_{13.2} = 0.49$; $r_{23.1} = -0.035$.

11.15 (b) (i) Not possible; (ii) Not possible; (iii) Is possible.

11.16 $r_{13.2} = -0.586$; $r_{42} = 0.874$, $r_{43} = 0.836$, $r_{43.2} = -0.586$.

11.17 (c) $R_{2.13} = \sqrt{(r_{23}^2 - r_{13}^2)/(1 - r_{13}^2)}$, which is not necessarily zero.

11.18 (b) $r_{12.3} = r_{13.2} = r_{23.1} = -1$.

11.20 $\hat{Y} = 2.08 + 0.64X - 0.1X^2$; 0.13



## Chapter 12, Pp. 503–510

### BJECTIVE

12.1 (c) $Y = 1.52 + 1.66X$

12.2 (c) $Y = 7.2 + 1.28X$. The calculated values are 7.20, 8.48, 9.76, 11.04, 12.32, 13.60, 14.88, 16.16, 17.44.

12.3 (b) $Y = 1.2 + 0.5X$; 5.0001.

12.4 (b) $Y = 11.25 + 0.70X$.

12.5 (a) $Y = 6.32 + 0.84X$ (b) $Y = 0.9 + 4.7X$ and the estimated value of $Y = 29.1$; and $X = 0.09 + 0.19Y$ and the fitted value of $Y = 31.1$.

12.6 (c) $Y = 0.83 + 1.60X - 0.71X^2$.

12.7 (b) $Y=1.172 + 0.056X + 2.28X^2$ (c) $Y=1.428 + 0.244X + 2.214X^2$

12.8 $Y = 1.04 - 0.20X + 0.24X^2$

12.9 $Y = -6.622 + 1.033X - 0.0056X^2$

12.11 (i) $Y = 123.2 + 8.29X$; 9008.4.

(ii) $Y = 246.66 + 29.99X - 0.29X^2$; 340.6.



12.12 (i) $Y = 0.32 + 1.13X$; 5.819;

(ii) $Y = 1.42 - 1.07X + 0.55X^2$; 1.584:

(iii) $Y = 1.03 + 1.725X - 1.40X^2 + 0.325X^3$; 0.063.

12.13 (b) $Y = 10 + 5X + 2X^2$.

12.14 (b) $Y = 74.04\ (1.22)^X$ and the estimated value of $Y = 244.13$.

12.15 $Y = 32.15\ (1.427)^X$ and the estimated value of $Y = 387.40$.

12.16 $Y = 0.8576\ (1.393)^X$ or $Y = 3.22\ (1.393)^X$ with origin at 4.

12.17 $Y = 8.478\ (1.195)^X$.

12.18 $Y = 2.39\ (1.19)^X$; 1954 for $X = 12$.

12.19 (a) $Y = 158.5\ (2.17)^X$ (b) $Y = 2.50 + 3.50\sqrt{X}$.

12.20 $Y = 2033\ (X)^{-1.7488}$.

12.21 $Y = 6483.1\ (X)^{-1.04}$ and the estimated value of $Y = 12.76$.

12.22 $a = 5.703$, $n = 1.02$;

12.23 $v = 9.998\ e^{-0.2t}$

12.24 (a) $Y = 9.88\ e^{1.002X}$

12.25 $\log A = 12.1669$; $\alpha = -2.3064$.

12.26 $\gamma = 1.42$.

12.27 (b) $\dfrac{1}{Y} = 0.10 + 0.025X$.

12.28 (b) $Y = 4.49 - 0.04X_1 + 0.64X_2$.

12.31 (b) $X = 0.034$, $Y = -0.305$

12.32 (a) $X = 1.162$, $Y = 2.262$; (b) $X = 1.02$, $Y = 2.50$; $\sum e_i^2 = 0.1292$

12.33 (a) $X = 0.9997$, $Y = 2.0010$; (b) $X = 2.31$, $Y = -1.02$.

12.34 $X = 1.17$, $Y = -0.75$, $Z = 2.08$.

12.35 (b) $X = 2.47$, $Y = 3.55$, $Z = 1.92$.

## Chapter 13, Pp. 543–552

### OBJECTIVE

(a) (i) F (long-term), (ii) F (four), (iii) F (seasonal variations),
(iv) F (cannot), (v) F (historigram), (vi) F (multiplication),
(vii) F (secular), (viii) T, (ix) F (300,000),
(x) T.



(b) (i) a, (ii) d, (iii) c, (iv) c,
(v) c, (vi) a, (vii) a, (viii) b,
(ix) a, (x) d.

### SUBJECTIVE

13.1 (b) (i) cyclical, (ii) long-term trend, (iii) seasonal, (iv) seasonal, (v) irregular.

13.2 (b) (i) Irregular, (ii) cyclical, (iii) seasonal, (iv) long-term trend, (v) seasonal, (vi) cyclical, (vii) seasonal, (viii) long-term trend, (ix) irregular, (x) long-term trend.

13.10 (ii) 132.5, 152.7, 172.9, 193.1, 213.3, 235.5, 253.7, 273.9, 294.1, 314.3, 334.5, 354.7.

13.11 127.6, 143.2, 160.8, 176.8, 192.8, 207.2.

13.12 (ii) 149.8, 142.0, 141.0, 139.8, 138.2, 140.0, 142.0, 137.6, 128.2, 126.2, 119.8, 113.0, 104.4, 102.2.

13.13 (a) 46.3, 46.7, 46.3, 46.7, 46.0, 46.1, 46.3, 46.9.

$\hat{Y}_t = 46.57 + 1.28\,X$ with origin at midpoint of Saturday and Sunday.

Trend values: 38, 40, 41, 42, 43, 45, 46, 47, 48, 50, 51, 52, 54, 55.

13.14 146.6, 138.0, 133.6, 130.7, 128.9, 129.6, 130.6, 133.3, 139.6, 145.1, 157.9, 176.3, 197.6, 220.3, 243.4.

13.15 (a) 6.9, 6.7, 6.3, 6.6, 7.6, 8.6, 8.8, 8.7, 8.6, 8.2, 8.7, 9.7, 10.6, 10.7, 10.3, 10.0, 9.7, 10.0, 10.8, 11.4.

(b) 27.25, 29.88, 32.12, 33.50, 34.12, 34.88.

13.16 83.1, 83.0, 82.6, 81.5, 79.8, 78.9, 79.2, 80.1, 80.9, 81.2, 81.8, 82.1, 82.6, 83.2, 84.0.

13.17 82.4, 89.6, 97.2, 117.1, 153.2, 188.1, 220.5, 269.6, 327.1, 379.8, 438.0, 528.8.

13.18 (a) 2-year *centred* moving averages are: 50.8, 64.8, 65.8, 66.2, 70.2, 74.2, 81.0, 83.8.
3-year weighted moving averages are: 50.8, 64.8, 65.8, 66.2, 70.2, 74.2, 81.0, 83.8.

(b) 50.2, 67.8, 64.5, 65.8, 71.2, 72.8, 82.0, 84.5.

13.19 (i) 3.8, 4.6, 5.4, 6.2, 7.0, 7.8, 8.6, 9.4, 10.2., (ii) 4.0, 6.0, 7.0, 7.0, 7.0, 8.0, 10.0.

(iii) $\hat{Y}_t = 7 + X$ with origin at 1972.

13.20 (i) 24.0, 36.0, 42.0, 42.0, 42.0, 48.7, 59.7.

(ii) $\hat{Y}_t = 41.89 + 5.95X$ with origin at 1974 and units of $X$ are 1 year.

13.21 (a) $\hat{Y}_t = 99.6 - 12.4X$ with origin at 1997 and units of X are 1 year.

(b) $\hat{Y}_t = 39.9 - 0.7673X$ with origin at 1953.

Trend values are: 43.74, 42.97, 42.20, 41.43, 40.67, 39.90, 39.13, 38.37, 37.60, 36.83, 36.06.



13.22 (i) $\hat{Y}_t = 107.14 + 6.38X$ with origin at 1980; estimated value of $\hat{Y}_t = 139.04$.

(ii) $\hat{Y}_t = 103.24 + 6.38X + 0.976X^2$ with origin at 1980.

13.23 $\hat{Y}_t = 133.09 + 30.25X + 0.155X^2$ with origin at 1970-71; estimated value of $\hat{Y} = 385.01$.

13.24 $\hat{Y} = 31.57 + 5.46X + 0.155X^2$ with origin at <u>2000</u> and units of $X$ are 1 year. Trend values are 18.52, 18.85, 20.32, 22.93, 26.68, 31.57, 37.60, 44.77, 53.08, 62.53, 73.12.

13.25 $\hat{Y} = 351.1 + 13.188X + 0.311X^2$ with origin at <u>Jan. 1, 2005</u> and X is measured in units of a half year

13.26 $\hat{Y} = 192 - 20X + 1.332X^2$ with origin at 1924. Estimated value of $\hat{Y} = 133$.

13.27 $\hat{Y} = 5.35\,(1.345)^X$ and estimated value of $\hat{Y} = 57.30$.

13.28 (b) 118.30, 93.70, 78.75, 108.95.

13.29 101.5, 98.6, 100, 100.

**13.30** (a) 135.44, 105.88, 59.75 and 98.92

(b) 125.3, 104.6, 60.3, 109.7

Deseasonalization of Data for <u>1997</u>: 75, 69.8, 68.0, 62.0

13.31 160.3, 105.8, 46.6, 87.2.

13.32 99.6, 100.9, 98.3, 101.2.

13.33 72.06, 85.76, 110.02, 132.18. Deseasonalization of Data for 1954: 111.02, 122.43, 136.[illegible] 147.53.

13.34 104.13, 107.24, 98.87, 89.77.

13.35 (a) $Y = 94.08 + 0.83X$.

(b) (i) 121.6, 90.2, 78.6, 109.5.

13.36 93.6, 106.6, 112.6, 87.2.

13.37 Seasonal Indices: 95.81, 109.46, 85.28, 109.45; and forecast: 73, 88, 72, 96.

13.38 Cyclical relatives: 97.70, 99.37, 99.47, 101.58, 103.61, 104.13, 103.42, 102.27, 100.19, 98.[illegible] 96.35, 95.41, 96.76, 100.33.

13.40 (b) $r_1 = 0.66$.

◆◆◆◆◆◆◆◆◆◆

# TABLES

# TABLES

## Table 1 – Logarithms



| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **10** | 0000 | 0043 | 0086 | 0128 | 0170 | | | | | | 5 | 9 | 13 | 17 | 21 | 26 | 30 | 34 | 38 |
| | | | | | | 0212 | 0253 | 0294 | 0334 | 0374 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| **11** | 0414 | 0453 | 0492 | 0531 | 0569 | | | | | | 4 | 8 | 12 | 16 | 20 | 23 | 27 | 31 | 35 |
| | | | | | | 0607 | 0645 | 0682 | 0719 | 0755 | 4 | 7 | 11 | 15 | 18 | 22 | 26 | 29 | 33 |
| **12** | 0792 | 0828 | 0864 | 0899 | 0934 | | | | | | 3 | 7 | 11 | 14 | 18 | 21 | 25 | 28 | 32 |
| | | | | | | 0969 | 1004 | 1038 | 1072 | 1106 | 3 | 7 | 10 | 14 | 17 | 20 | 24 | 27 | 31 |
| **13** | 1139 | 1173 | 1206 | 1239 | 1271 | | | | | | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| | | | | | | 1303 | 1335 | 1367 | 1399 | 1430 | 3 | 7 | 10 | 13 | 16 | 19 | 22 | 25 | 29 |
| **14** | 1461 | 1492 | 1523 | 1553 | 1584 | | | | | | 3 | 6 | 9 | 12 | 15 | 19 | 22 | 25 | 28 |
| | | | | | | 1614 | 1644 | 1673 | 1703 | 1732 | 3 | 6 | 9 | 12 | 14 | 17 | 20 | 23 | 26 |
| **15** | 1761 | 1790 | 1818 | 1847 | 1875 | | | | | | 3 | 6 | 9 | 11 | 14 | 17 | 20 | 23 | 26 |
| | | | | | | 1903 | 1931 | 1959 | 1987 | 2014 | 3 | 6 | 8 | 11 | 14 | 17 | 19 | 22 | 25 |
| **16** | 2041 | 2068 | 2095 | 2122 | 2148 | | | | | | 3 | 6 | 8 | 11 | 14 | 16 | 19 | 22 | 24 |
| | | | | | | 2175 | 2201 | 2227 | 2253 | 2279 | 3 | 5 | 8 | 10 | 13 | 16 | 18 | 21 | 23 |
| **17** | 2304 | 2330 | 2355 | 2380 | 2405 | | | | | | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| | | | | | | 2430 | 2455 | 2480 | 2504 | 2529 | 3 | 5 | 8 | 10 | 12 | 15 | 17 | 20 | 22 |
| **18** | 2553 | 2577 | 2601 | 2625 | 2648 | | | | | | 2 | 5 | 7 | 9 | 12 | 14 | 17 | 19 | 21 |
| | | | | | | 2672 | 2695 | 2718 | 2742 | 2765 | 2 | 4 | 7 | 9 | 11 | 14 | 16 | 18 | 21 |
| **19** | 2788 | 2810 | 2833 | 2856 | 2878 | | | | | | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| | | | | | | 2900 | 2923 | 2945 | [illegible] | 2989 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| **20** | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| **21** | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **22** | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| **23** | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| **24** | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| **25** | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| **26** | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| **27** | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| **28** | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **29** | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| **30** | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| **31** | 4914 | 4928 | 4942 | 4955 | 4969 | 4083 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| **32** | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| **33** | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| **34** | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| **35** | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| **36** | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| **37** | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| **38** | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| **39** | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| **40** | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| **41** | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **42** | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **43** | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **44** | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **45** | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **46** | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| **47** | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| **48** | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **49** | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |

## Table 1 – *(Continued)* Logarithms

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **50** | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **51** | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **52** | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| **53** | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **54** | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **55** | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **56** | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **57** | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **58** | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| **59** | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| **60** | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| **61** | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| **62** | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| **63** | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **64** | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **65** | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **66** | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **67** | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **68** | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| **69** | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| **70** | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| **71** | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **72** | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **73** | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **74** | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **75** | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| **76** | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| **77** | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **78** | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **79** | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **80** | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **81** | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **82** | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **83** | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **84** | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **85** | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **86** | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **87** | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **88** | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **89** | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **90** | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **91** | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **92** | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **93** | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **94** | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **95** | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **96** | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **97** | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **98** | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **99** | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |


https://stat9943.blogspot.com

# Table 2 – Antilogarithms

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| .00 | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .07 | 1175 | 1178 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .10 | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .15 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .17 | 1479 | 1483 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .18 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| .20 | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| .21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| .22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1687 | 1690 | 1694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| .23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .26 | 1820 | 1824 | 1828 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| .27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| .28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .30 | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .33 | 2138 | 2143 | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .34 | 2188 | 2193 | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2286 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| .40 | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| .43 | 2692 | 2698 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2924 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| .49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |

## Table 2 – *(Continued)* Antilogarithms



| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **.50** | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| **.51** | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **.52** | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **.53** | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **.54** | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **.55** | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| **.56** | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **.57** | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **.58** | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **.59** | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| **.60** | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| **.61** | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.62** | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.63** | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.64** | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.65** | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.66** | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| **.67** | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| **.68** | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| **.69** | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| **.70** | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| **.71** | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| **.72** | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| **.73** | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| **.74** | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| **.75** | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| **.76** | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| **.77** | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| **.78** | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| **.79** | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| **.80** | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| **.81** | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **.82** | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **.83** | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| **.84** | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| **.85** | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| **.86** | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| **.87** | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| **.88** | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| **.89** | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| **.90** | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| **.91** | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| **.92** | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| **.93** | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **.94** | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **.95** | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| **.96** | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| **.97** | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| **.98** | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| **.99** | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

## Table 3 – Squares and Square Roots



| $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ | $n$ | $n^2$ | $\sqrt{n}$ | $\sqrt{10n}$ |
|---|---|---|---|---|---|---|---|
| 1.0 | 1.00 | 1.000 | 3.162 | 5.5 | 30.25 | 2.345 | 7.416 |
| 1.1 | 1.21 | 1.049 | 3.317 | 5.6 | 31.36 | 2.366 | 7.483 |
| 1.2 | 1.44 | 1.095 | 3.464 | 5.7 | 32.49 | 2.387 | 7.550 |
| 1.3 | 1.69 | 1.140 | 3.606 | 5.8 | 33.64 | 2.408 | 7.616 |
| 1.4 | 1.96 | 1.183 | 3.742 | 5.9 | 34.81 | 2.429 | 7.681 |
| 1.5 | 2.25 | 1.225 | 3.873 | 6.0 | 36.00 | 2.449 | 7.746 |
| 1.6 | 2.56 | 1.265 | 4.000 | 6.1 | 37.21 | 2.470 | 7.810 |
| 1.7 | 2.89 | 1.304 | 4.123 | 6.2 | 38.44 | 2.490 | 7.874 |
| 1.8 | 3.24 | 1.342 | 4.243 | 6.3 | 39.69 | 2.510 | 7.937 |
| 1.9 | 3.61 | 1.378 | 4.359 | 6.4 | 40.96 | 2.530 | 8.000 |
| 2.0 | 4.00 | 1.414 | 4.472 | 6.5 | 42.25 | 2.550 | 8.062 |
| 2.1 | 4.41 | 1.449 | 4.583 | 6.6 | 43.56 | 2.569 | 8.124 |
| 2.2 | 4.84 | 1.483 | 4.690 | 6.7 | 44.89 | 2.558 | 8.185 |
| 2.3 | 5.29 | 1.517 | 4.796 | 6.8 | 46.24 | 2.608 | 8.246 |
| 2.4 | 5.76 | 1.549 | 48.99 | 6.9 | 47.61 | 2.627 | 8.307 |
| 2.5 | 6.25 | 1.581 | 5.000 | 7.0 | 49.00 | 2.646 | 8.367 |
| 2.6 | 6.76 | 1.612 | 5.099 | 7.1 | 50.41 | 2.665 | 8.426 |
| 2.7 | 7.29 | 1.643 | 5.196 | 7.2 | 51.84 | 2.683 | 8.485 |
| 2.8 | 7.84 | 1.673 | 5.292 | 7.3 | 53.29 | 2.702 | 8.544 |
| 2.9 | 8.41 | 1.703 | 5.385 | 7.4 | 54.76 | 2.720 | 8.602 |
| 3.0 | 9.00 | 1.732 | 5.477 | 7.5 | 56.25 | 2.739 | 8.660 |
| 3.1 | 9.61 | 1.761 | 5.568 | 7.6 | 57.76 | 2.757 | 8.718 |
| 3.2 | 10.24 | 1.789 | 5.657 | 7.7 | 59.29 | 2.775 | 8.775 |
| 3.3 | 10.89 | 1.817 | 5.745 | 7.8 | 60.84 | 2.793 | 8.832 |
| 3.4 | 11.56 | 1.844 | 5.831 | 7.9 | 62.41 | 2.811 | 8.888 |
| 3.5 | 12.25 | 1.871 | 5.916 | 8.0 | 64.00 | 2.828 | 8.944 |
| 3.6 | 12.96 | 1.897 | 6.000 | 8.1 | 65.61 | 2.846 | 9.000 |
| 3.7 | 13.69 | 1.924 | 6.083 | 8.2 | 67.24 | 2.864 | 9.055 |
| 3.8 | 14.44 | 1.949 | 6.164 | 8.3 | 68.89 | 2.881 | 9.110 |
| 3.9 | 15.21 | 1.975 | 6.245 | 8.4 | 70.56 | 2.898 | 9.165 |
| 4.0 | 16.00 | 2.000 | 6.325 | 8.5 | 72.25 | 2.915 | 9.220 |
| 4.1 | 16.81 | 2.025 | 6.403 | 8.6 | 73.96 | 2.933 | 9.274 |
| 4.2 | 17.64 | 2.049 | 6.481 | 8.7 | 75.69 | 2.950 | 9.327 |
| 4.3 | 18.49 | 2.074 | 6.557 | 8.8 | 77.44 | 2.966 | 9.381 |
| 4.4 | 19.36 | 2.098 | 6.633 | 8.9 | 79.21 | 2.983 | 9.434 |
| 4.5 | 20.25 | 2.121 | 6.708 | 9.0 | 81.00 | 3.000 | 9.487 |
| 4.6 | 21.16 | 2.145 | 6.782 | 9.1 | 82.81 | 3.017 | 9.539 |
| 4.7 | 22.09 | 2.168 | 6.856 | 9.2 | 84.64 | 3.033 | 9.592 |
| 4.8 | 23.04 | 2.191 | 6.928 | 9.3 | 86.49 | 3.050 | 9.644 |
| 4.9 | 24.01 | 2.214 | 7.000 | 9.4 | 88.36 | 3.066 | 9.695 |
| 5.0 | 25.00 | 2.236 | 7.071 | 9.5 | 90.25 | 3.082 | 9.747 |
| 5.1 | 26.01 | 2.258 | 7.141 | 9.6 | 92.16 | 3.098 | 9.798 |
| 5.2 | 27.04 | 2.280 | 7.211 | 9.7 | 94.09 | 3.114 | 9.849 |
| 5.3 | 28.09 | 2.302 | 7.280 | 9.8 | 96.04 | 3.130 | 9.899 |
| 5.4 | 29.16 | 2.324 | 7.348 | 9.9 | 98.01 | 3.146 | 9.950 |

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